

Exterior Calculus (cse274)

Albert Chern

Exterior Calculus

- Think of **multivariable calculus** geometrically

$$\begin{array}{c} f(x, y, z) \\ \nabla f \\ \frac{\partial f}{\partial y} \end{array}$$

$$\begin{array}{c} f : M \rightarrow \mathbb{R} \\ df : TM \xrightarrow{\text{linear}} \mathbb{R} \\ \text{if } TM \text{ is equipped with } \langle \cdot, \cdot \rangle \quad \swarrow \quad \downarrow \text{if } T^*M \text{ has a basis } dx, dy, dz \\ \nabla f = df^\sharp \end{array}$$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

Exterior Calculus

- Think of **multivariable calculus** geometrically
- Embrace **differential forms**

$$f : M \rightarrow \mathbb{R}$$

$$df : TM \xrightarrow{\text{linear}} \mathbb{R}$$

if TM is equipped
with $\langle \cdot, \cdot \rangle$

if T^*M has a basis
 dx, dy, dz

$$\nabla f = df^\sharp$$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

Exterior Calculus

- Think of **multivariable calculus** geometrically
- Embrace **differential forms**

vector algebra

$$\mathbf{a} \cdot \mathbf{b}$$

$$\mathbf{a} \times \mathbf{b}$$

$$\mathbf{a}^\top$$

vector calculus

$$\nabla f$$

$$\nabla \times \mathbf{u}$$

$$\nabla \cdot \mathbf{u}$$

$$\mathbf{u} \cdot \nabla \mathbf{v}$$

exterior algebra

$$i_{\mathbf{a}}\beta$$

$$\alpha \wedge \beta$$

$$\alpha^\sharp$$

$$f^*\alpha$$

$$\star\alpha$$

exterior calculus

$$d\alpha$$

$$d\star\alpha$$

$$di_{\mathbf{u}}\beta$$

$$i_{\mathbf{u}}d\beta$$

History of Differential Forms

- 1739 Alexis Clairaut: $A dx + B dy = df$ if and only if $\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} = 0$
 $\sum_i g_i dx_i = df$ if and only if $\frac{\partial g_i}{\partial x_j} - \frac{\partial g_j}{\partial x_i} = 0$ for every i, j
- 1788 Lagrange: k-forms $u dydz + v dzdx + w dxdy$ $dxdy = -dydx$
- Lagrange (1762), Gauss (1813), Ostrogradsky (1826), Green (1828), Poisson (1829):

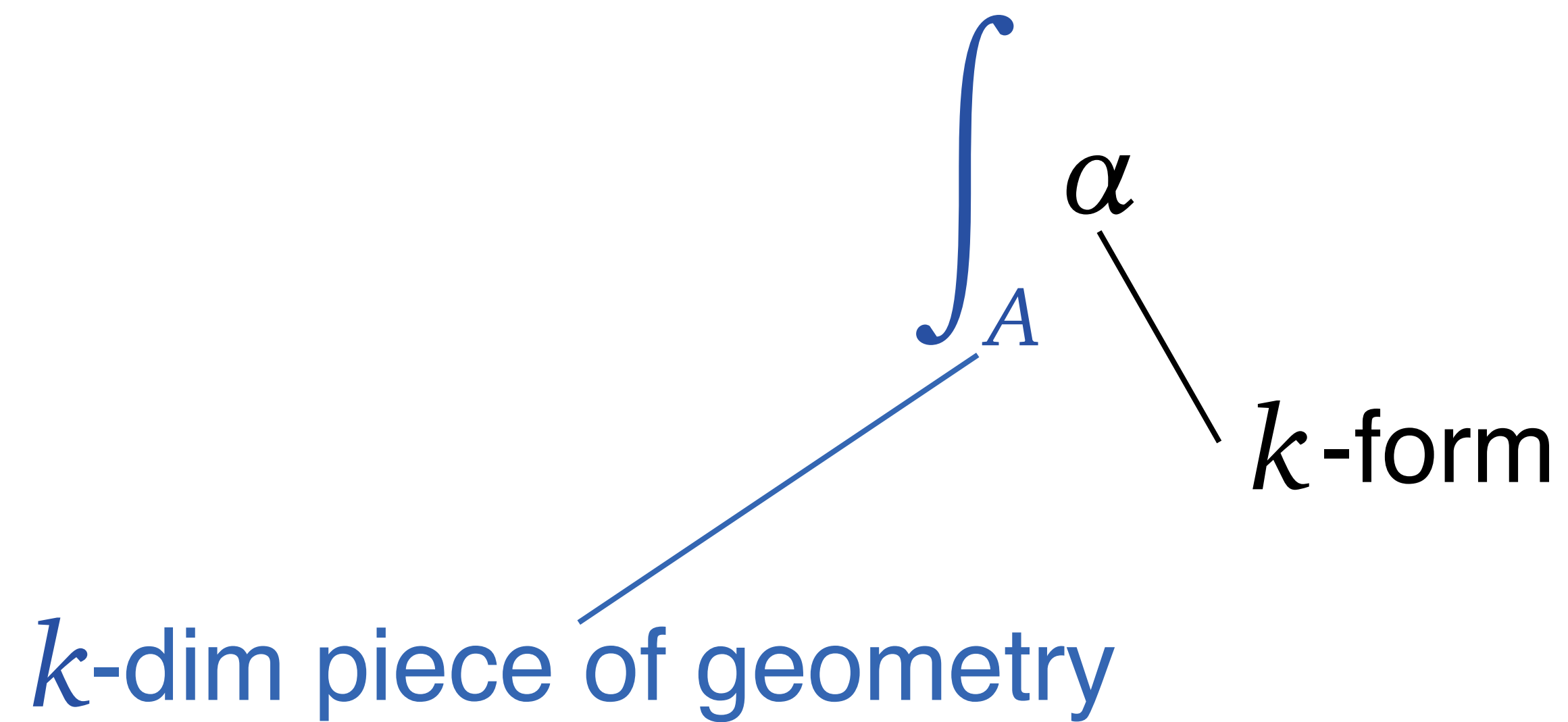
$$\oint_{\partial V} u dydz + v dzdx + w dxdy = \iiint_V \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) dxdydz$$

- 1830's Ostrogradsky: change of variable for multiple integrals
- 1840's Cauchy, Thomson (Kelvin), Stokes: curl theorem
- Hamilton (1840's), Grassmann (1860's): quaternions, ∇ , exterior algebra, modern linear algebra
- Cauchy, Riemann, Thomson, Stokes, Helmholtz, Betti, Poincaré (1830–1880's): non-simply-connected domains, topology
- 1873 Maxwell: preliminary chapter of electromagnetism

History of Differential Forms

- Weierstrass, Cantor (later half of 19th century): No more infinitesimals in calculus, use limits.
- Gibbs, Heaviside (1880–90's): Vector Analysis. No more quaternions, use dot and cross product.
- Elie Cartan (1899): Symbolic understanding of differential forms using Grassmann algebra.
- Goursat (1917): Generalized Stokes Theorem for forms.
- De Rham (1930's): k -chains. Betti, Poincaré & Mayer's homology theory.
- Cartan, de Rham, Weyl, Whitney, Chevalley (1930–50): Definition for differential form settled.
- Flender (1964): *Differential forms with applications to physical sciences*.

To-be-integrated Objects



The diagram shows the mathematical expression $\int_A \alpha$. A blue line points from the text " k -dim piece of geometry" to the subscript A . A black line points from the text " k -form" to the symbol α .

$$\int_A \alpha$$

k -dim piece of geometry

k -form

To-be-integrated Objects

evaluation
of a function

$$\int_p f = f(p)$$

value
per point

p

circulation

$$\int_C \eta$$

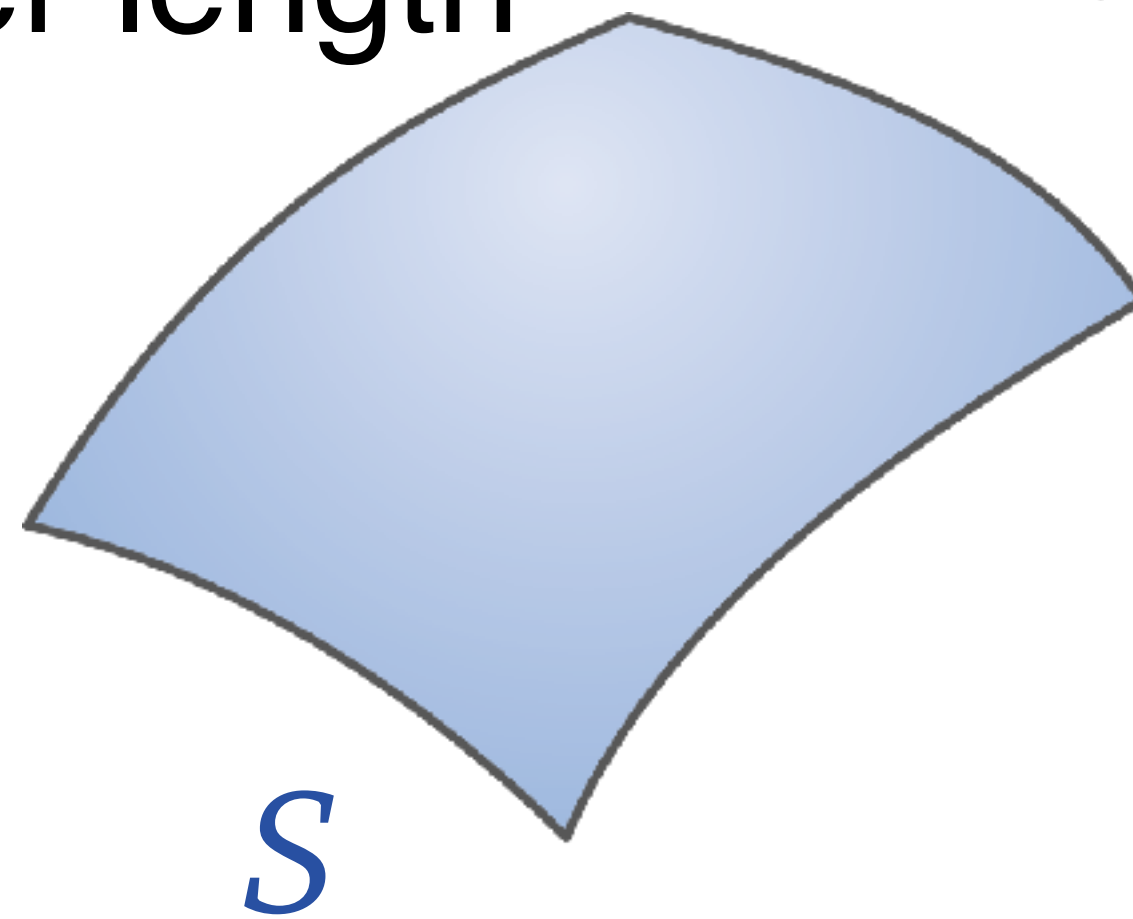
circulation
per length



flux

$$\int_S \omega$$

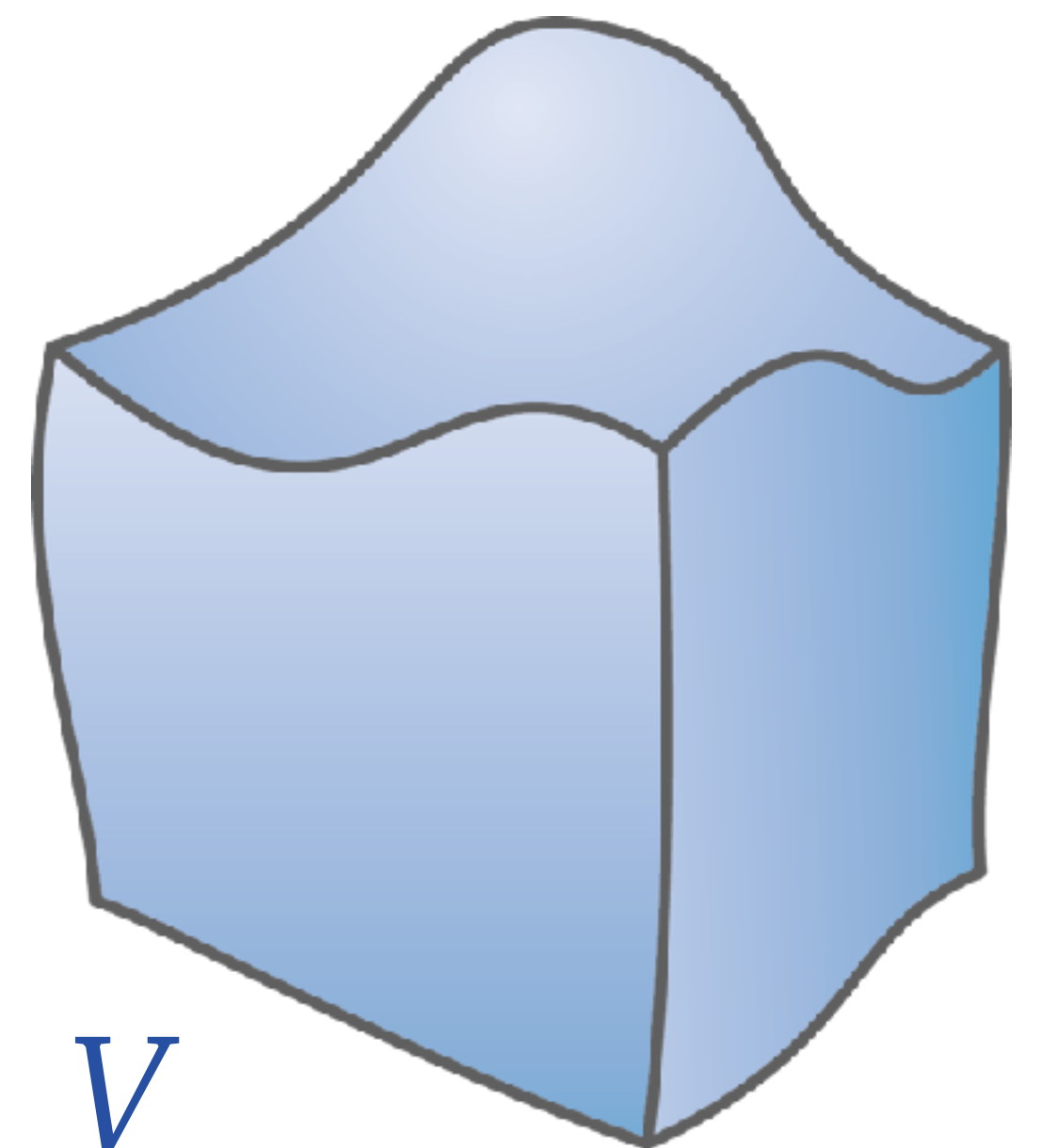
flux per
area



mass

$$\int_V \rho$$

mass per
volume



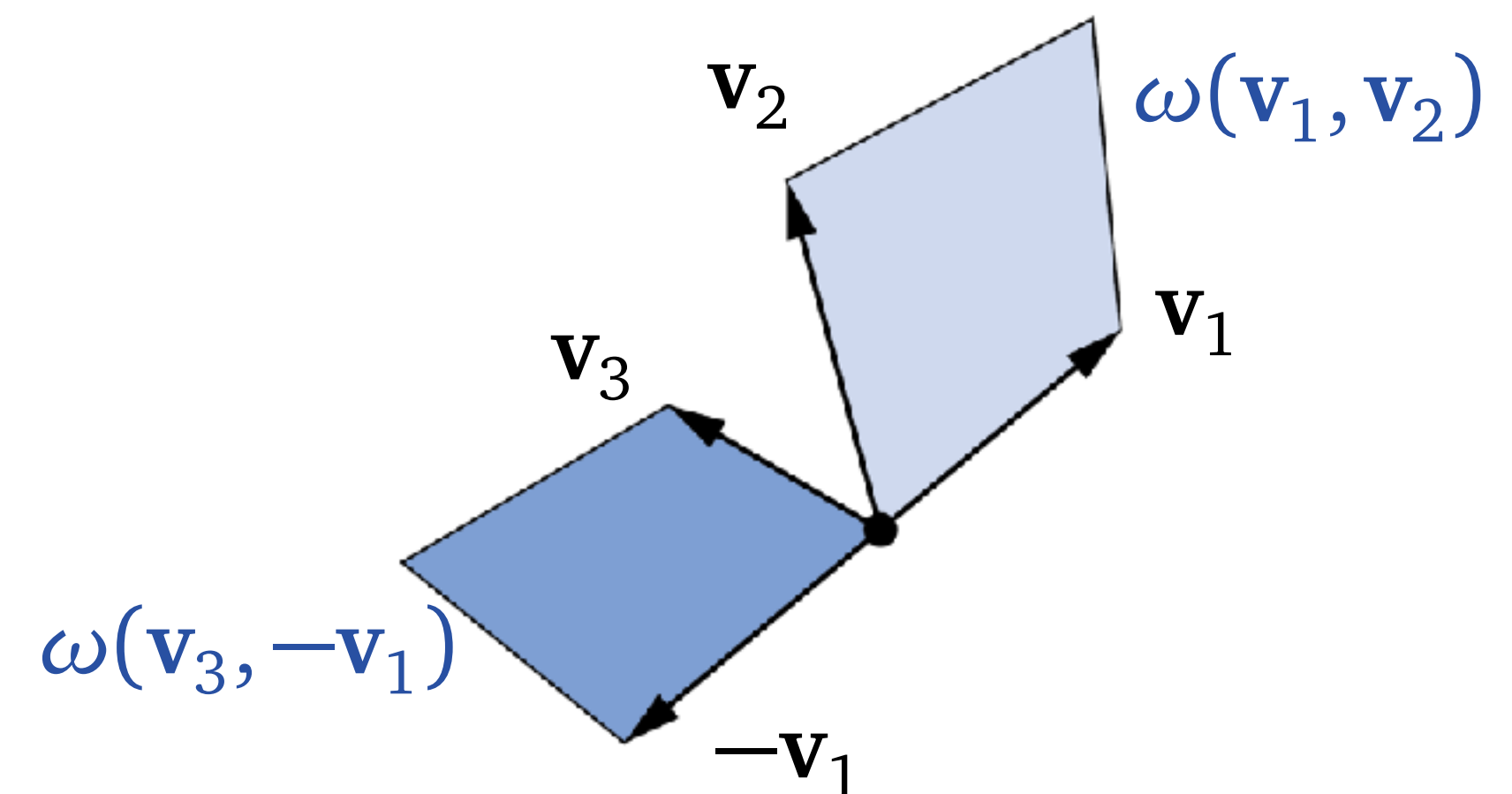
Alternating multilinear forms

$$\omega|_p \in \bigwedge^k T_p^* M$$

$$\omega|_p : \underbrace{T_p M \times \cdots \times T_p M}_k \xrightarrow{\text{multilinear}} \mathbb{R}$$

$\omega|_p(\mathbf{v}_1, \dots, \mathbf{v}_k)$ represents the “flux” over the infinitesimal parallelepiped spanned by $\mathbf{v}_1, \dots, \mathbf{v}_k$

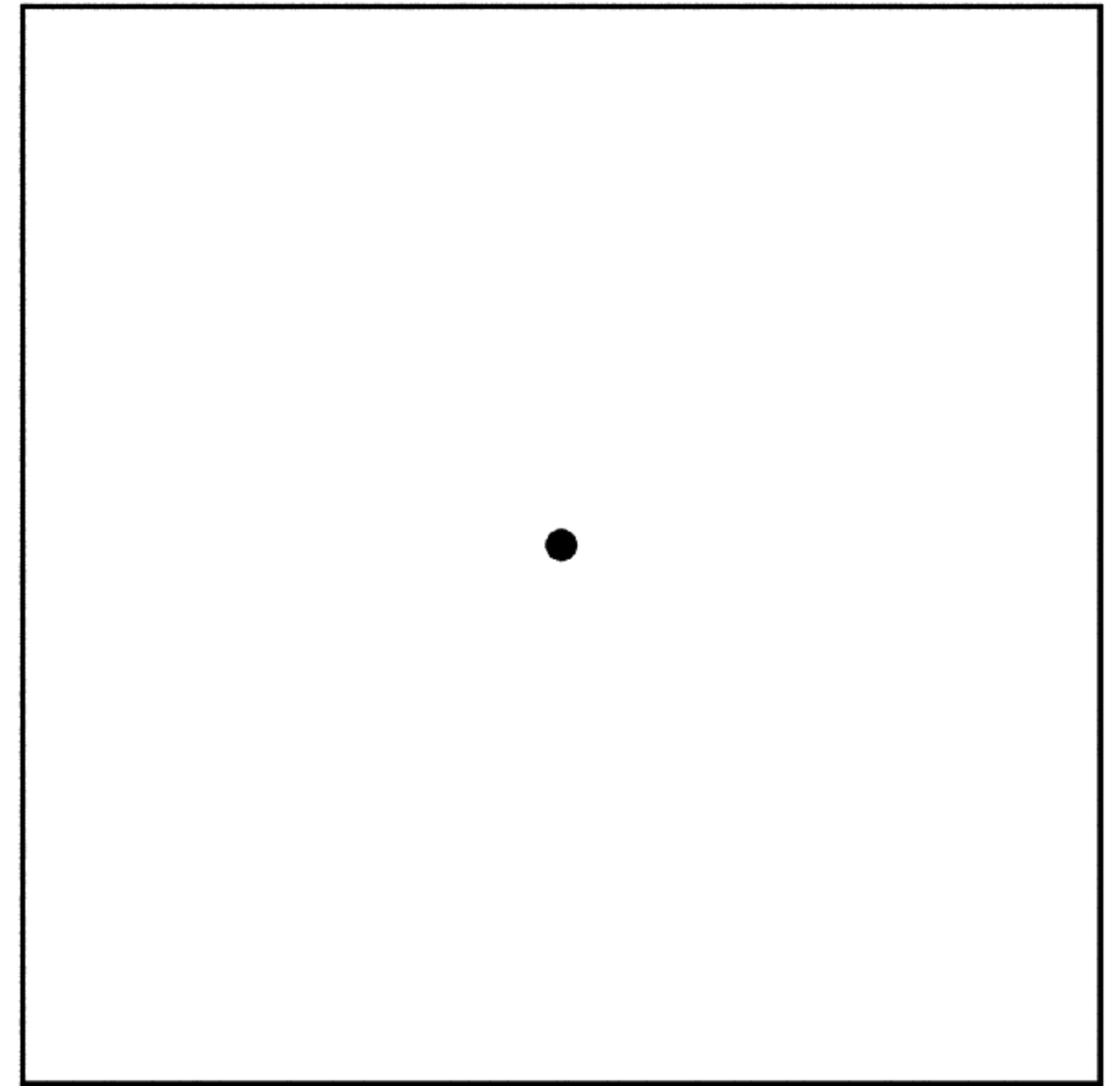
$$\begin{aligned} \omega(\dots, \mathbf{v}_i, \dots, \mathbf{v}_j, \dots) \\ = -\omega(\dots, \mathbf{v}_j, \dots, \mathbf{v}_i, \dots) \end{aligned}$$



1-form

V vector space

$V^* = \{\alpha: V \xrightarrow{\text{linear}} \mathbb{R}\}$ dual space



V

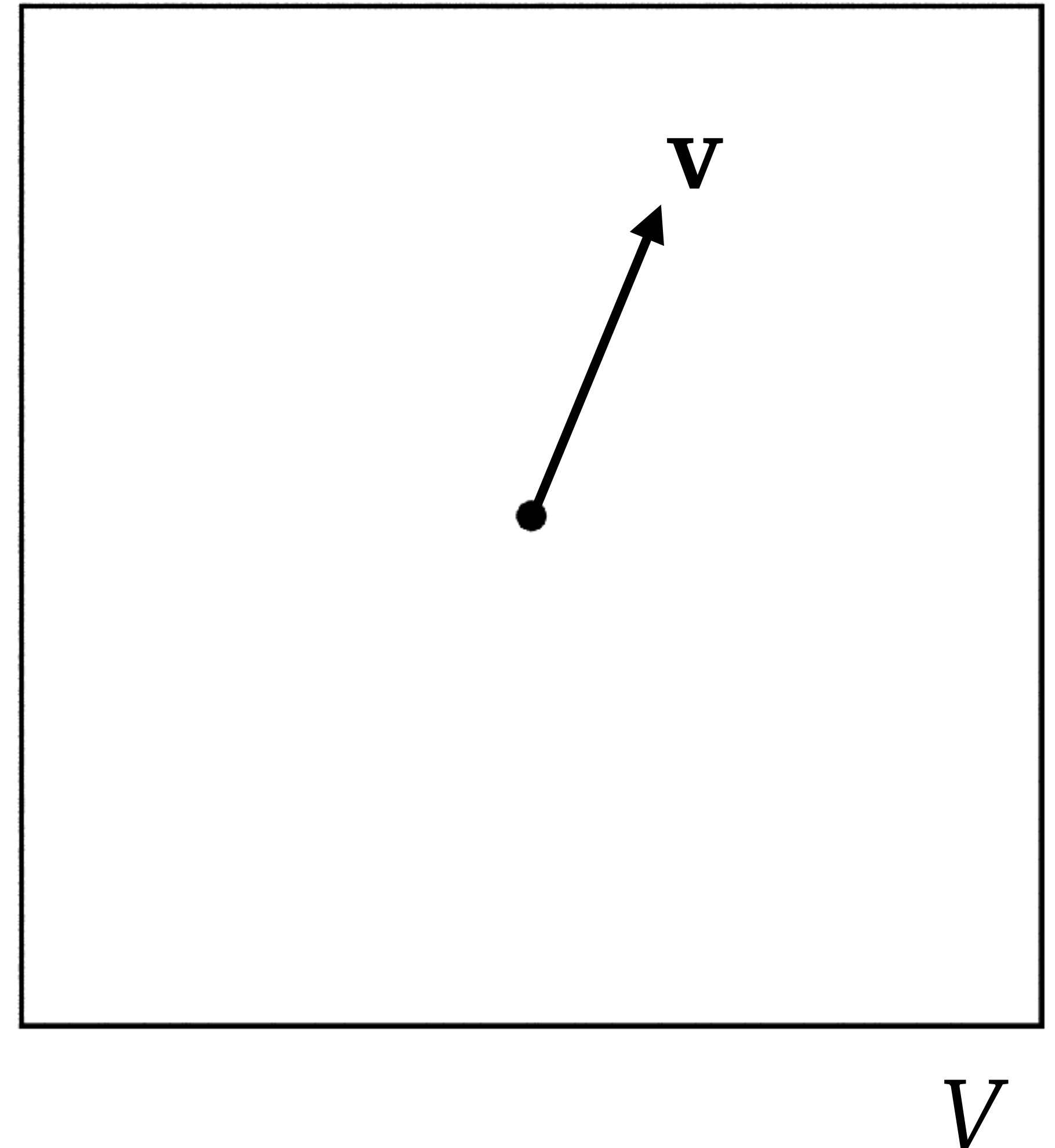
1-form

V vector space

$V^* = \{\alpha: V \xrightarrow{\text{linear}} \mathbb{R}\}$ dual space

$\mathbf{v} \in V$

vector
point
arrow



1-form

V vector space

$V^* = \{\alpha: V \xrightarrow{\text{linear}} \mathbb{R}\}$ dual space

$\alpha \in V^*$

covector

linear functional

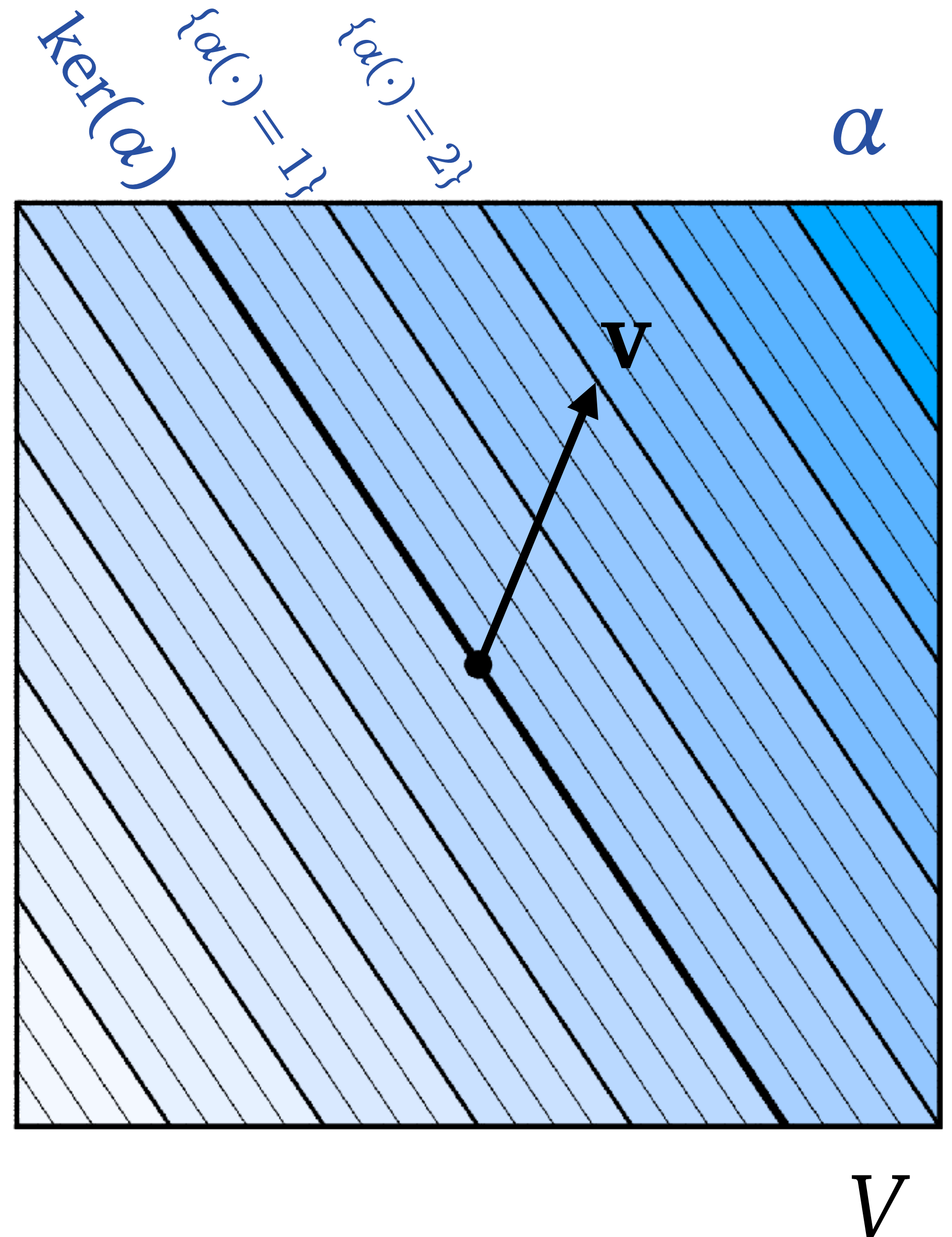
parallel hyperplanes

$\mathbf{v} \in V$

vector

point

arrow



1-form

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$\alpha \in V^*$

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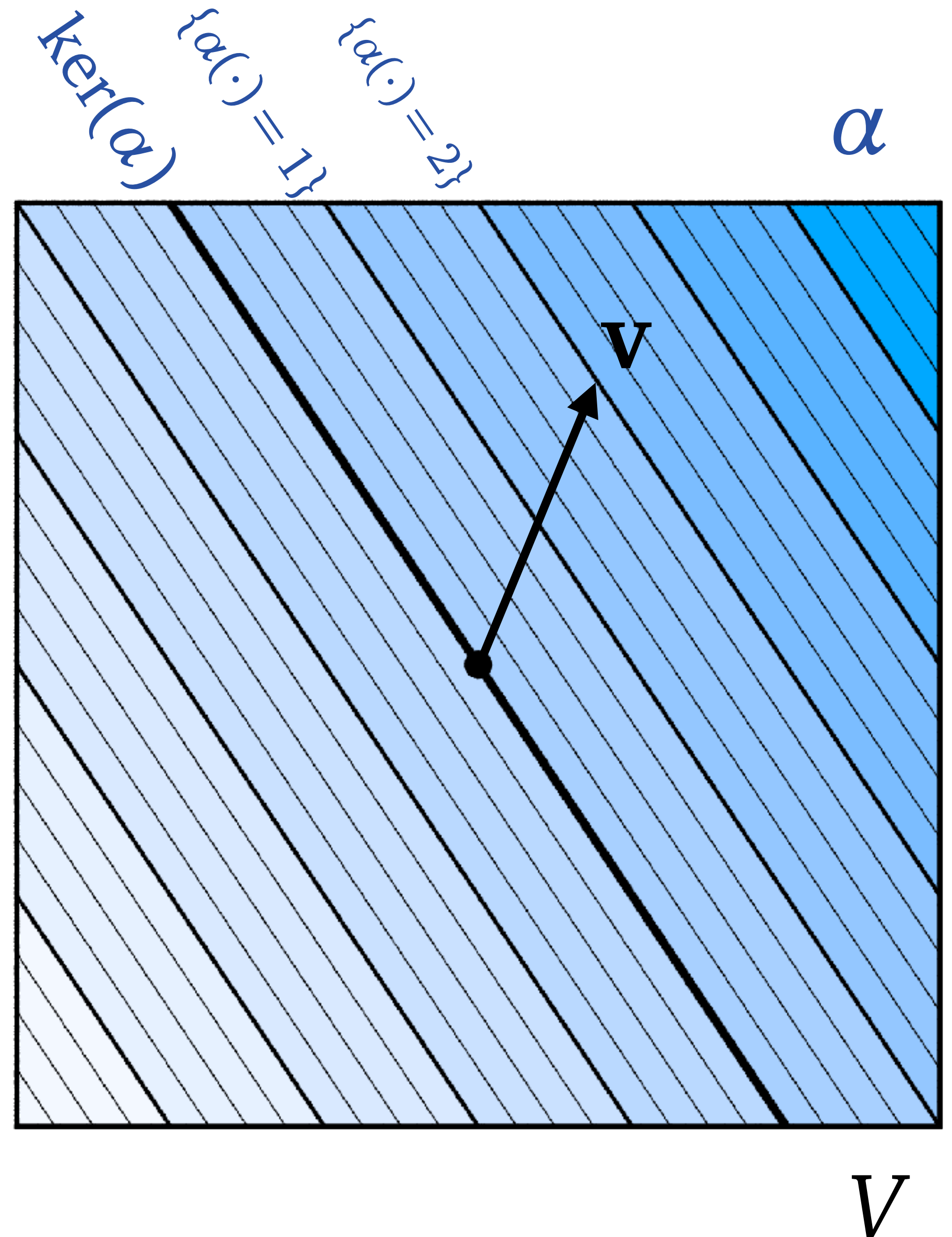
arrow

$$\langle \alpha | \mathbf{v} \rangle = 2.0$$

dual pairing

function evaluation

intersections of arrow & planes



1-form (when there is metric)

$$(V, \langle \cdot, \cdot \rangle)$$

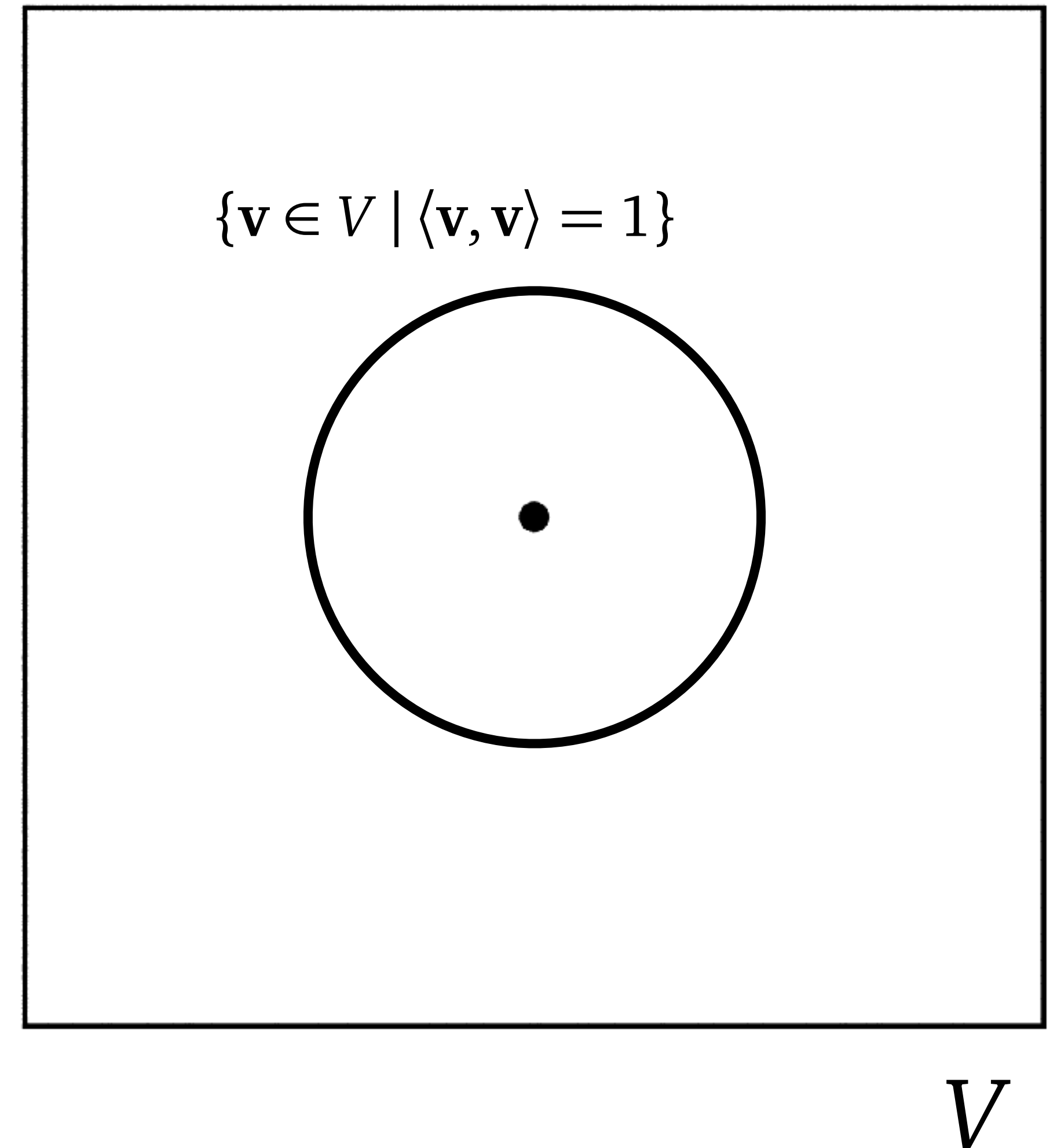
$$\langle \cdot, \cdot \rangle : V \times V \xrightarrow{\text{symmetric bilinear}} \mathbb{R}$$

“musical” correspondence

$$V \begin{array}{c} \xrightarrow{b} \\ \xleftarrow{\#} \end{array} V^*$$

$$\langle \alpha | \cdot \rangle = \langle \alpha^\#, \cdot \rangle$$

$$\langle \mathbf{v}^\flat | \cdot \rangle = \langle \mathbf{v}, \cdot \rangle$$



1-form (when there is metric)

$$(V, \langle \cdot, \cdot \rangle)$$

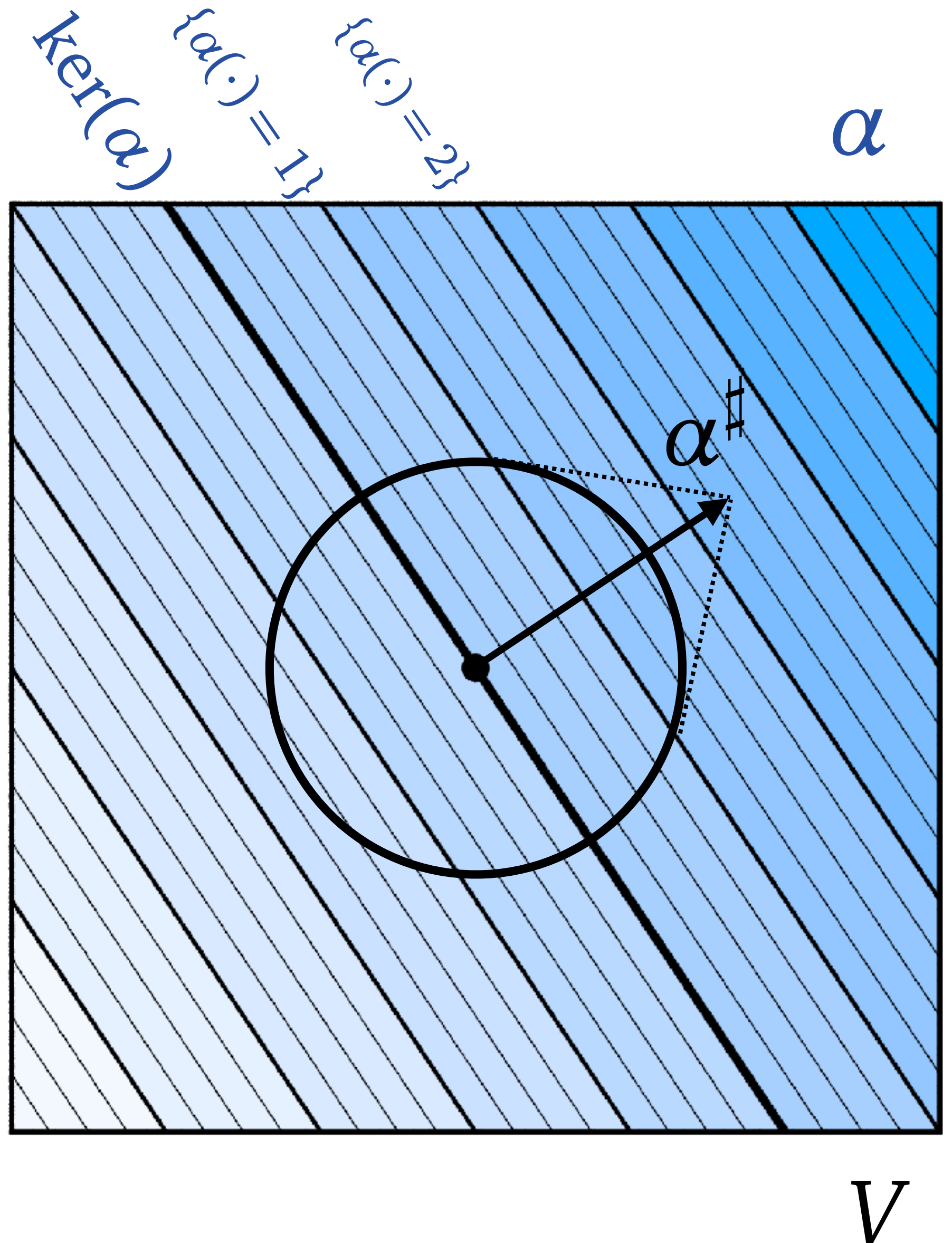
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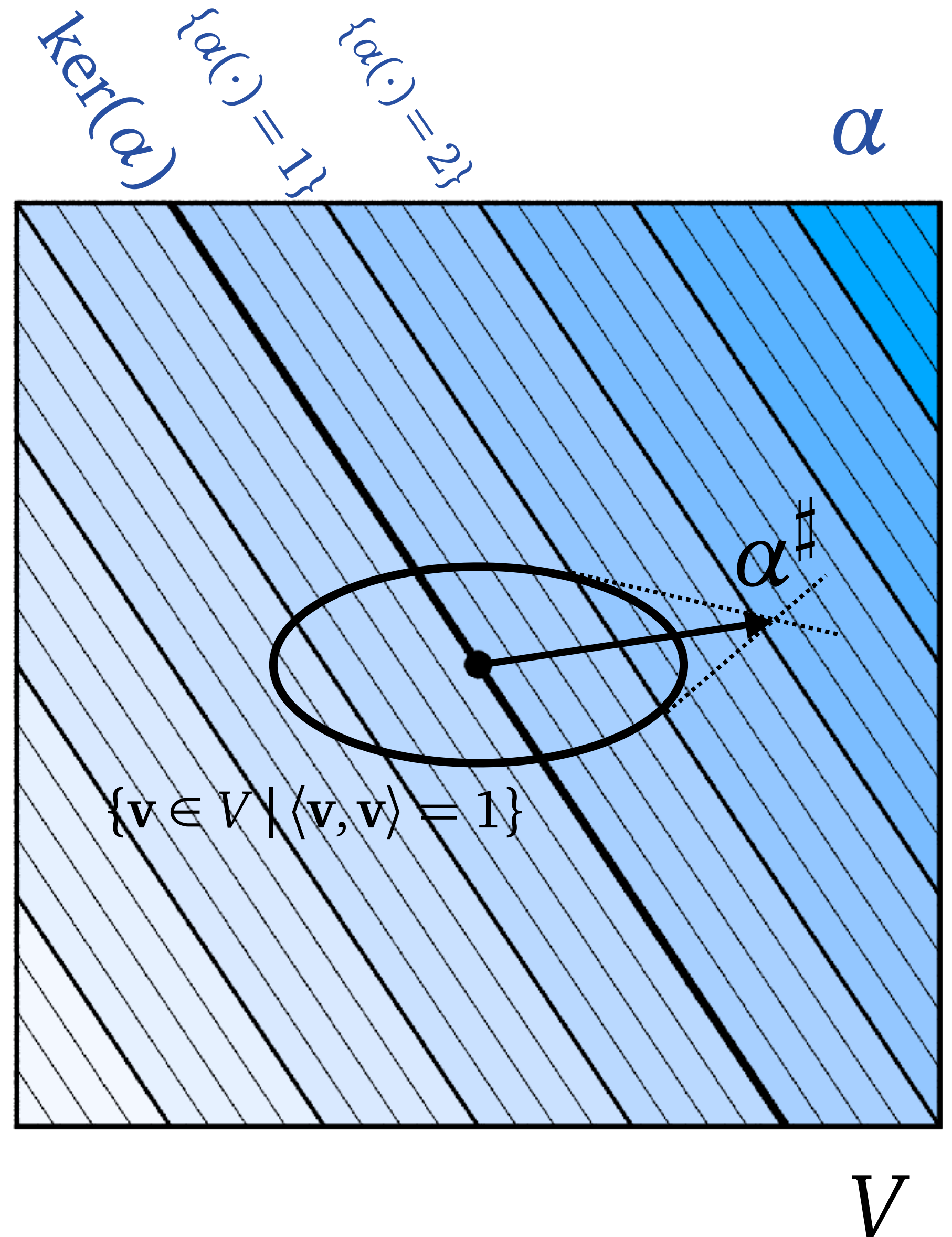
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1-form (when there is metric)

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$$\langle \cdot, \cdot \rangle : V \times V \xrightarrow{\text{symmetric bilinear}} \mathbb{R}$$

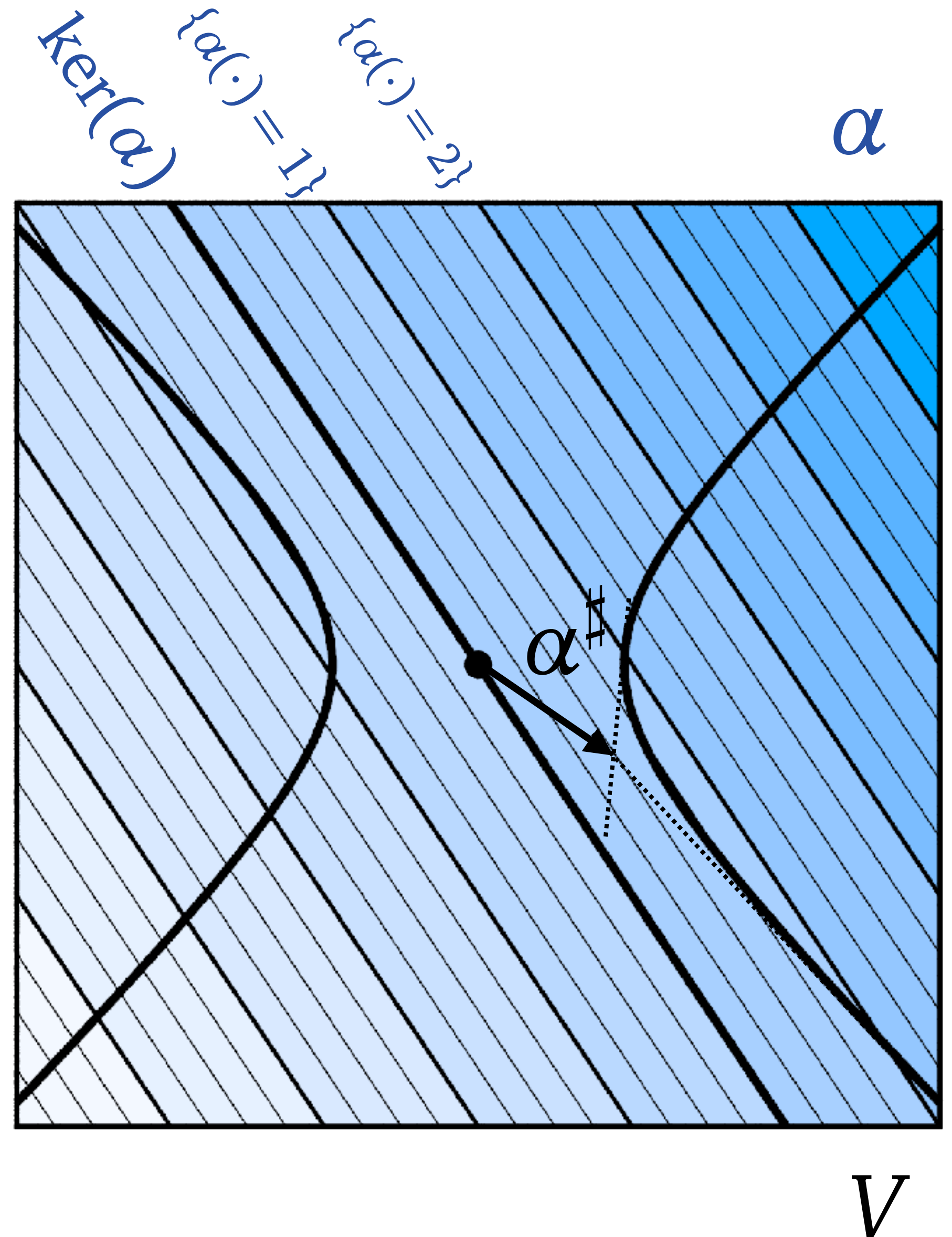
“musical” correspondence

$$V \begin{array}{c} \xrightarrow{b} \\ \xleftarrow{\#} \end{array} V^*$$

$$\langle \alpha | \cdot \rangle = \langle \alpha^\sharp, \cdot \rangle$$

$$\langle \mathbf{v}^\flat | \cdot \rangle = \langle \mathbf{v}, \cdot \rangle$$

$$\{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{v} \rangle = 1\}$$



1-form (when there is metric)

$$(V, \langle \cdot, \cdot \rangle)$$

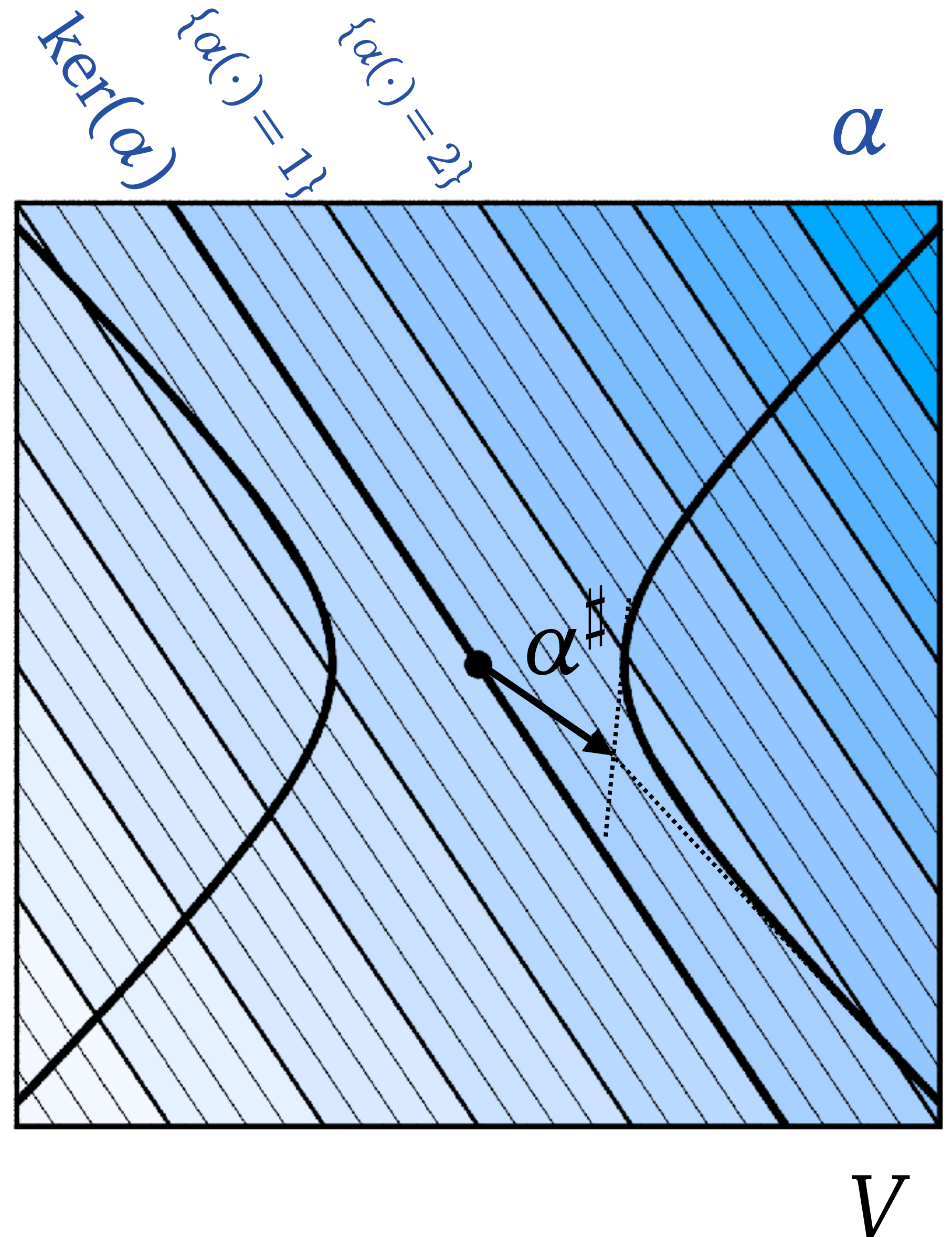
$$\langle \cdot, \cdot \rangle : V \times V \xrightarrow{\text{symmetric bilinear}} \mathbb{R}$$

“musical” correspondence

$$V \begin{array}{c} \xrightarrow{b} \\ \xleftarrow{\#} \end{array} V^*$$

$$\langle \alpha, \beta \rangle_{V^*} := \langle \alpha^\#, \beta^\# \rangle_V$$

$$\{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{v} \rangle = 1\}$$

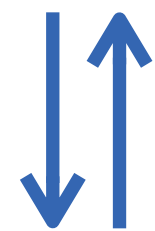


Dual basis

V vector space

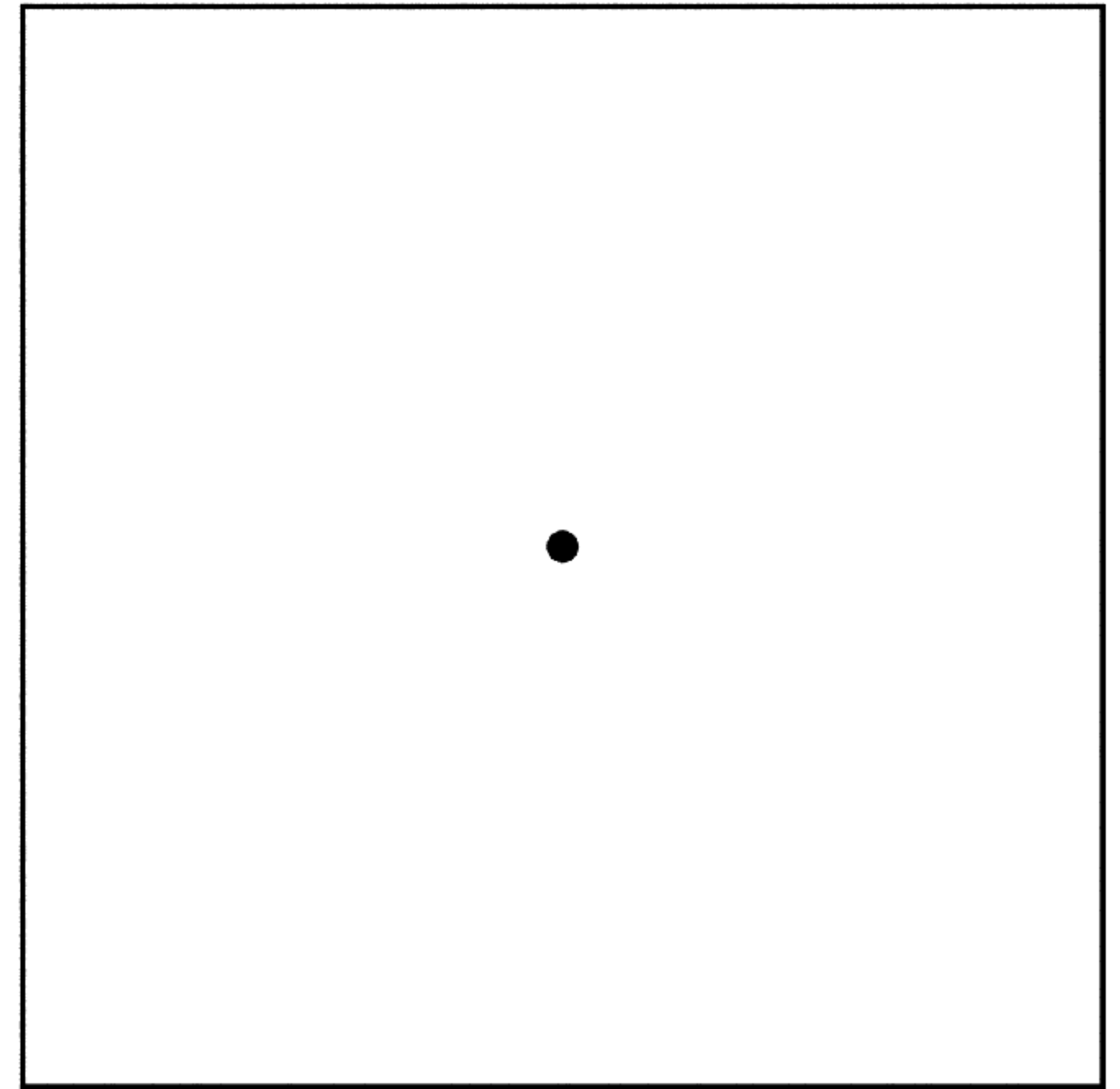
**a basis
determines a dual basis**

$\mathbf{v}_1, \dots, \mathbf{v}_n$



$\alpha_1, \dots, \alpha_n$

$$\langle \alpha_i | \mathbf{v}_j \rangle = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$



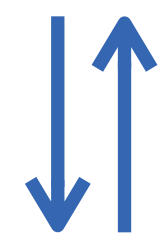
V

Dual basis

V vector space

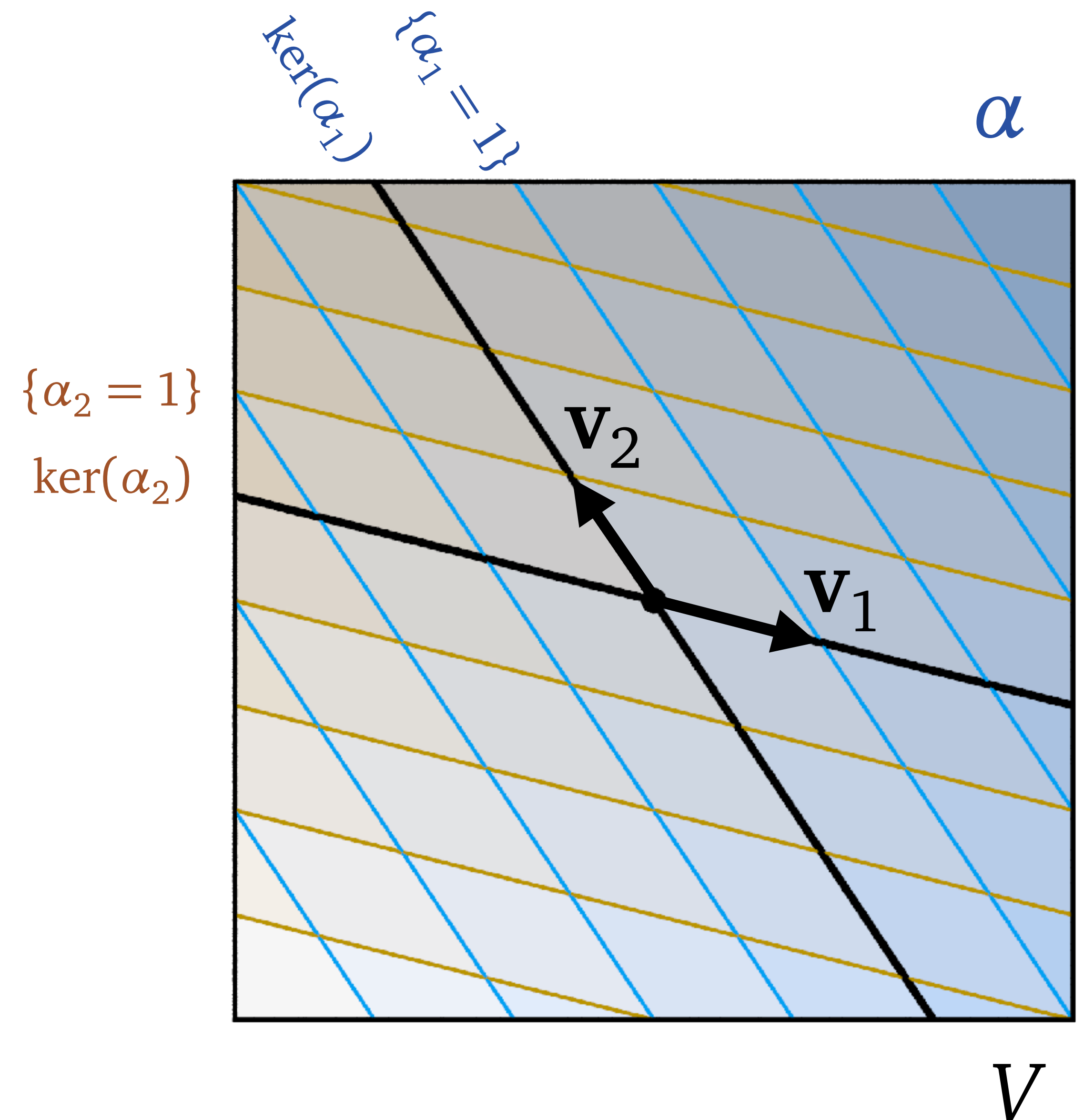
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$\alpha_1, \dots, \alpha_n$

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Derivative of a function

M manifold

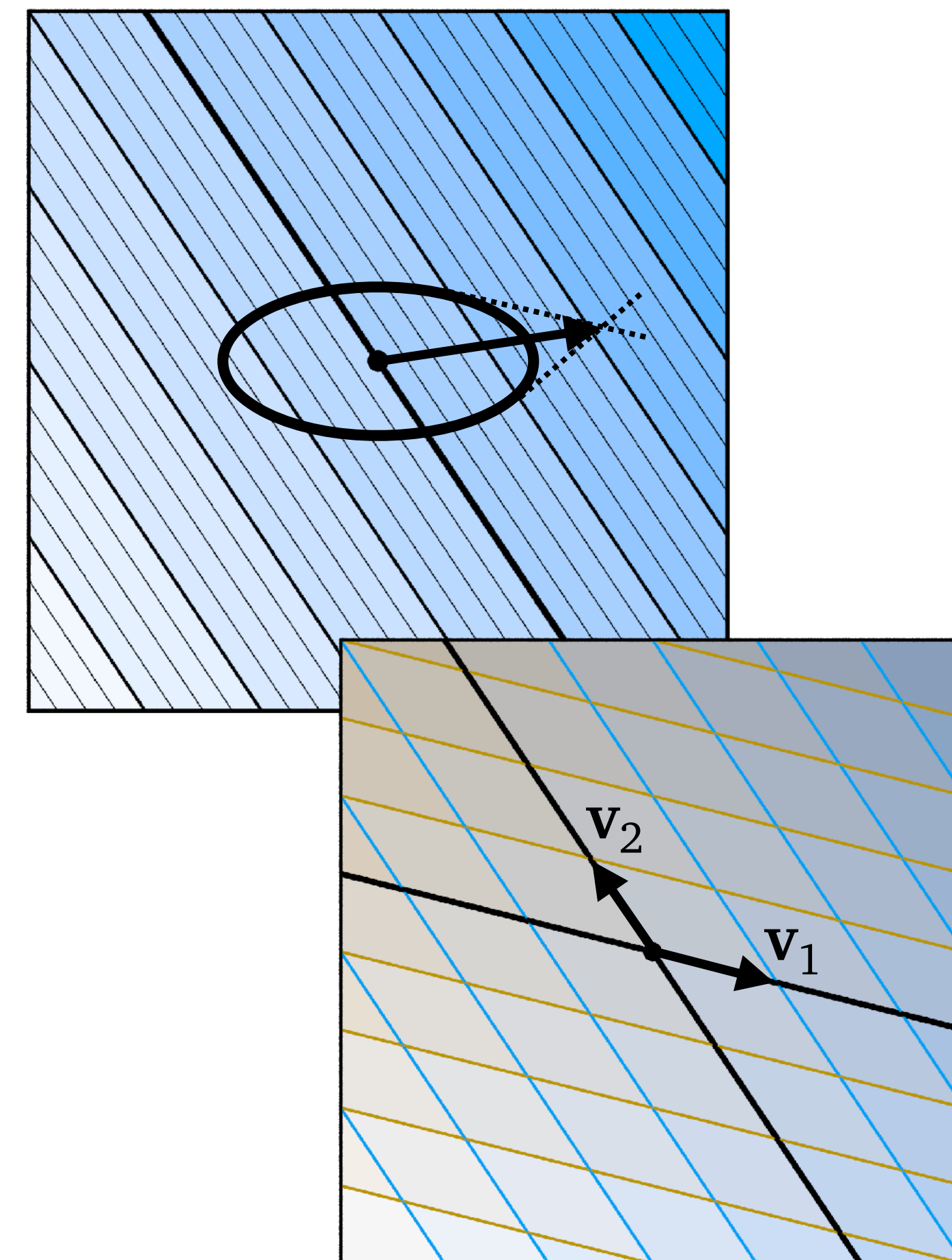
$$f : M \rightarrow \mathbb{R}$$

$df|_p \in T_p^*M$ is a covector (1-form) field

$\langle df | \mathbf{v} \rangle$ is the rate of change of f
as we move at velocity $\mathbf{v}$

- If we have a metric $\nabla f = df^\sharp$
- If we have a covector basis $dx_1, \dots, dx_n$

$$df = \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_n} dx_n$$



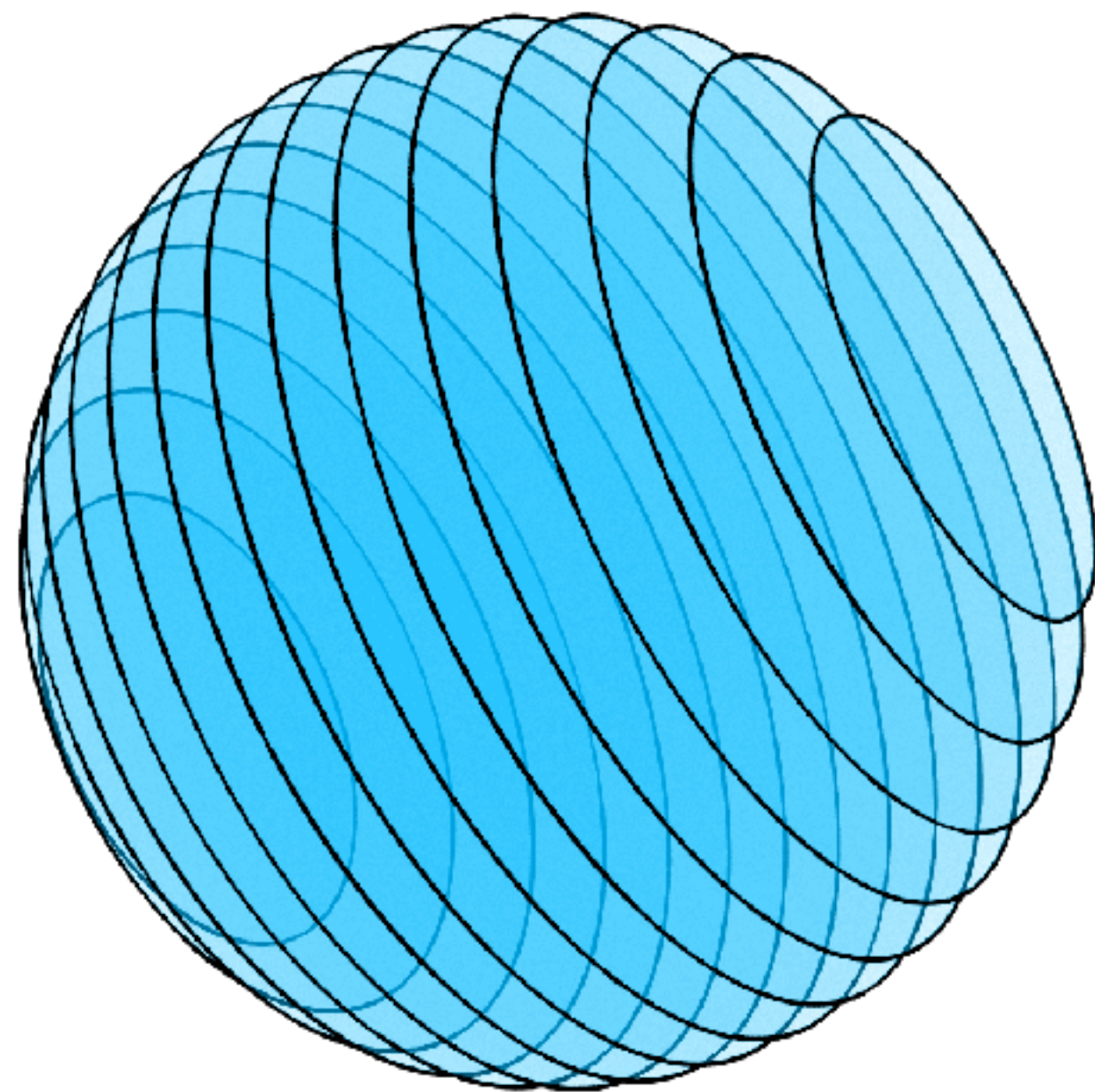
Wedge product

$$(\cdot) \wedge (\cdot): \bigwedge^k V^* \times \bigwedge^\ell V^* \rightarrow \bigwedge^{k+\ell} V^*$$

- Associative $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$
- For 1-forms $\alpha_1, \dots, \alpha_k$

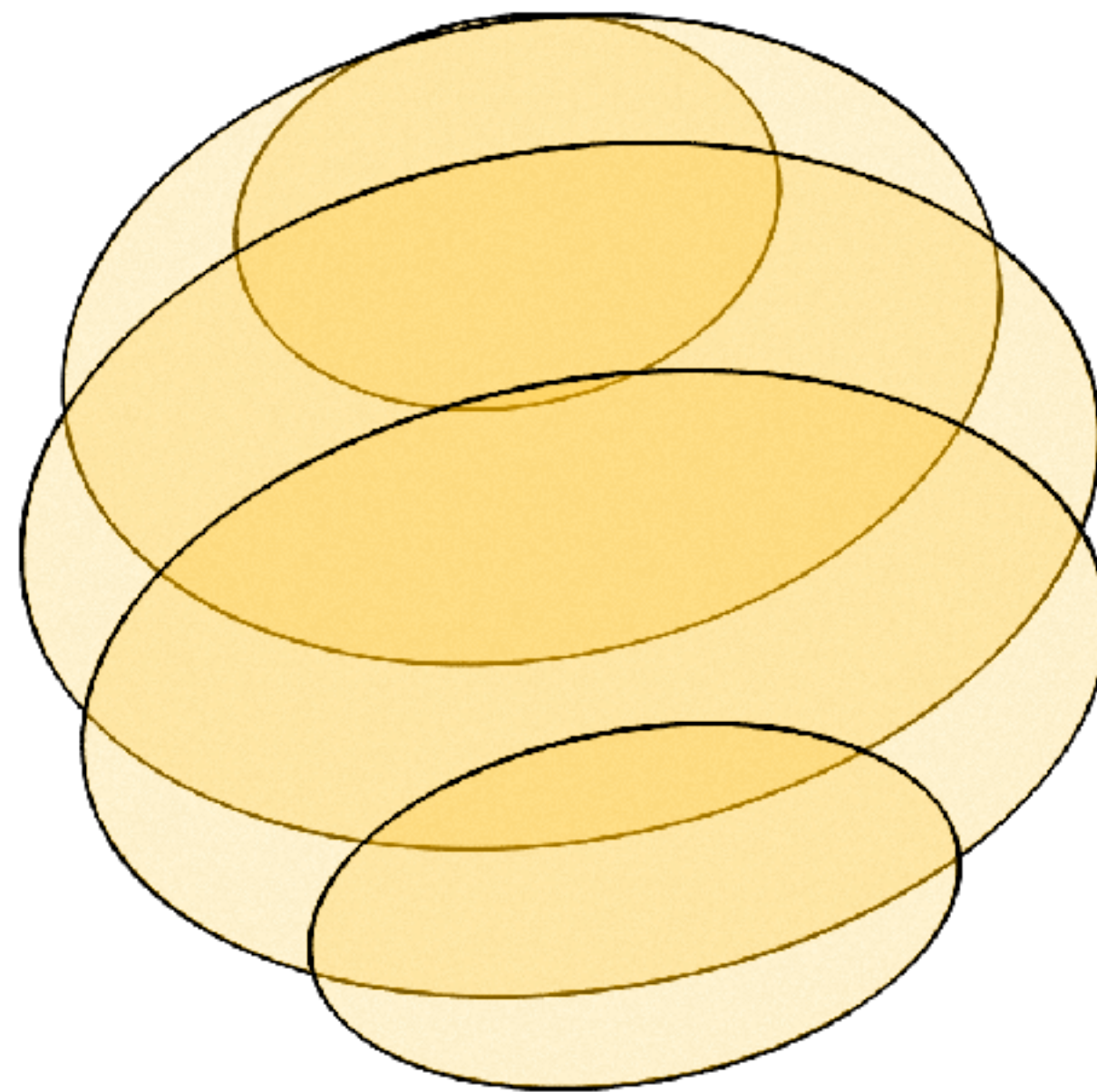
$$(\alpha_1 \wedge \dots \wedge \alpha_k)(\mathbf{v}_1, \dots, \mathbf{v}_k) = \det \begin{pmatrix} \langle \alpha_1 | \mathbf{v}_1 \rangle & \cdots & \langle \alpha_1 | \mathbf{v}_k \rangle \\ \vdots & & \vdots \\ \langle \alpha_k | \mathbf{v}_1 \rangle & \cdots & \langle \alpha_k | \mathbf{v}_k \rangle \end{pmatrix}$$

Wedge product



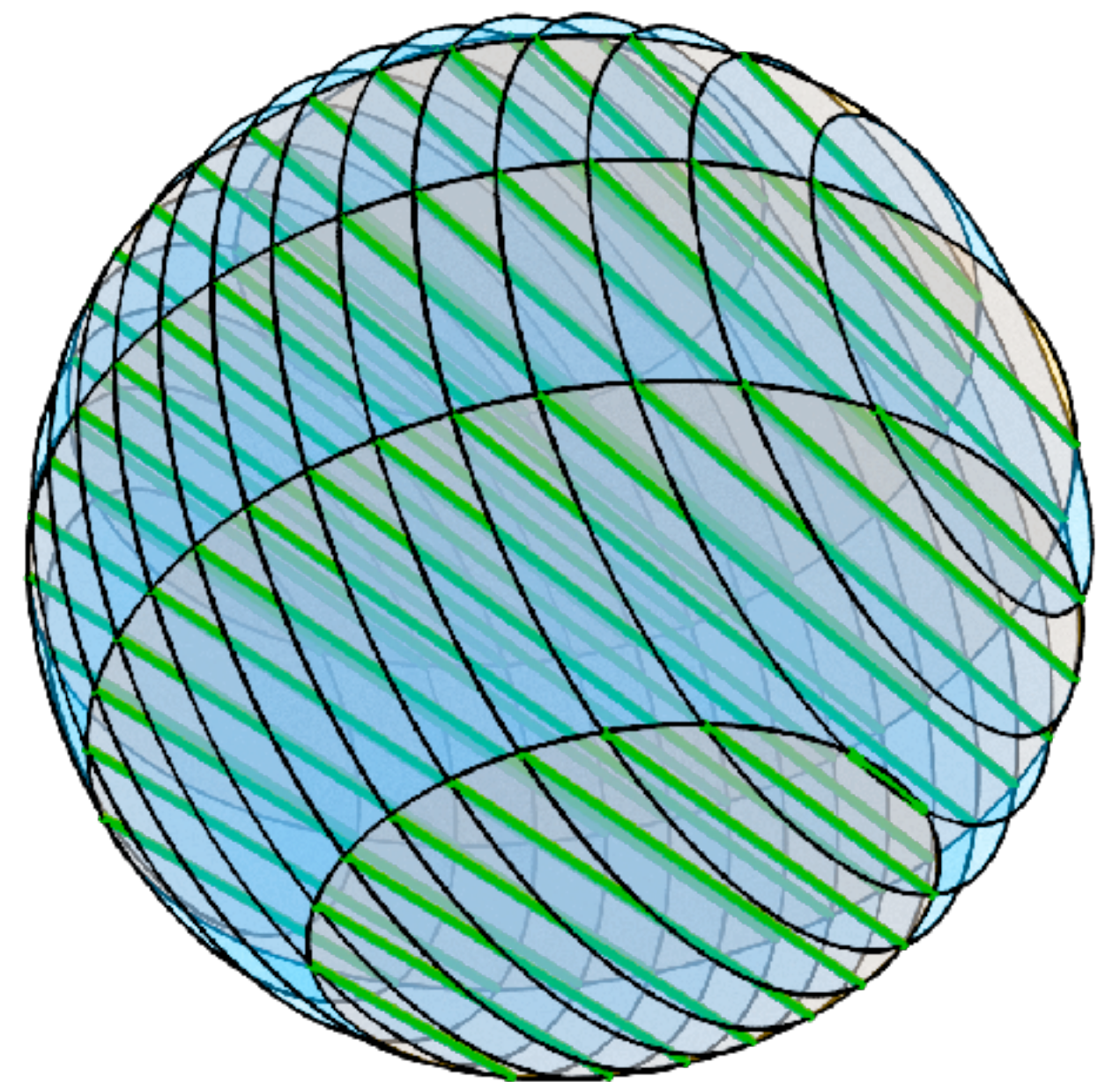
α
1-form

$\wedge$



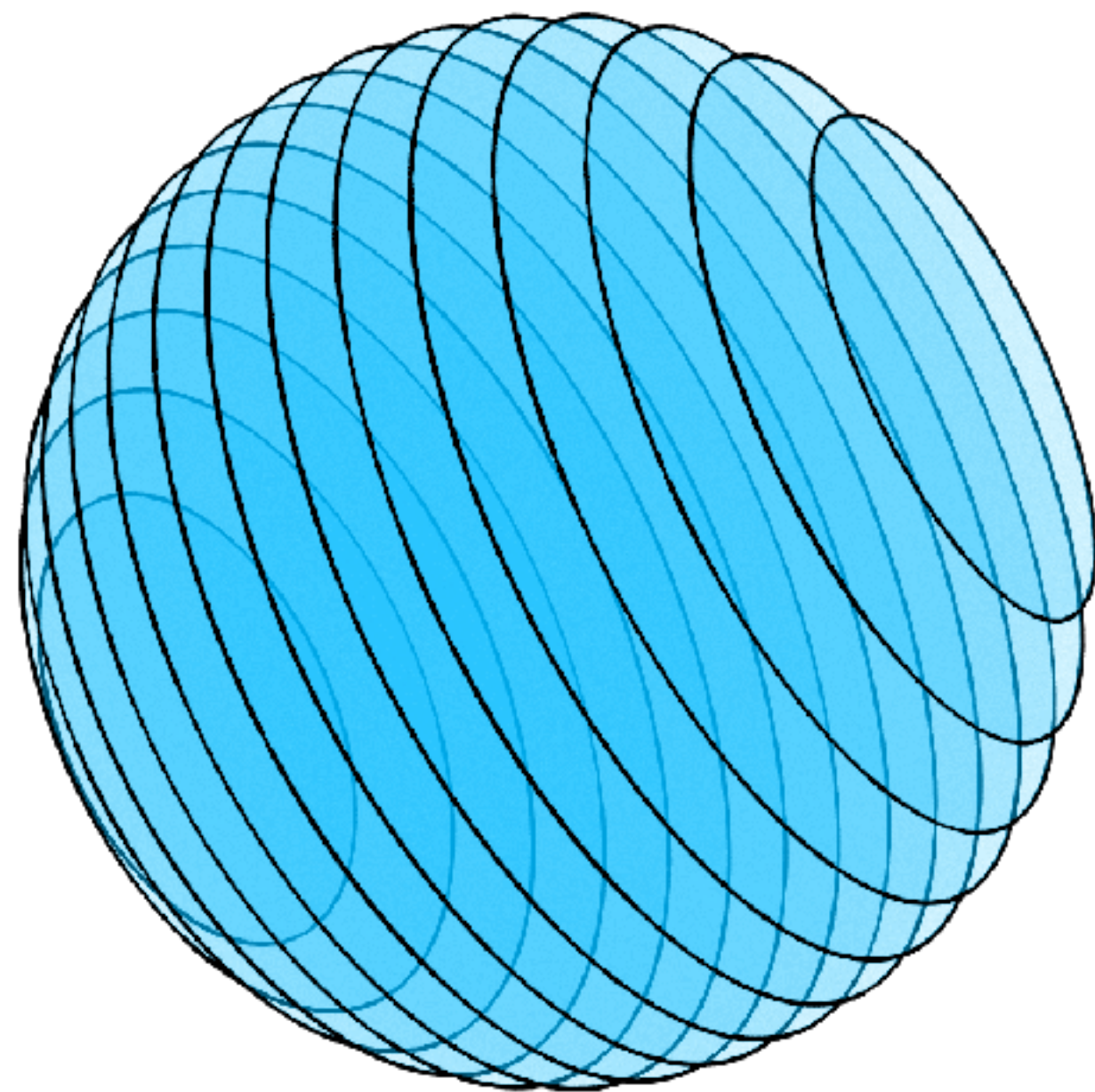
β
1-form

$=$



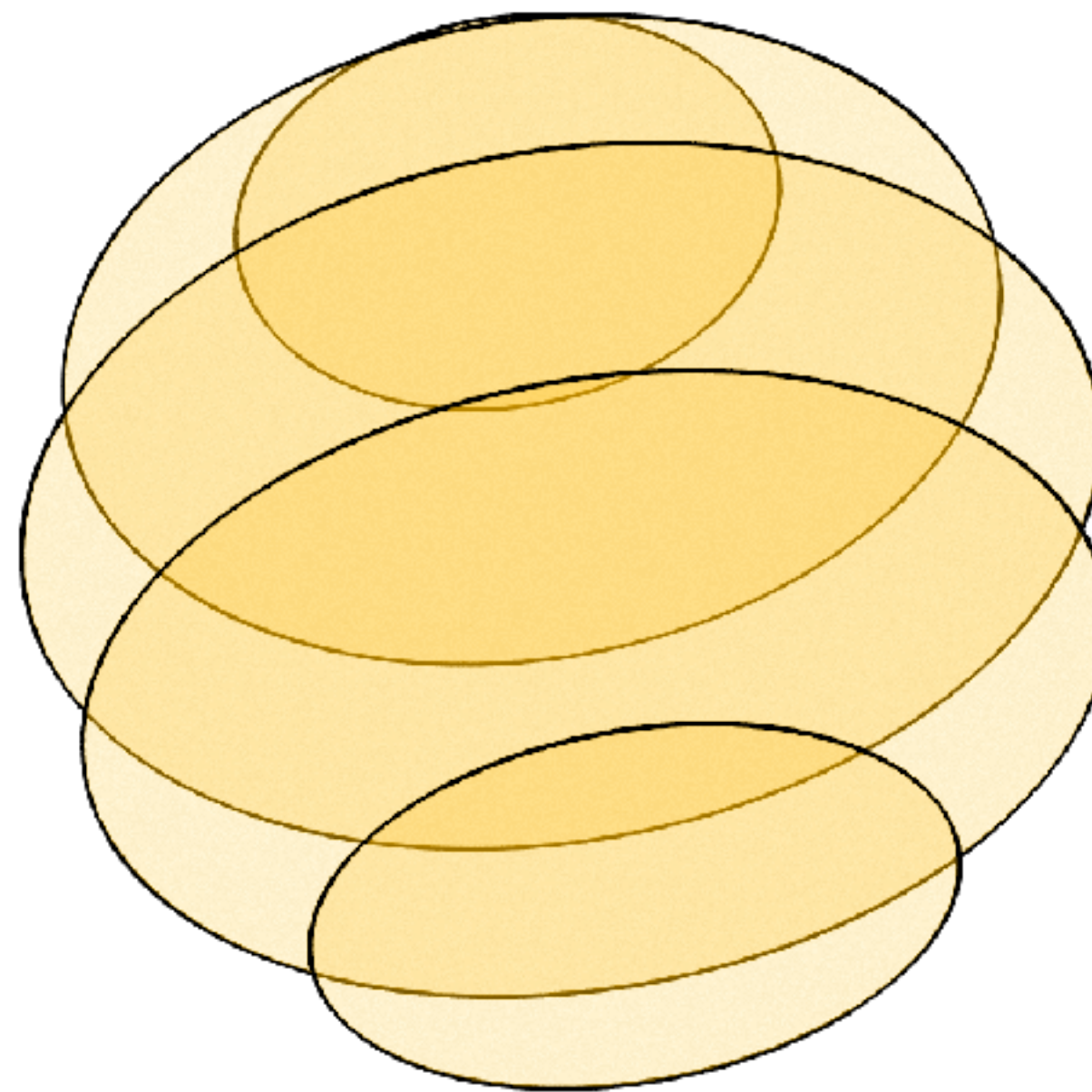
$\alpha \wedge \beta$
2-form

Wedge product



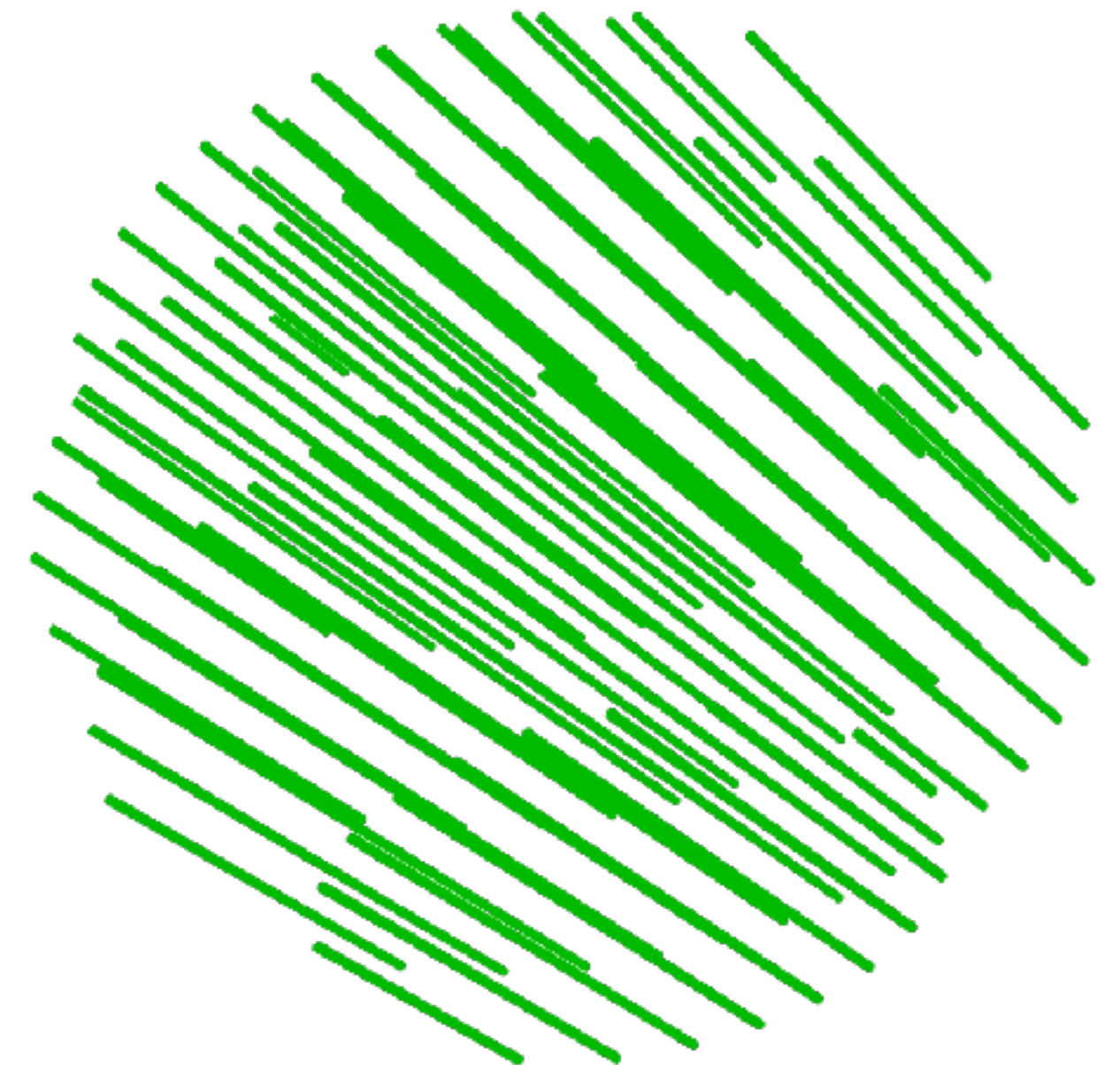
α
1-form

$\wedge$



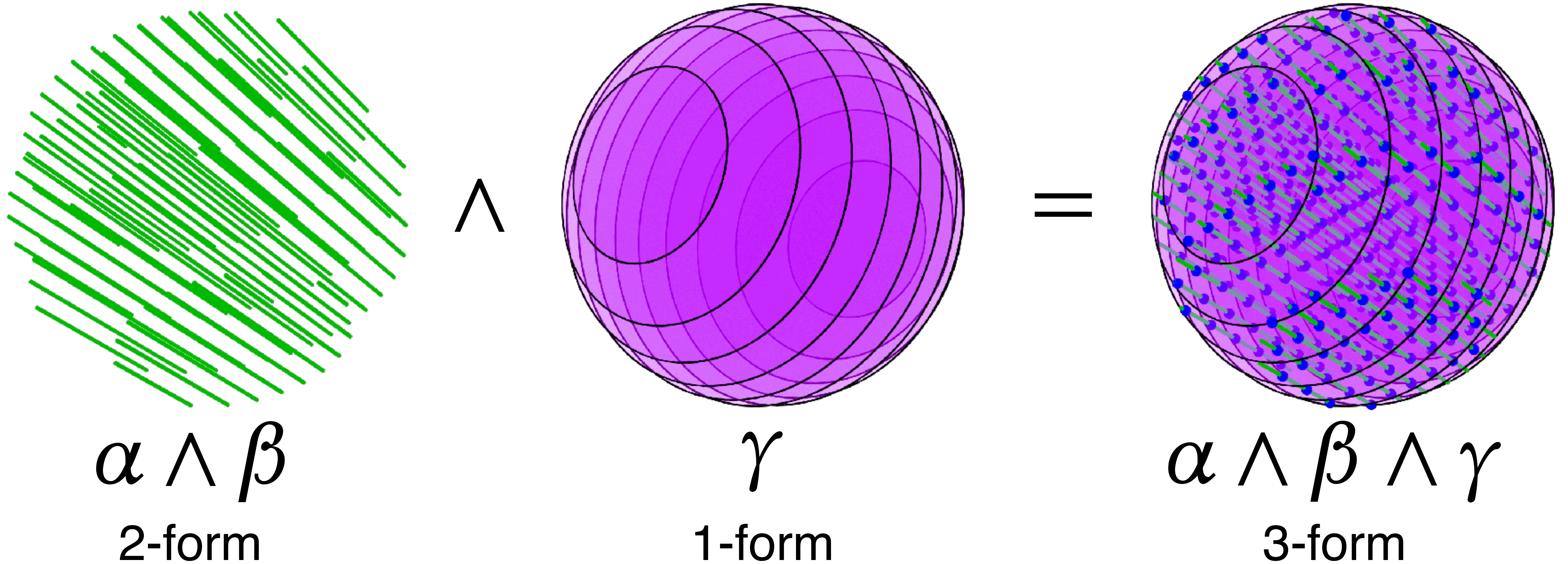
β
1-form

$=$

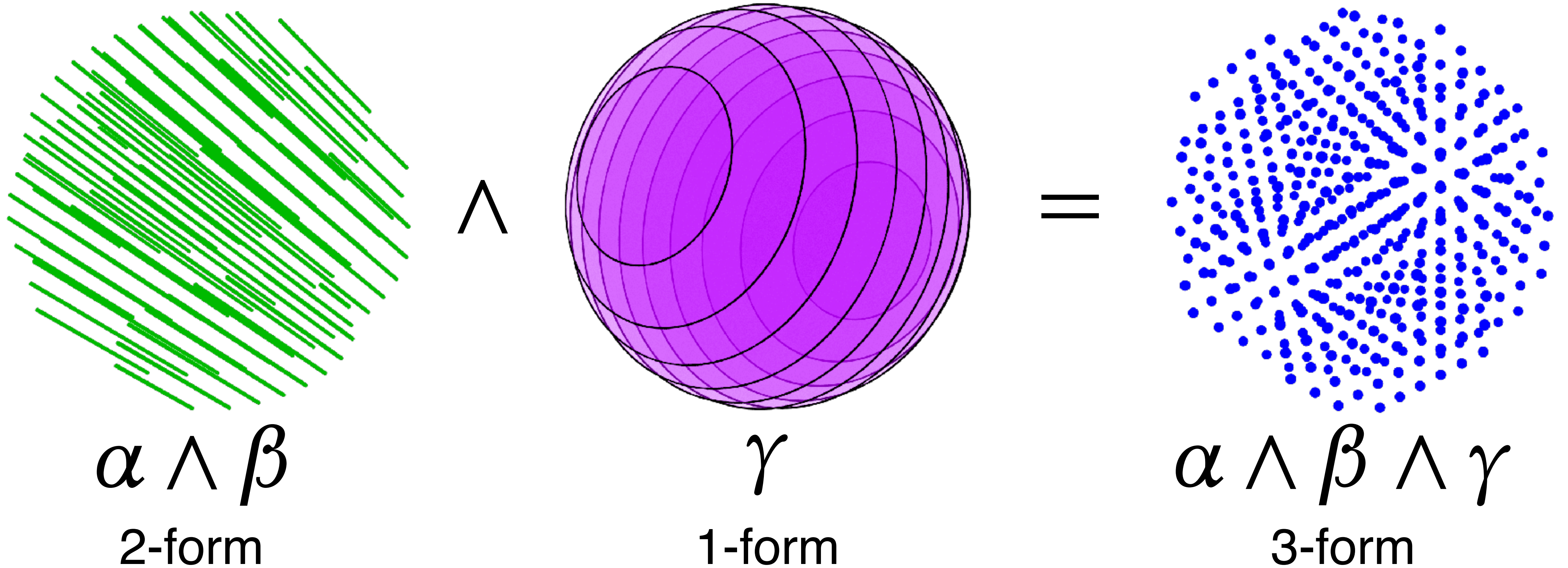


$\alpha \wedge \beta$
2-form

Wedge product



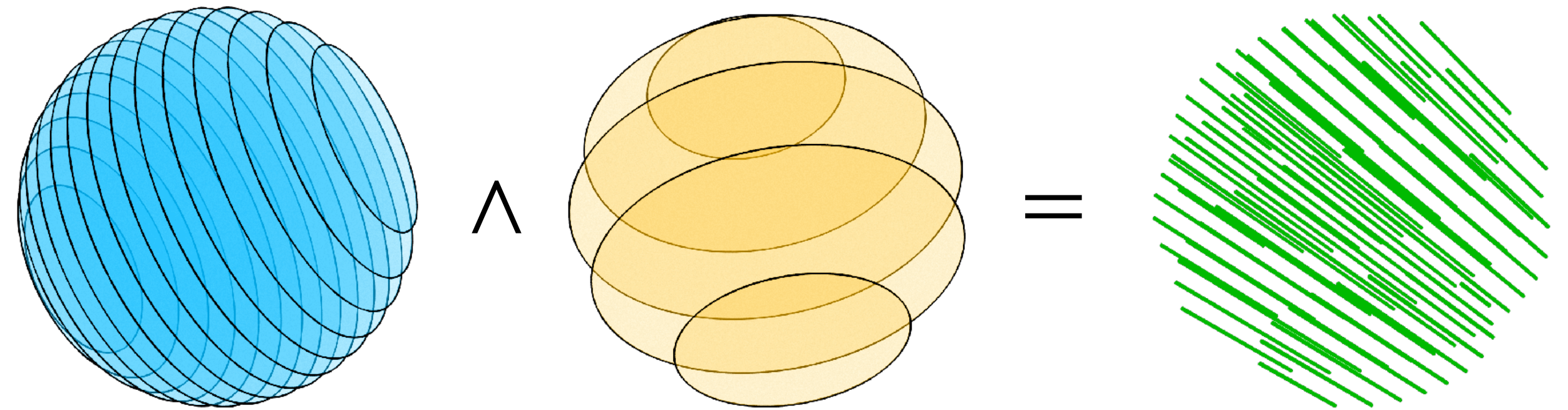
Wedge product



Wedge product

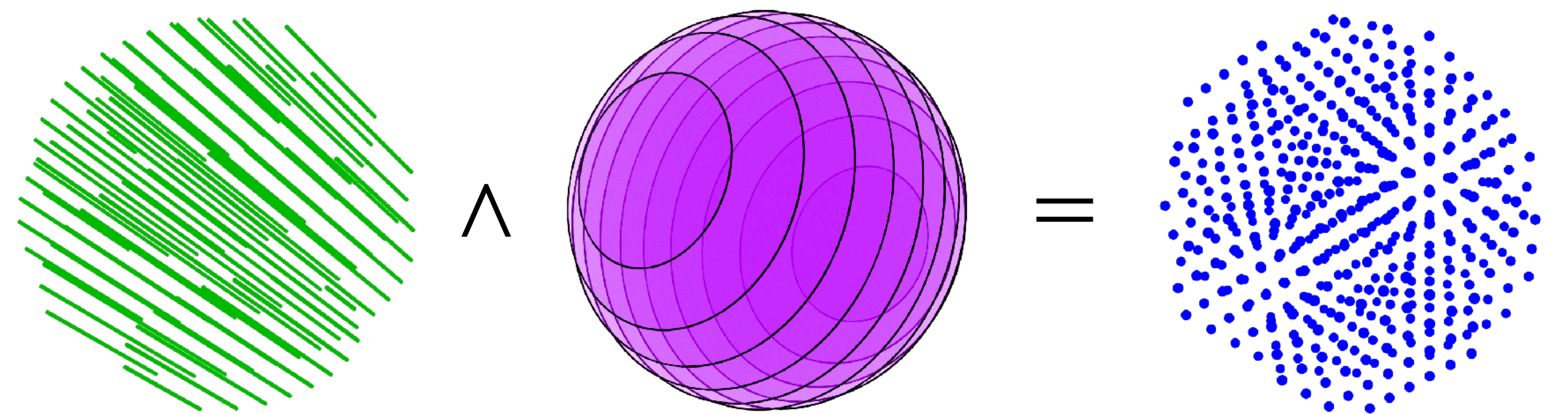
cross product

$$\begin{aligned} & (a_1 dx + a_2 dy + a_3 dz) \\ & \wedge (b_1 dx + b_2 dy + b_3 dz) \\ & = (a_2 b_3 - a_3 b_2) dy \wedge dz \\ & \quad + (a_3 b_1 - a_1 b_3) dz \wedge dx \\ & \quad + (a_1 b_2 - a_2 b_1) dx \wedge dy \end{aligned}$$



dot product

$$\begin{aligned} & (w_1 dy \wedge dz + w_2 dz \wedge dx + w_3 dx \wedge dy) \\ & \wedge (c_1 dx + c_2 dy + c_3 dz) \\ & = (w_1 c_1 + w_2 c_2 + w_3 c_3) dx dy dz \end{aligned}$$



Interior product

$$\mathbf{v} \in V$$

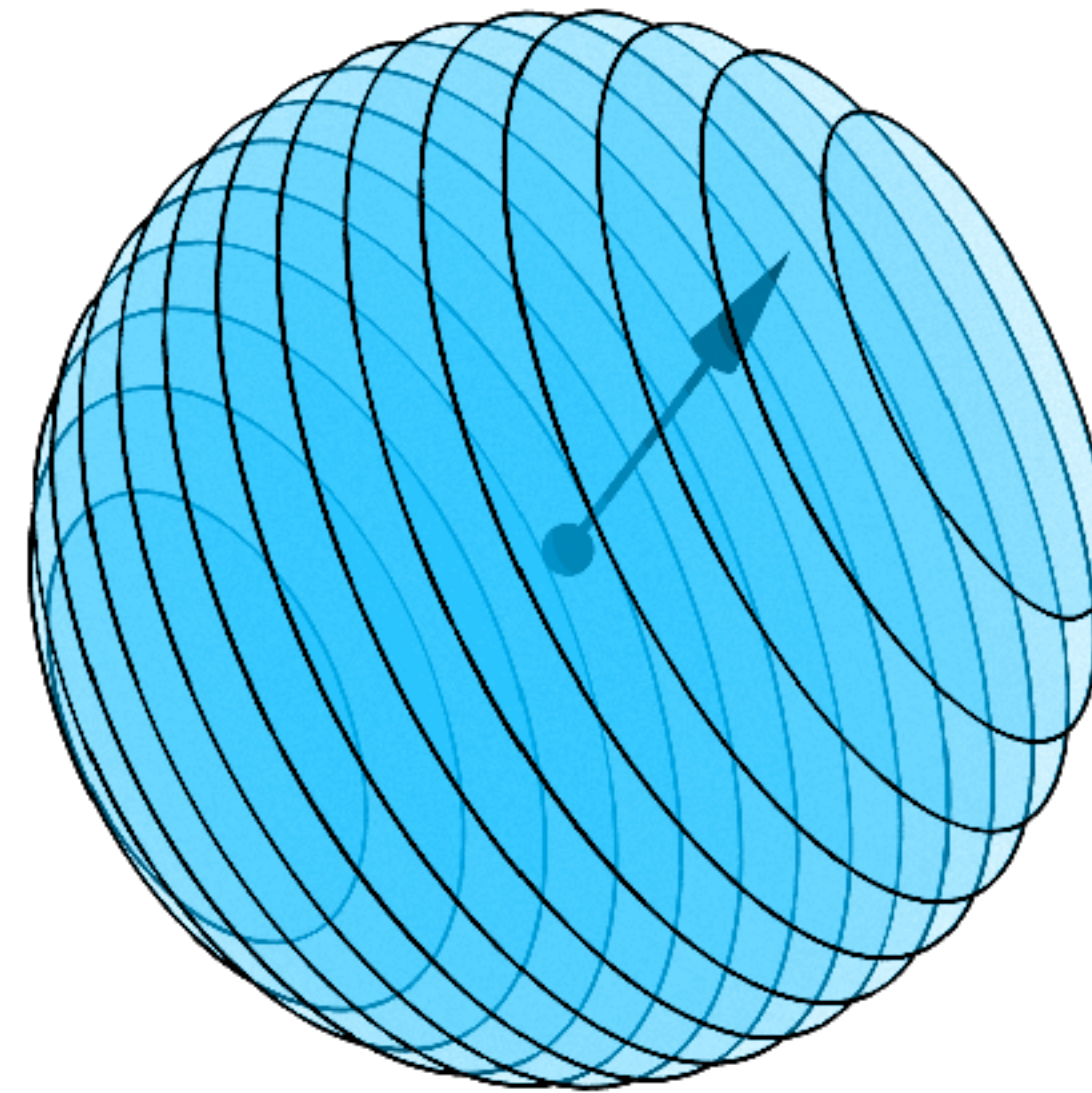
$$i_{\mathbf{v}}: \bigwedge^k V^{\star} \xrightarrow{\text{linear}} \bigwedge^{k-1} V^{\star}$$

- For 1-form $i_{\mathbf{v}}\alpha = \langle \alpha | \mathbf{v} \rangle$

- Leibniz rule

$$i_{\mathbf{v}}(\omega \wedge \eta) = (i_{\mathbf{v}}\omega) \wedge \eta + (-1)^k \omega \wedge (i_{\mathbf{v}}\eta)$$

where ω is a k -form

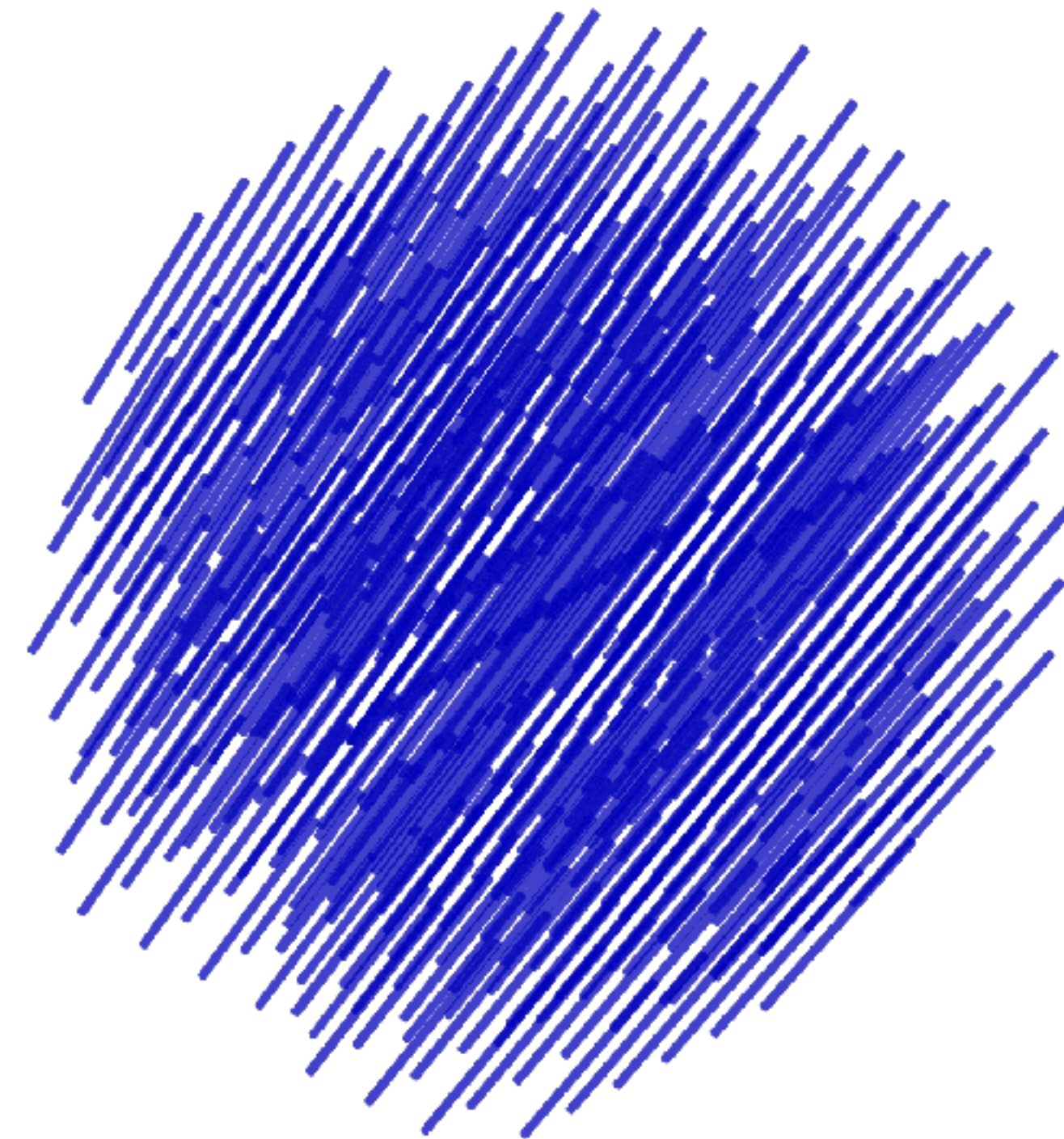
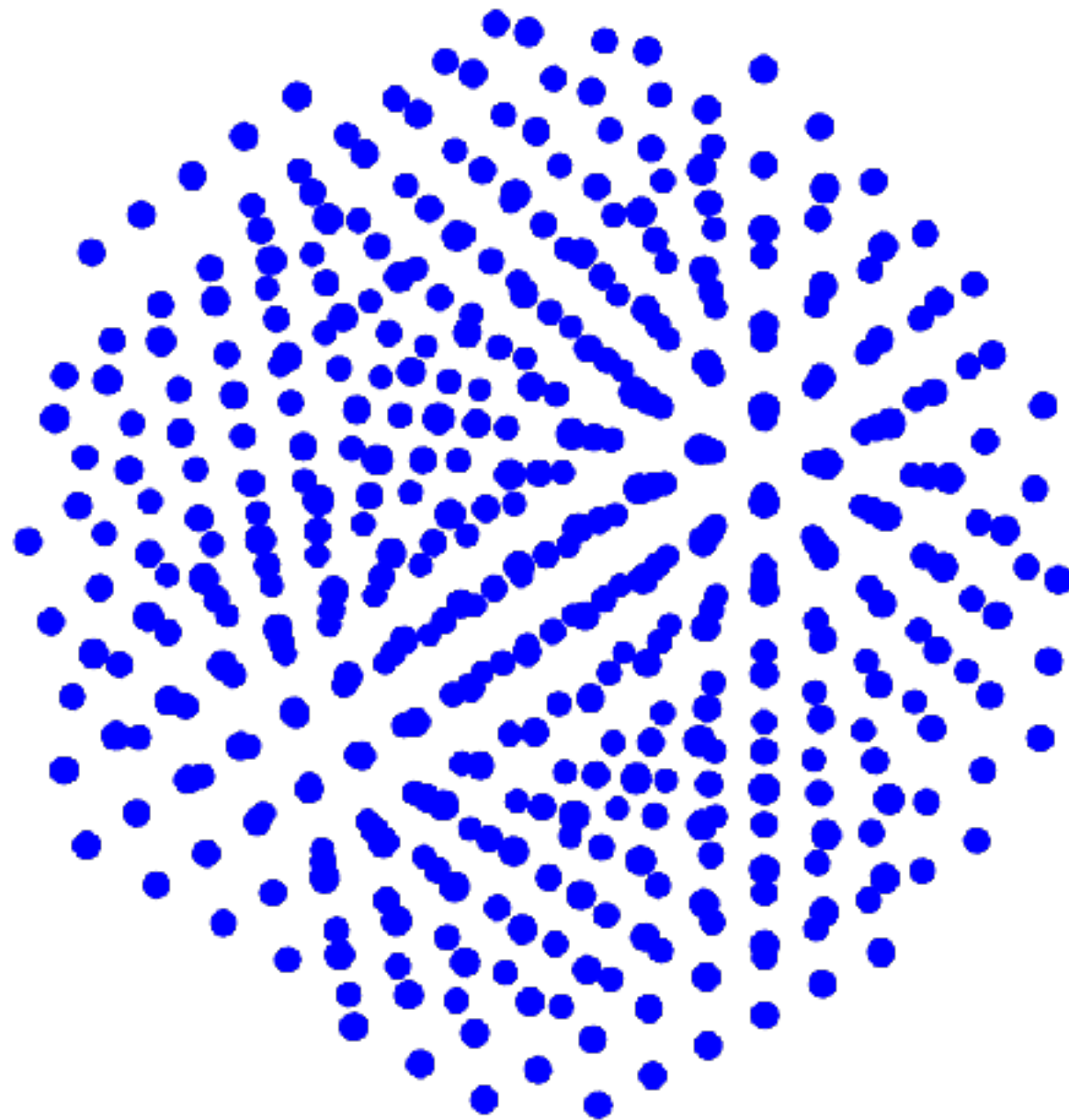
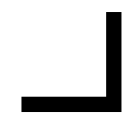


Interior product

Also known as the **insertion**

$$(i_{\mathbf{v}}\omega)(\mathbf{u}_1, \dots, \mathbf{u}_{k-1}) = \omega(\mathbf{v}, \mathbf{u}_1, \dots, \mathbf{u}_{k-1})$$

Interior product



$\mathbf{v}$

vector

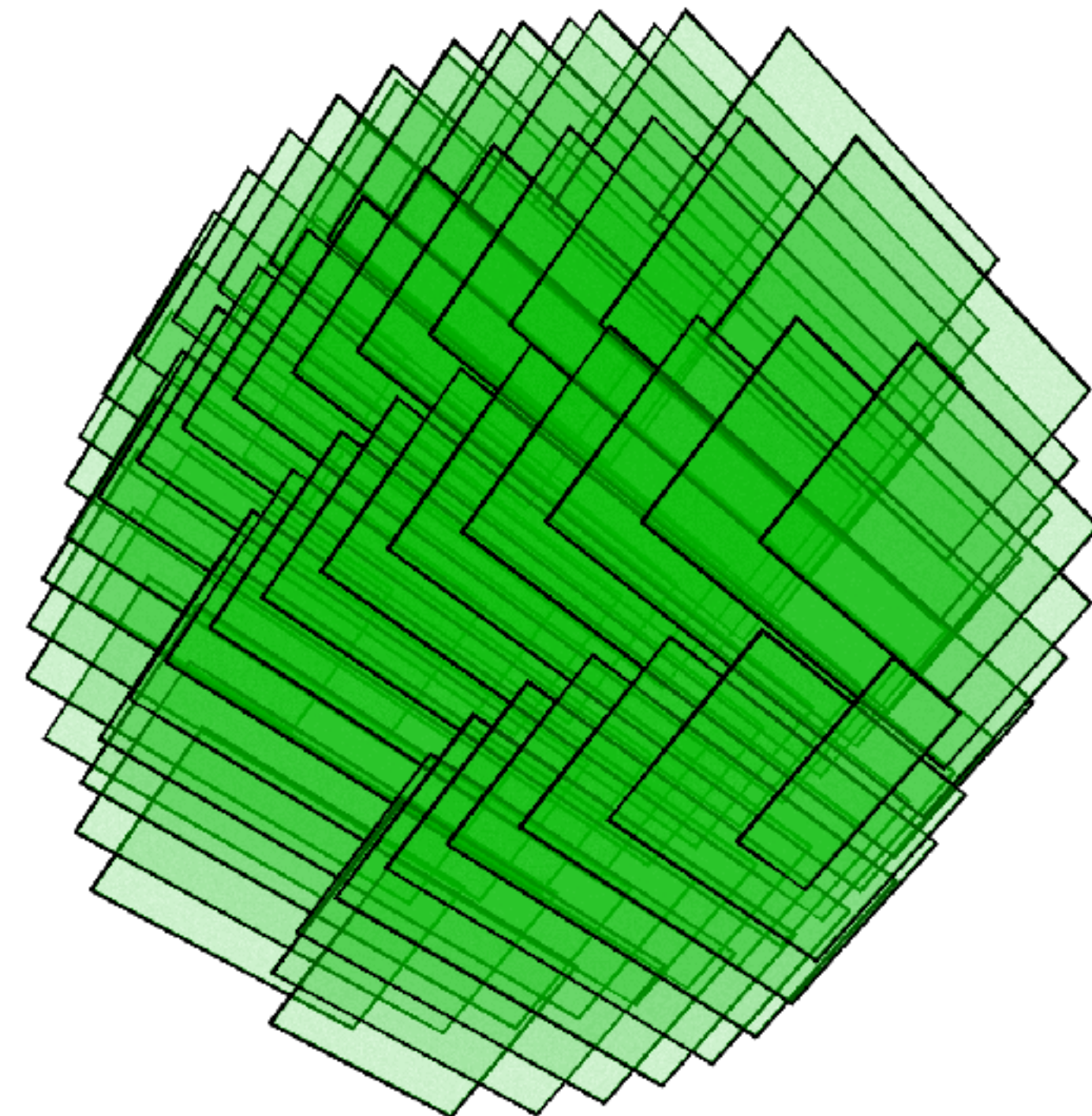
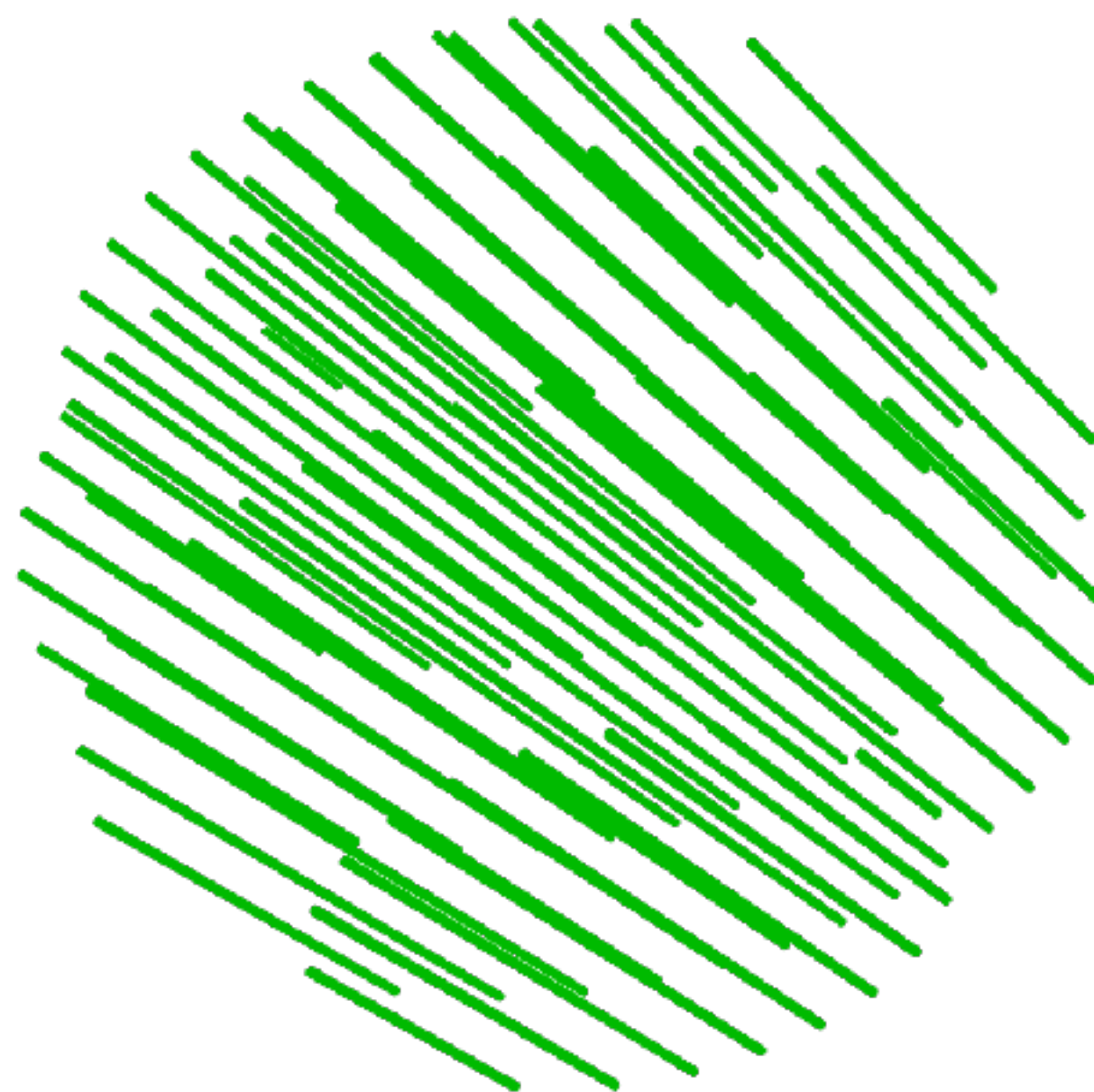
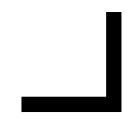
ρ

3-form

$i_{\mathbf{v}}\rho$

2-form

Interior product



$\mathbf{V}$

vector

ω

2-form

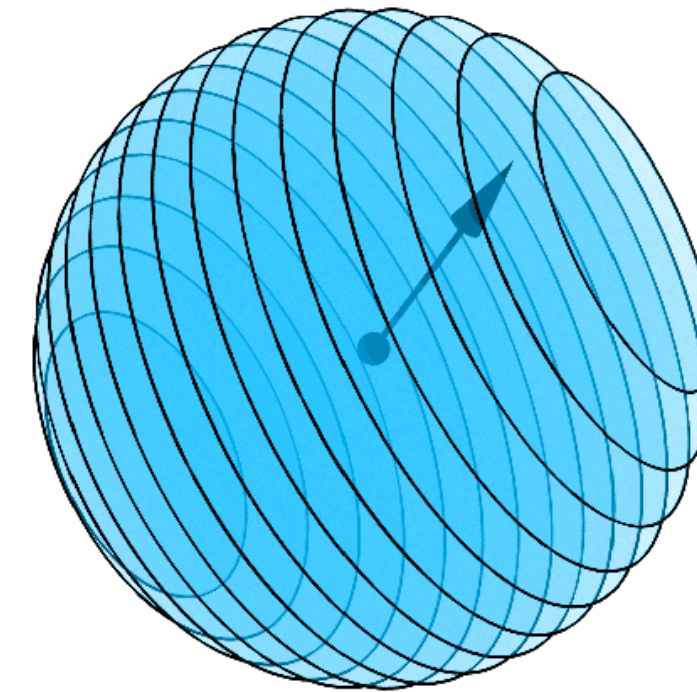
$i_{\mathbf{V}}\omega$

1-form

Interior product

dot product

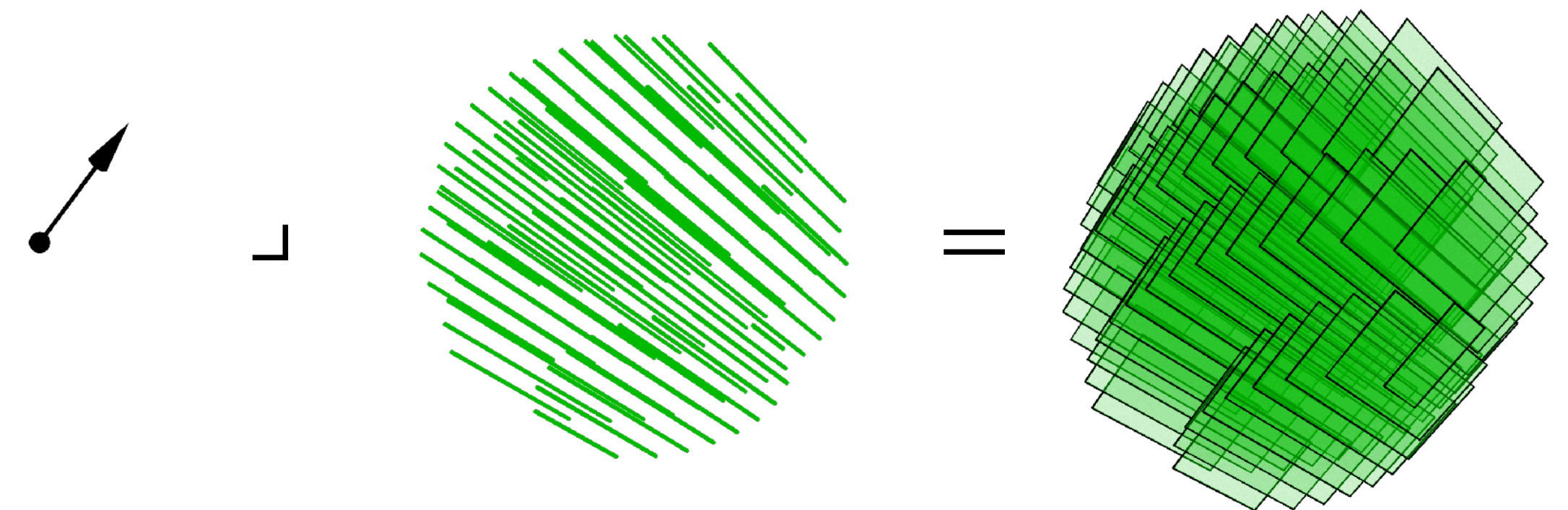
$$\begin{aligned} i_v(a_1 dx + a_2 dy + a_3 dz) \\ = a_1 v_1 + a_2 v_2 + a_3 v_3 \end{aligned}$$



cross product

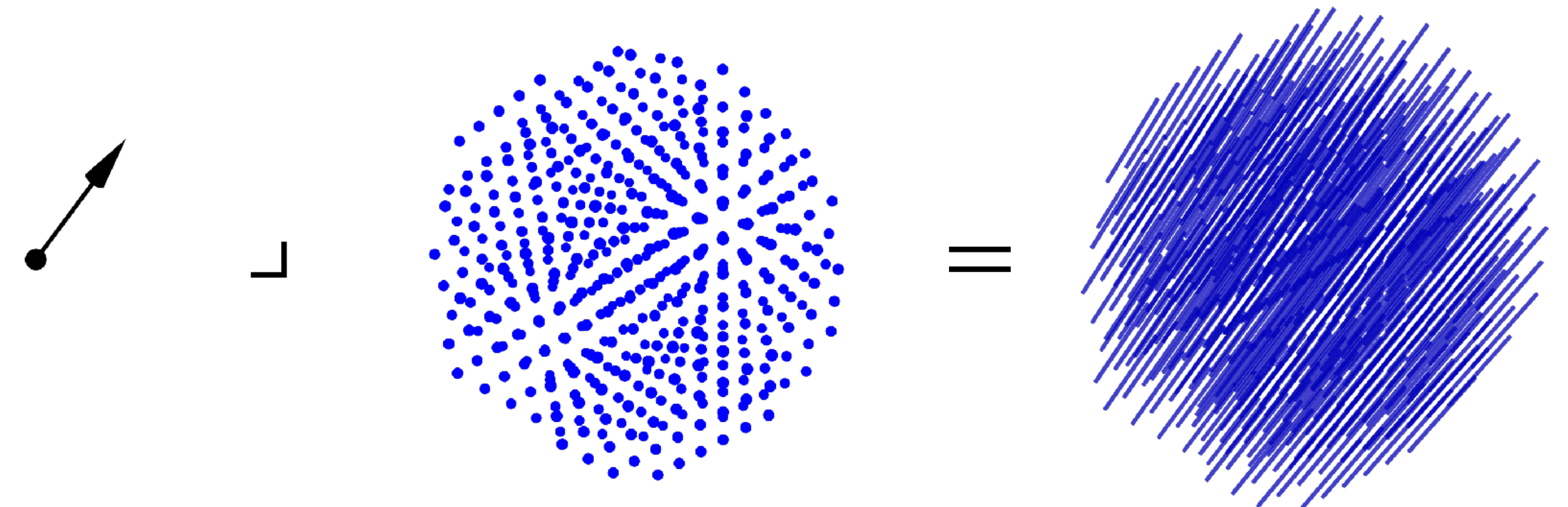
$$\begin{aligned} i_v(w_1 dy \wedge dz + w_2 dz \wedge dx + w_3 dx \wedge dy) \\ = u_1 dx + u_2 dy + u_3 dz \end{aligned}$$

where $\mathbf{u} = \mathbf{w} \times \mathbf{v}$



vector · density = flux

$$\begin{aligned} i_v(q dx \wedge dy \wedge dz) \\ = qv_1 dy \wedge dz + qv_2 dz \wedge dx + qv_3 dx \wedge dy \end{aligned}$$



Example

$$\alpha = a_1 dx + a_2 dy + a_3 dz$$

$$\beta = b_1 dx + b_2 dy + b_3 dz$$

Leibniz rule

$$i_v(\alpha \wedge \beta) = (i_v \alpha) \beta - \alpha i_v \beta$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{v}$$

$$(\mathbf{v} \cdot \mathbf{a}) \mathbf{b}$$

$$(\mathbf{v} \cdot \mathbf{b}) \mathbf{a}$$

Hodge star

$(V, \langle \cdot, \cdot \rangle)$ vector space with metric

$$\star: \bigwedge^k V^\star \rightarrow \bigwedge^{n-k} V^\star$$

- In 3D, suppose dx, dy, dz are orthonormal. Then

$$\star 1 = dx \wedge dy \wedge dz$$

$$\star dx = dy \wedge dz, \quad \star dy = dz \wedge dx, \quad \star dz = dx \wedge dy$$

$$\star(dy \wedge dz) = dx, \quad \star(dz \wedge dx) = dy, \quad \star(dx \wedge dy) = dz$$

$$\star(dx \wedge dy \wedge dz) = 1$$

Hodge star

$(V, \langle \cdot, \cdot \rangle)$ vector space with metric

$$\star: \bigwedge^k V^\star \rightarrow \bigwedge^{n-k} V^\star$$

- In 3D, suppose dx, dy, dz are orthonormal. Then

$$\mathbf{v}^\flat = v_1 dx + v_2 dy + v_3 dz$$

$$\star \mathbf{v}^\flat = v_1 dy \wedge dz + v_2 dz \wedge dx + v_3 dx \wedge dy$$

Hodge star

$(V, \langle \cdot, \cdot \rangle)$ vector space with metric

$$\star: \bigwedge^k V^\star \rightarrow \bigwedge^{n-k} V^\star$$

- The formula that relates wedge product and interior product

$$i_{\mathbf{v}} \star \omega = (-1)^k \star (\mathbf{v}^\flat \wedge \omega)$$

where ω is a k -form

Hodge star

$(V, \langle \cdot, \cdot \rangle)$ vector space with metric

$$\star: \bigwedge^k V^\star \rightarrow \bigwedge^{n-k} V^\star$$

- In general $\alpha, \beta \in \bigwedge^k V^\star$

$$\alpha \wedge \star \beta = \langle \alpha, \beta \rangle \star 1$$

Example: spherical coordinate

$d\theta, d\phi$ basis covectors

$$\langle d\theta, d\theta \rangle = 1$$

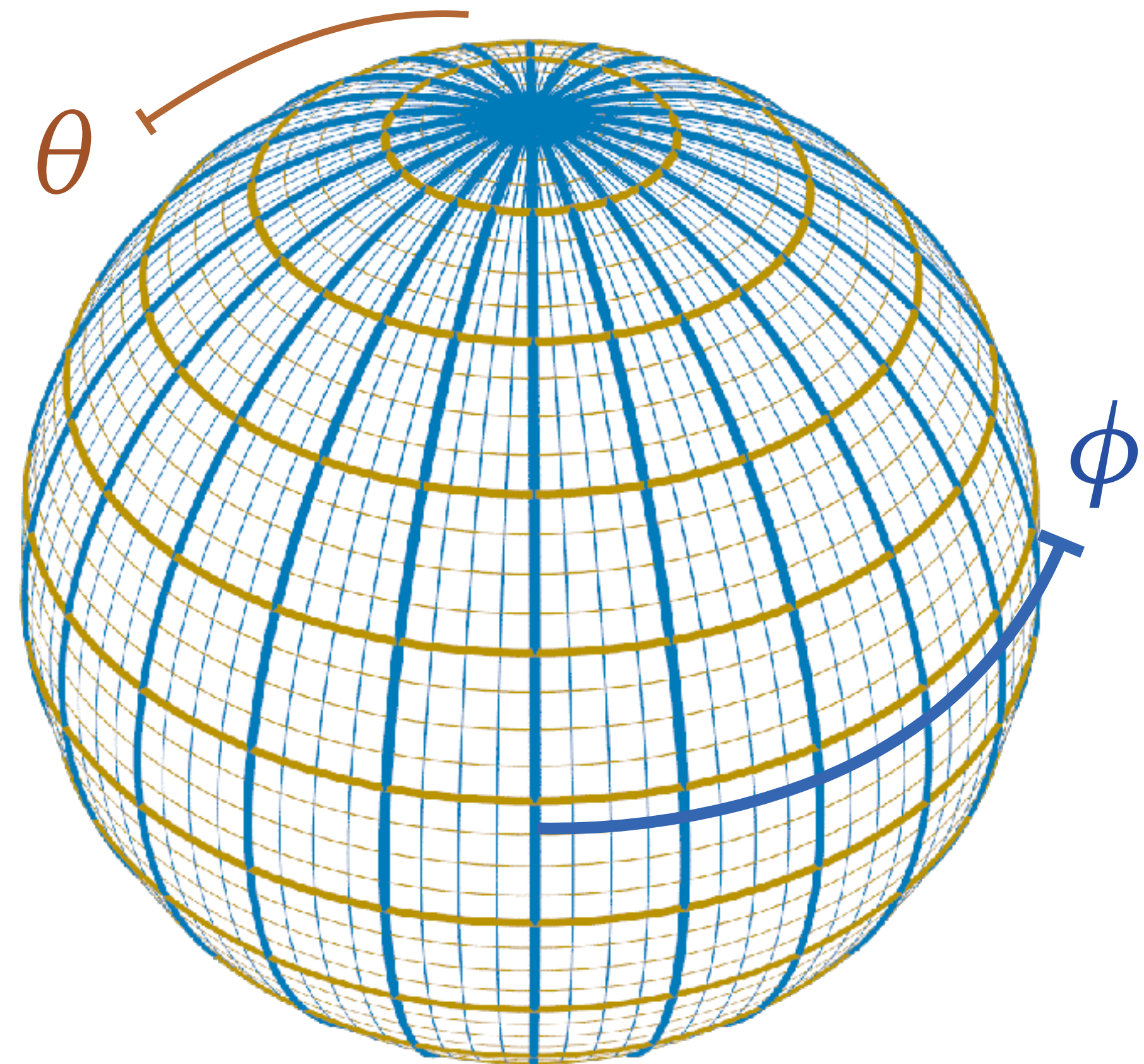
$$\langle d\theta, d\phi \rangle = 0$$

$$\langle d\phi, d\phi \rangle = \frac{1}{\sin^2 \theta}$$

$$\star 1 = \sin \theta \, d\theta \wedge d\phi$$

$$\Rightarrow \star d\theta = \sin \theta \, d\phi$$

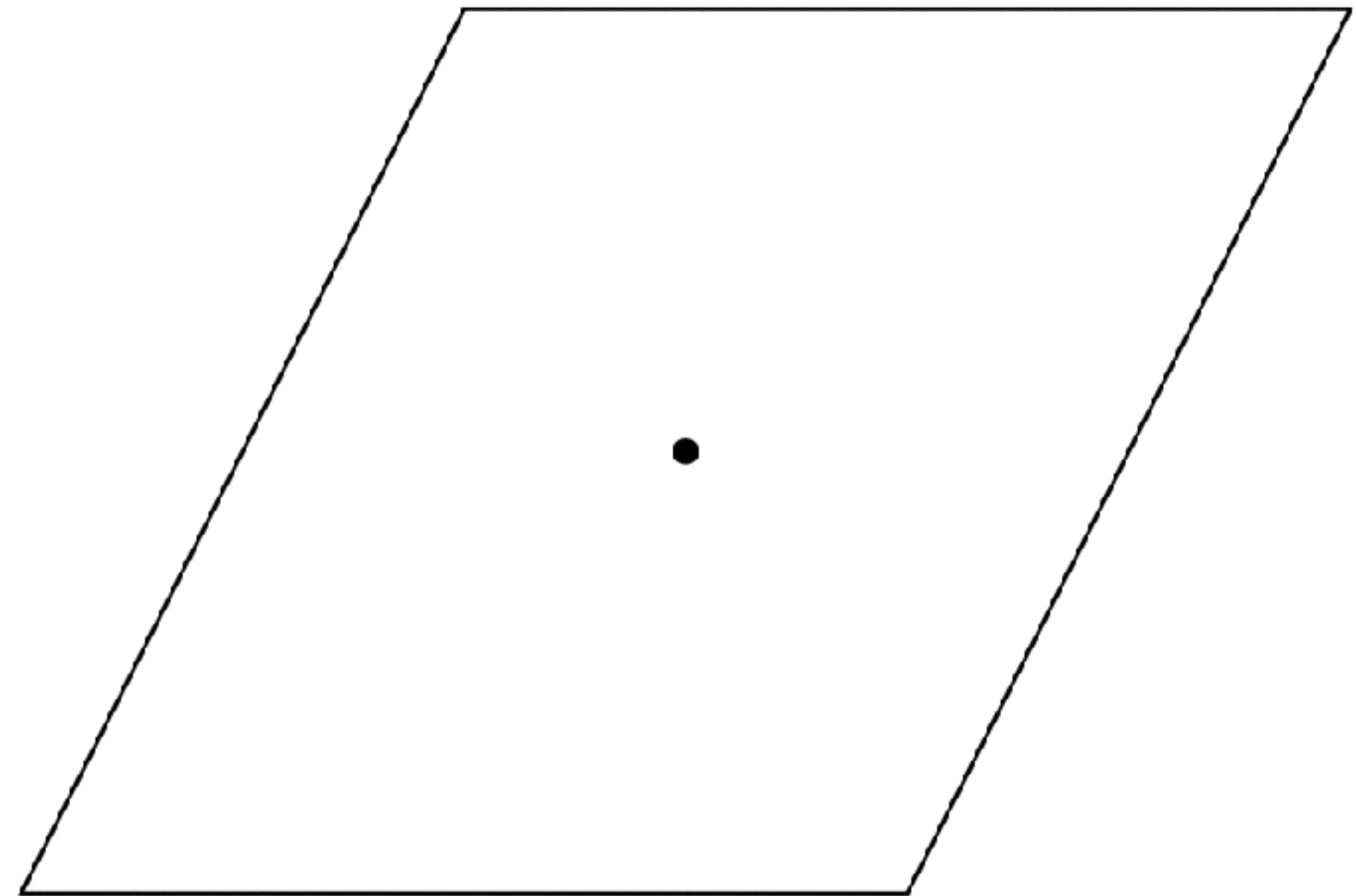
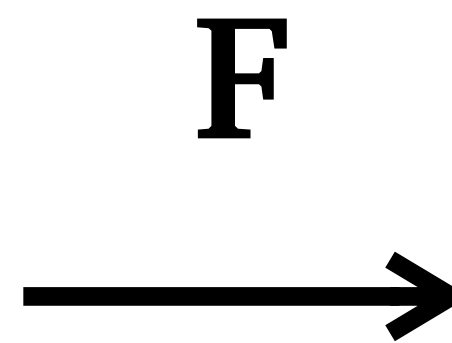
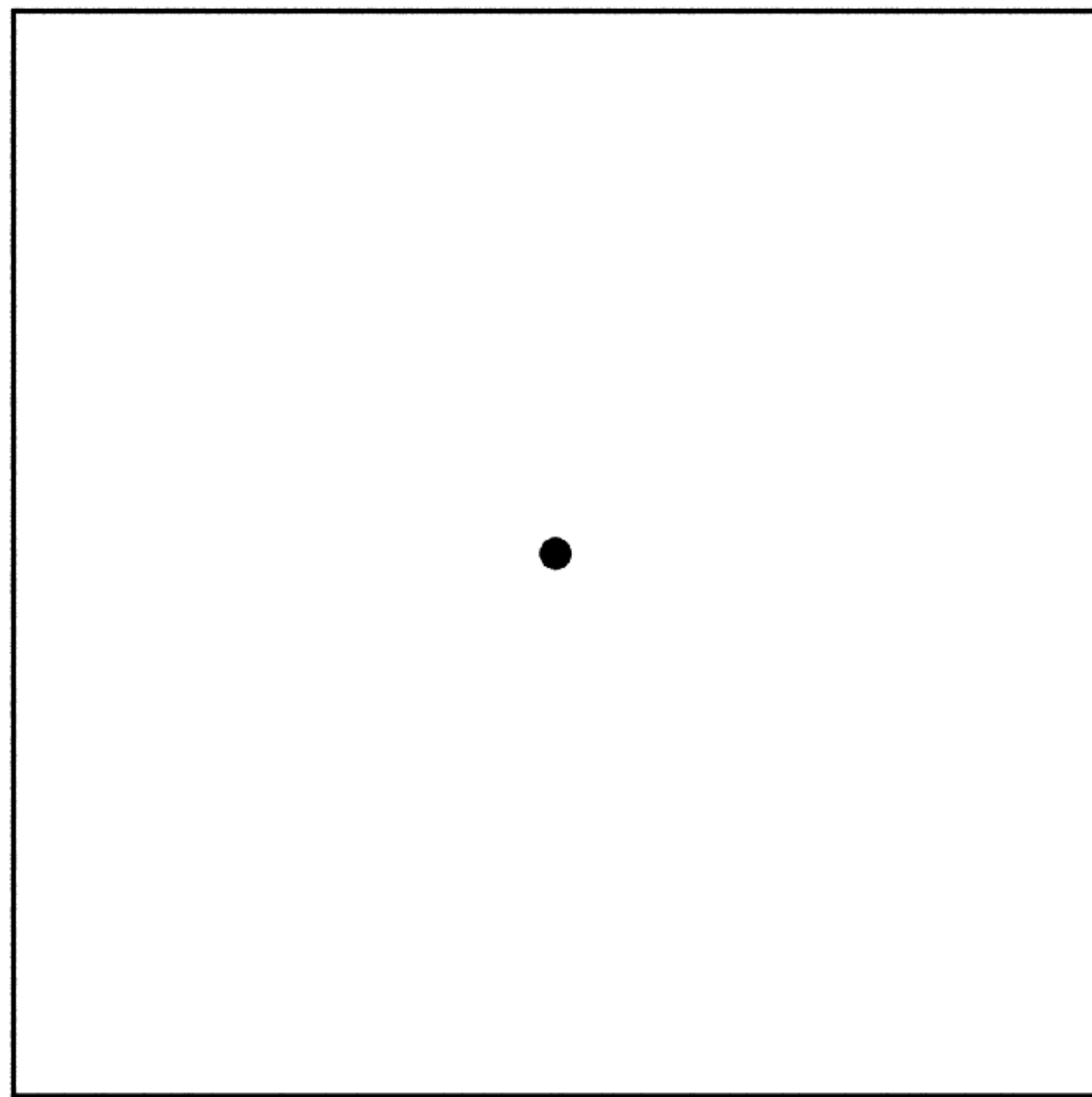
$$\star d\phi = -\frac{1}{\sin \theta} d\theta$$



Pullback

$$\mathbf{F}: V \xrightarrow{\text{linear}} W$$

$$\mathbf{F}^*: \bigwedge^k W^* \rightarrow \bigwedge^k V^*$$

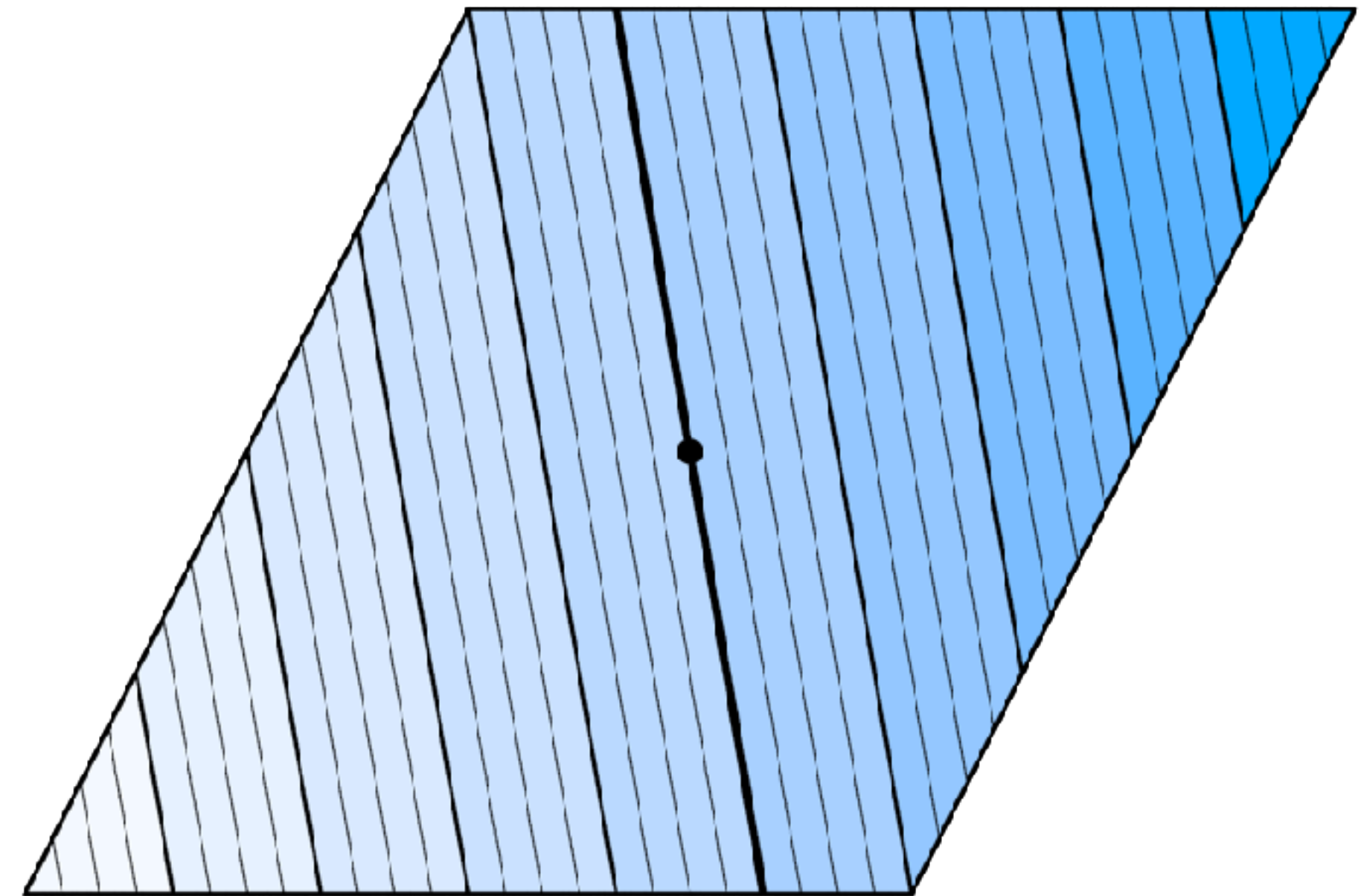
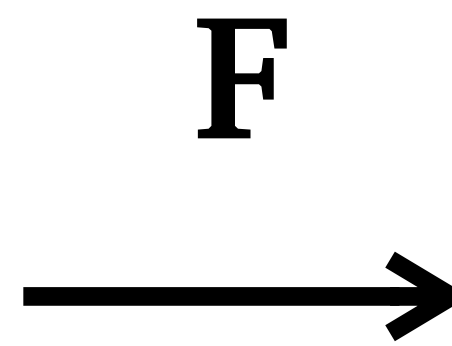
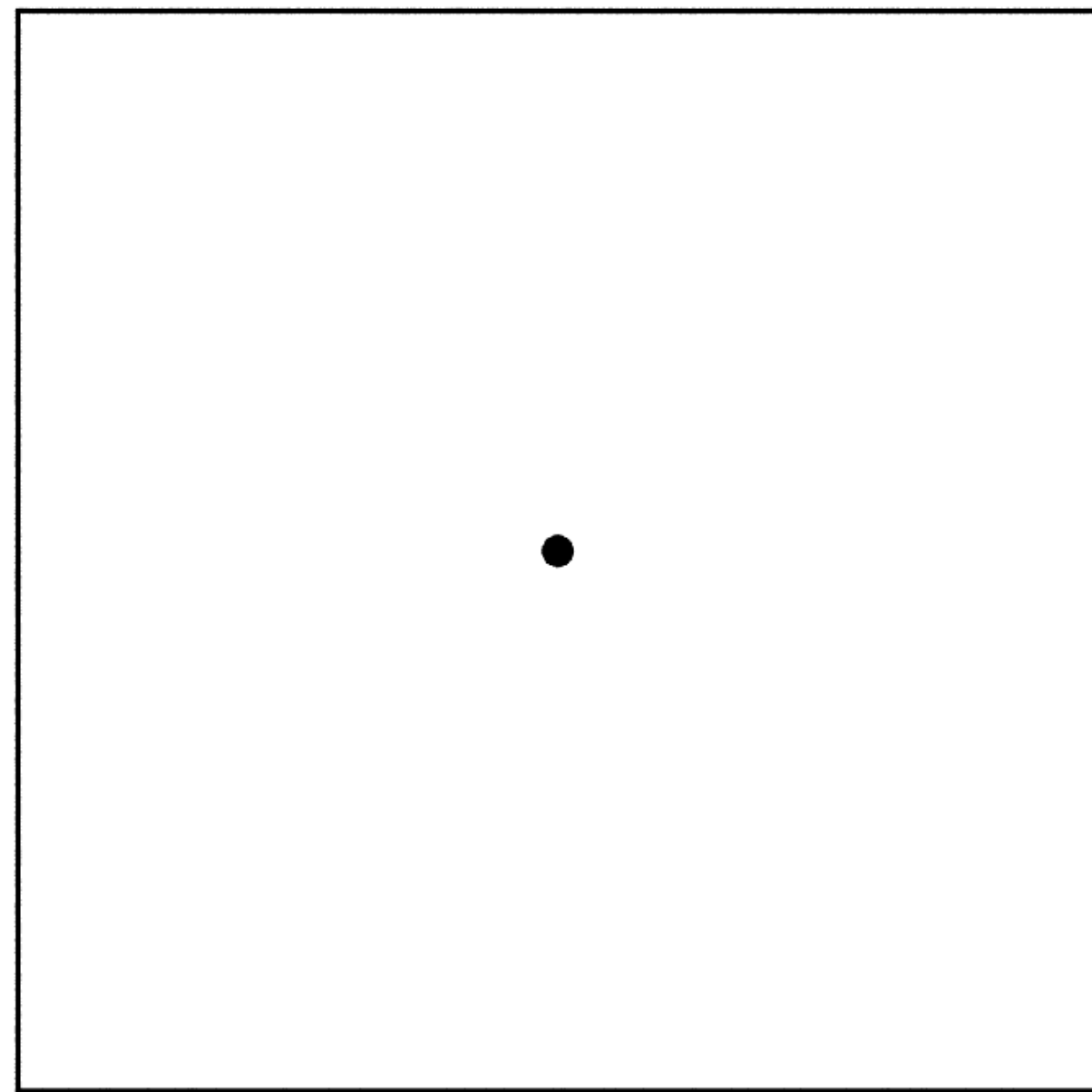


Pullback

$$\mathbf{F}: V \xrightarrow{\text{linear}} W$$

$$\mathbf{F}^*: \bigwedge^k W^\star \rightarrow \bigwedge^k V^\star$$

$$\omega \in \bigwedge^k W^\star$$



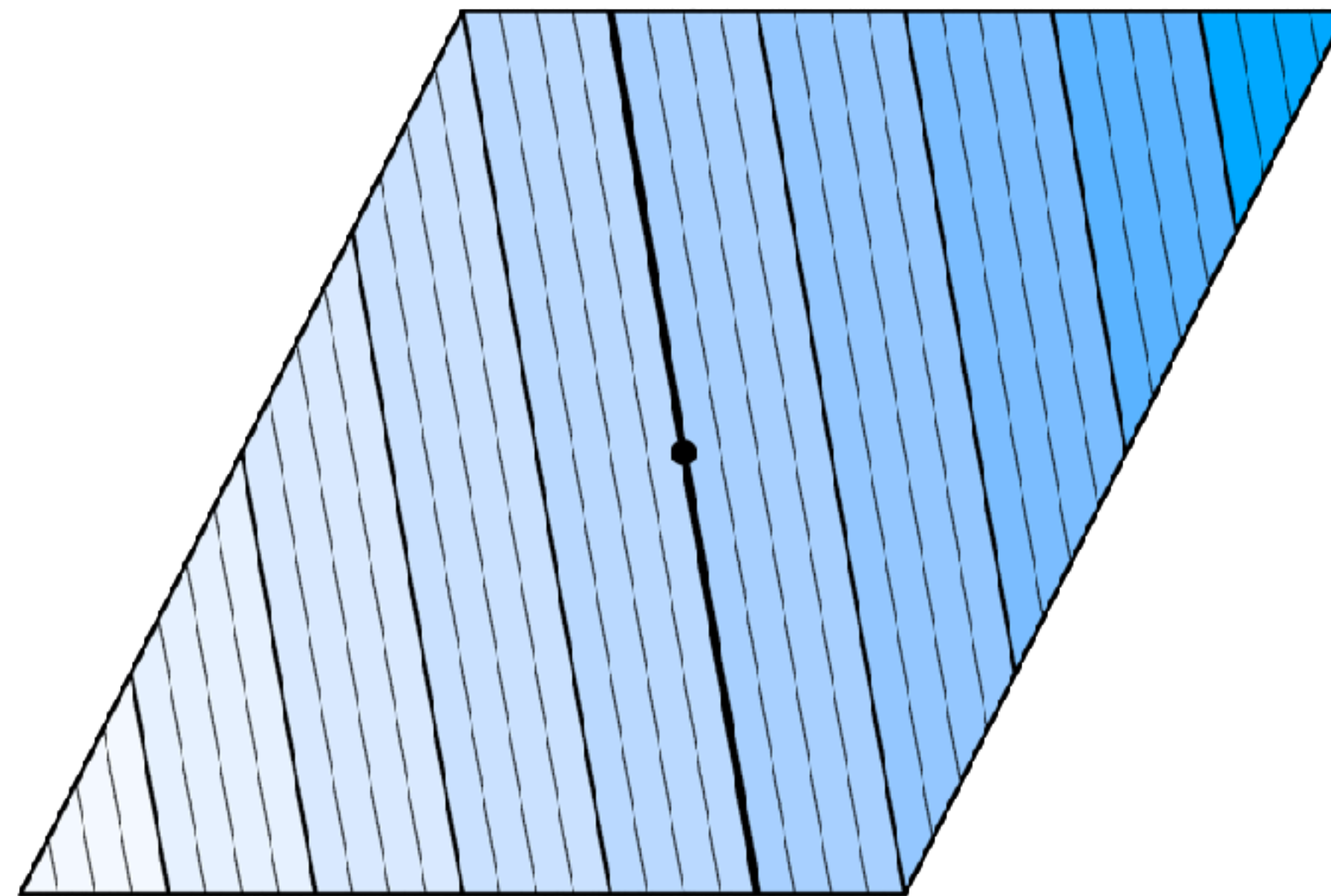
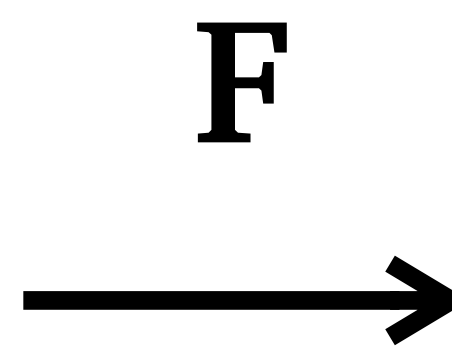
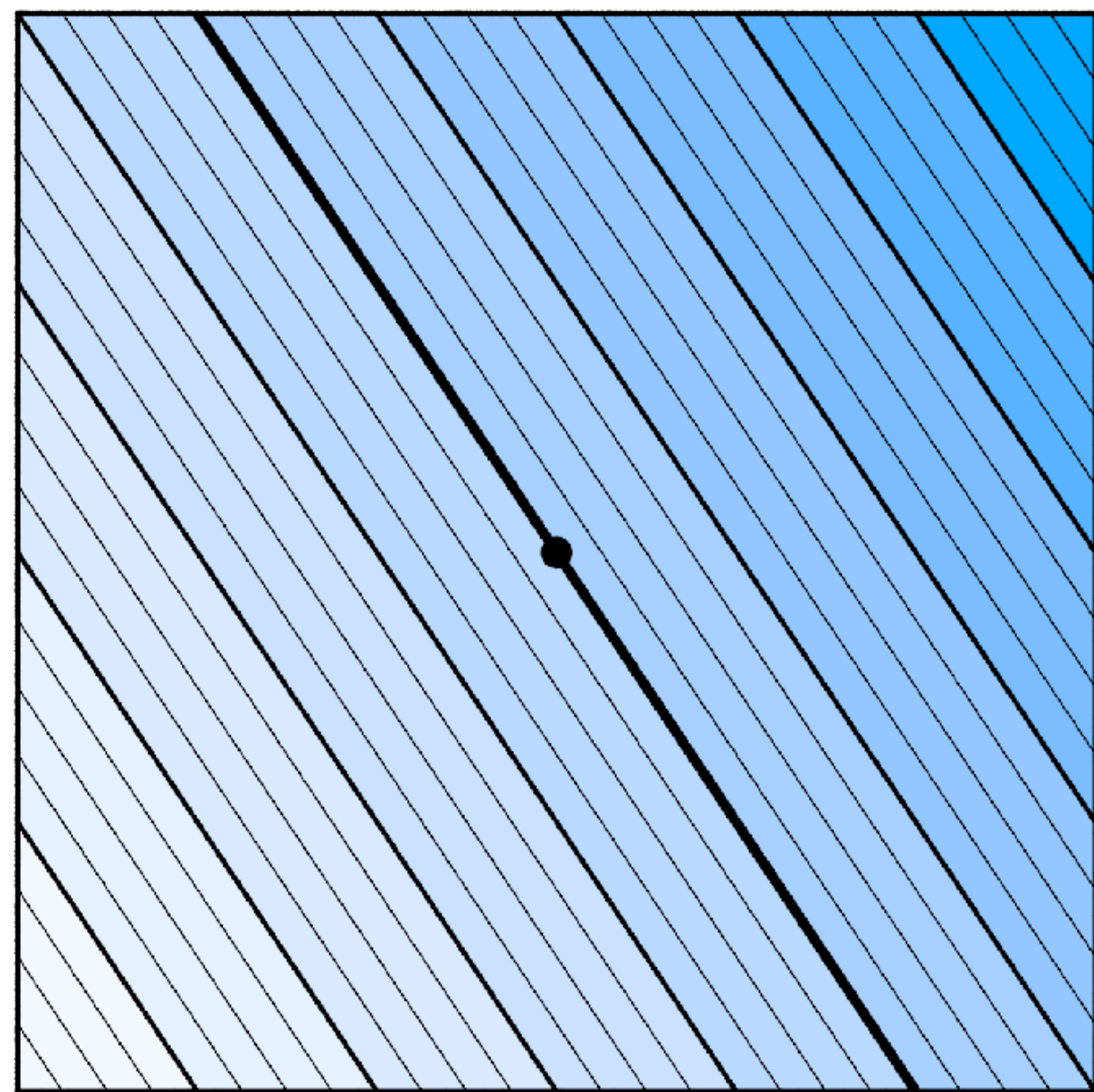
Pullback

$$\mathbf{F}: V \xrightarrow{\text{linear}} W$$

$$\mathbf{F}^*: \bigwedge^k W^* \rightarrow \bigwedge^k V^*$$

$$\mathbf{F}^* \omega \in \bigwedge^k V^*$$

$$\omega \in \bigwedge^k W^*$$



$$(\mathbf{F}^* \omega)(\mathbf{v}_1, \dots, \mathbf{v}_k) = \omega(\mathbf{F}\mathbf{v}_1, \dots, \mathbf{F}\mathbf{v}_k)$$

Pullback

$$(\mathbf{F}^* \omega)(\mathbf{v}_1, \dots, \mathbf{v}_k) = \omega(\mathbf{F}\mathbf{v}_1, \dots, \mathbf{F}\mathbf{v}_k)$$

Pullback distributes over wedges

$$\mathbf{F}^*(\omega \wedge \eta) = (\mathbf{F}^* \omega) \wedge (\mathbf{F}^* \eta)$$

Pullback passes through insertion

$$\mathbf{F}^*(i_{\mathbf{v}} \omega) = i_{\mathbf{F}^{-1}\mathbf{v}}(\mathbf{F}^* \omega)$$

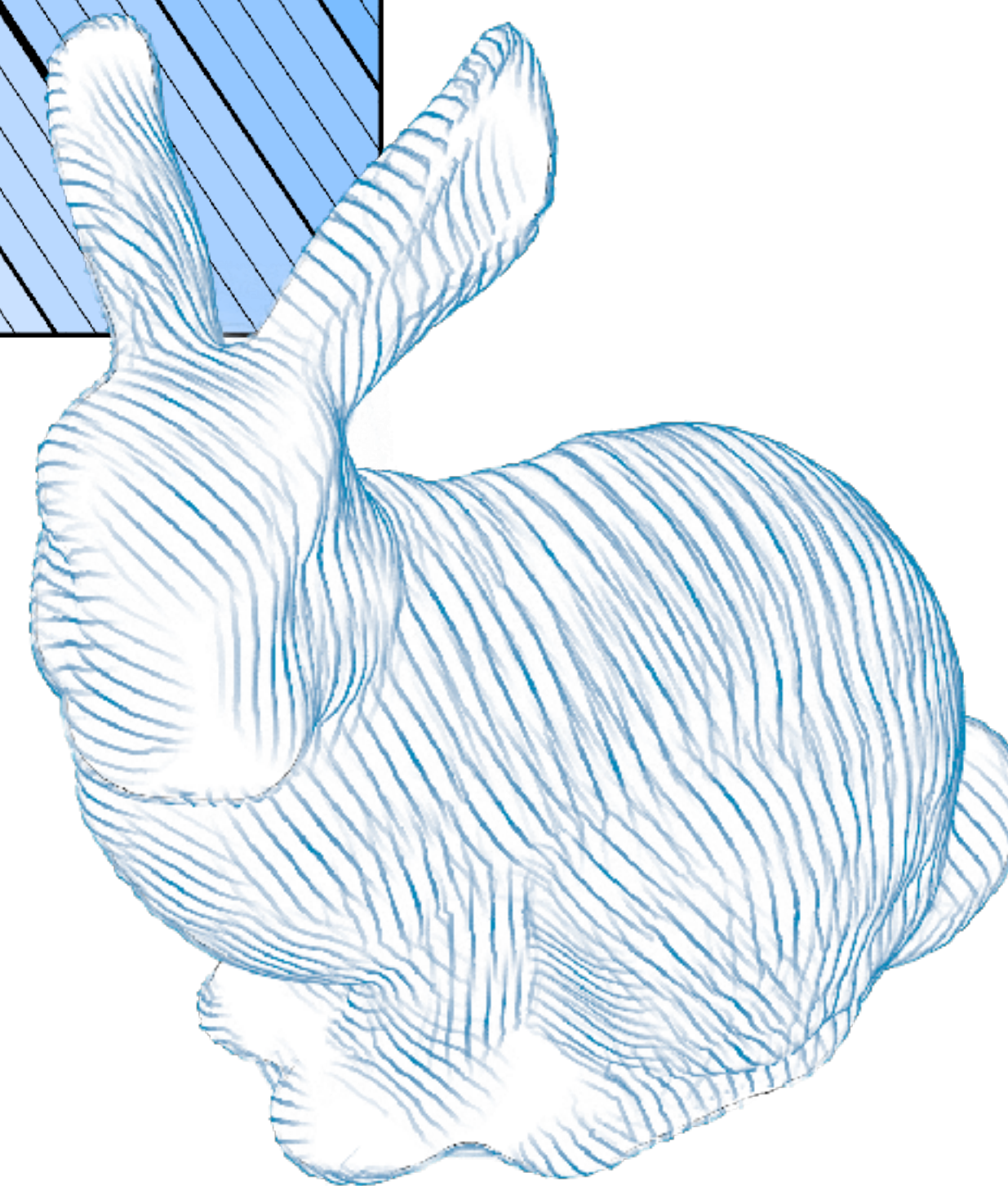
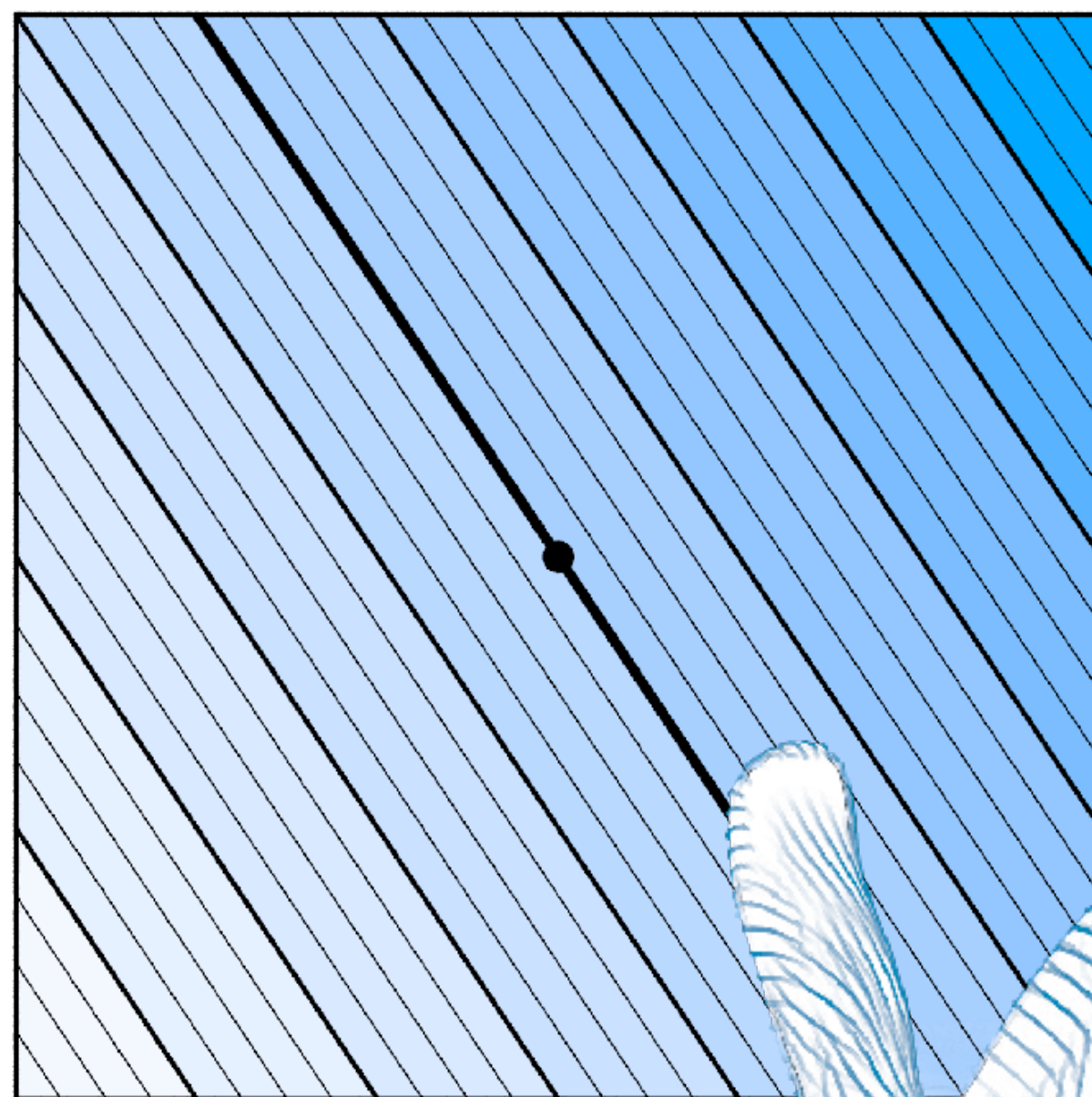
Pullback doesn't pass through Hodge star unless the map is isometric

$$\mathbf{F}^*(\star \omega) \neq \star \mathbf{F}^* \omega$$

k-form field

$$\omega|_p \in \bigwedge^k T_p^* M$$

$$\omega \in \Omega^k(M)$$

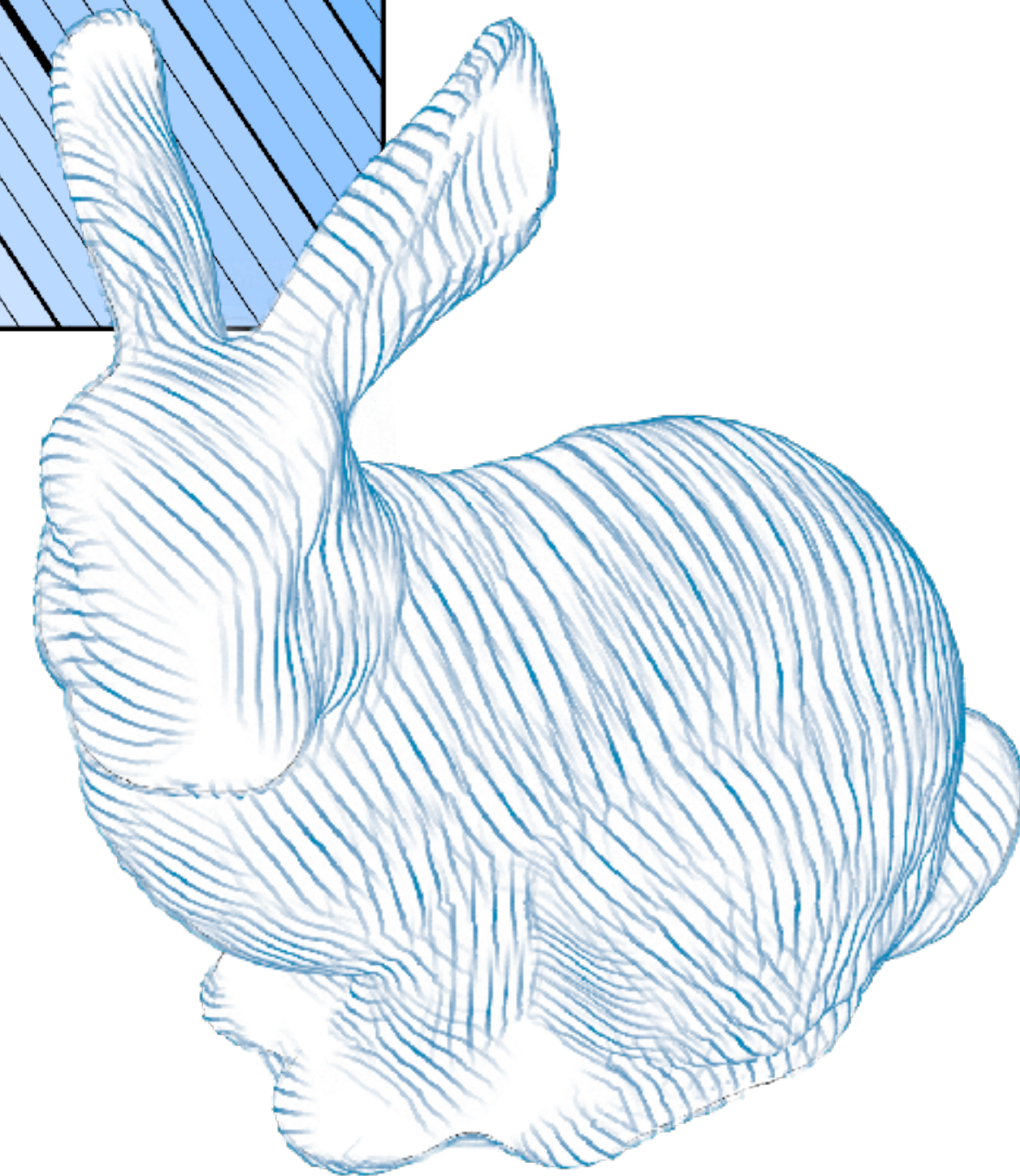
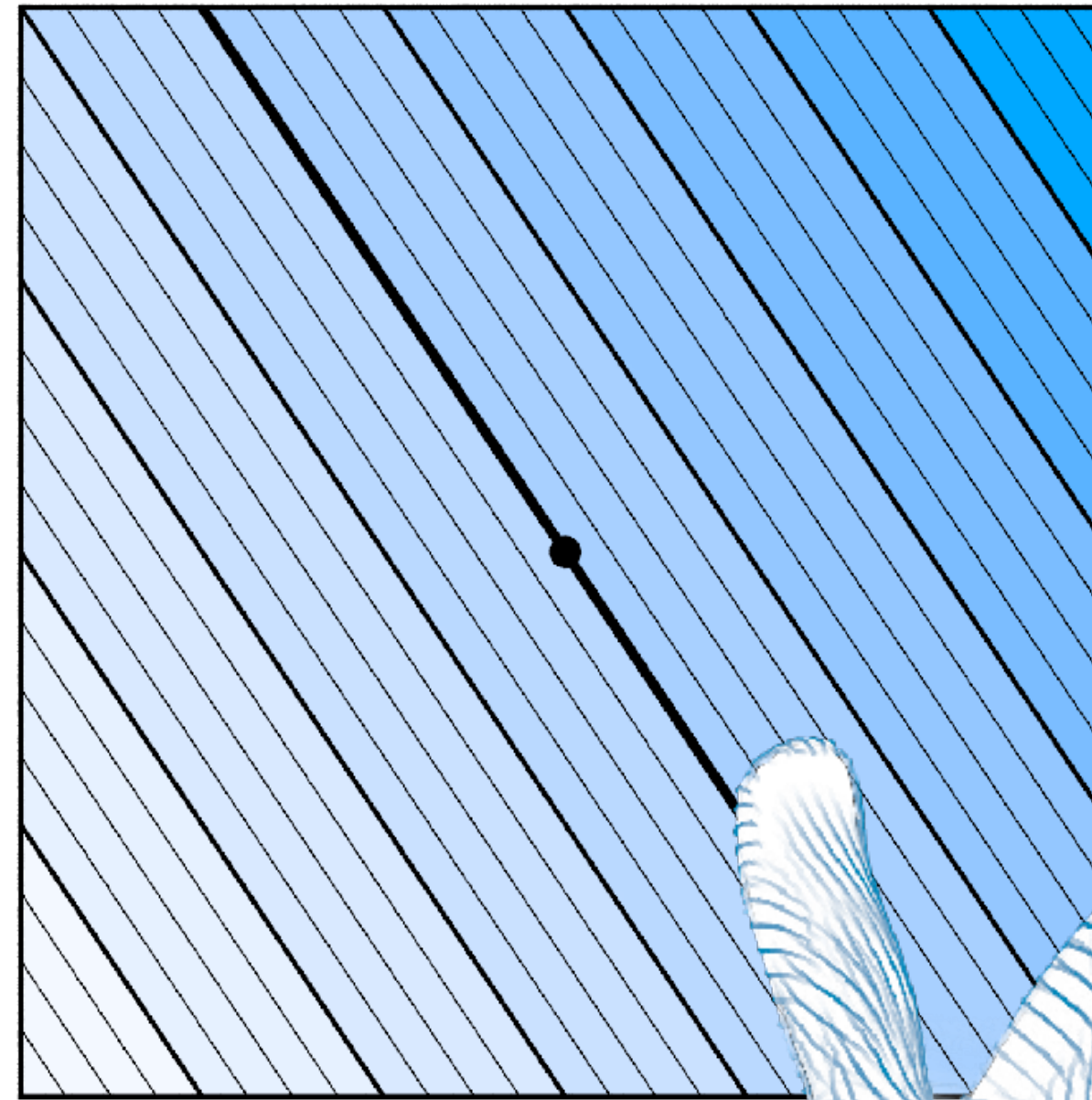


k-form field

$$\omega|_p \in \bigwedge^k T_p^* M$$

$$\omega \in \Omega^k(M)$$

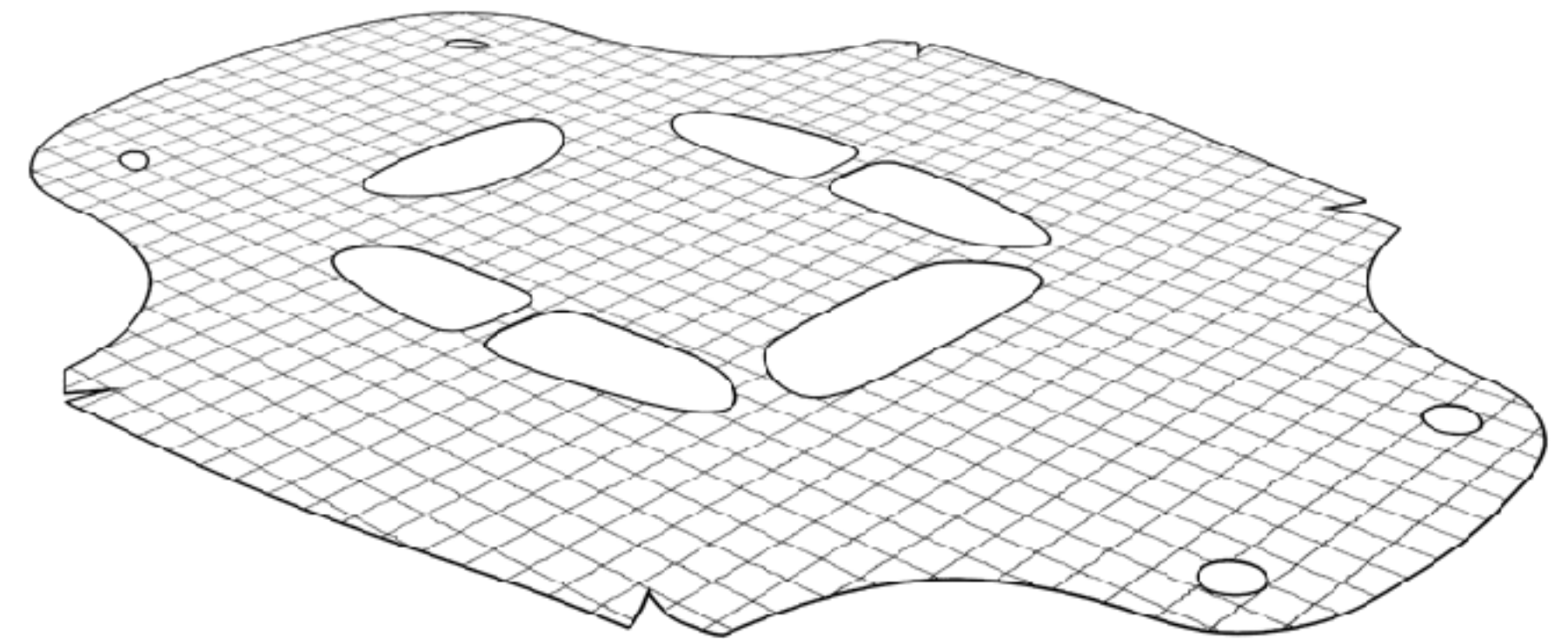
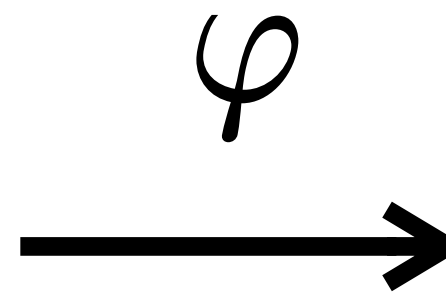
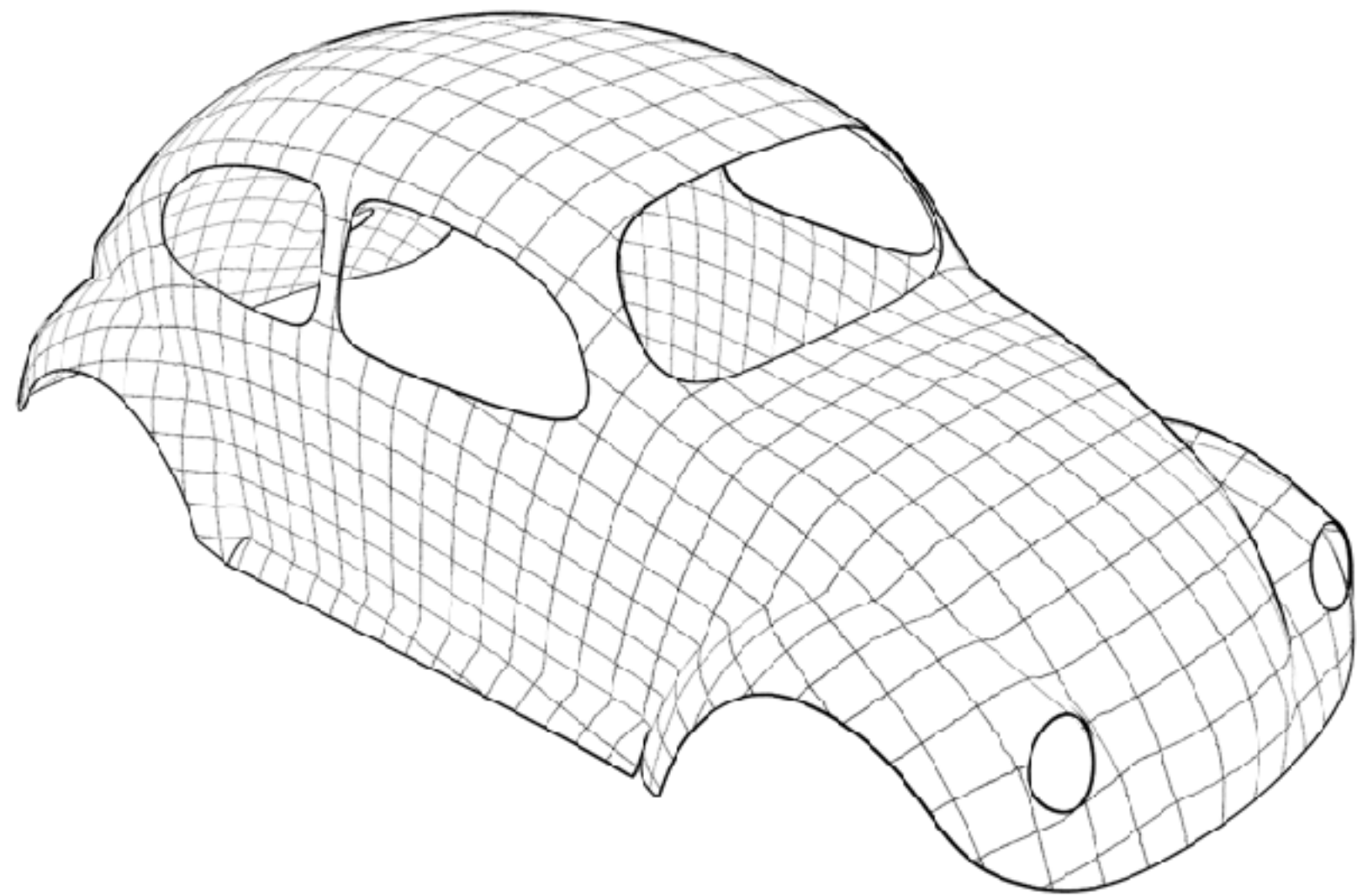
- **Wedge, interior product, Hodge star** are operated pointwise.



Pullback

$$\varphi : M \rightarrow N$$

$$\varphi^* : \Omega^k(N) \rightarrow \Omega^k(M)$$

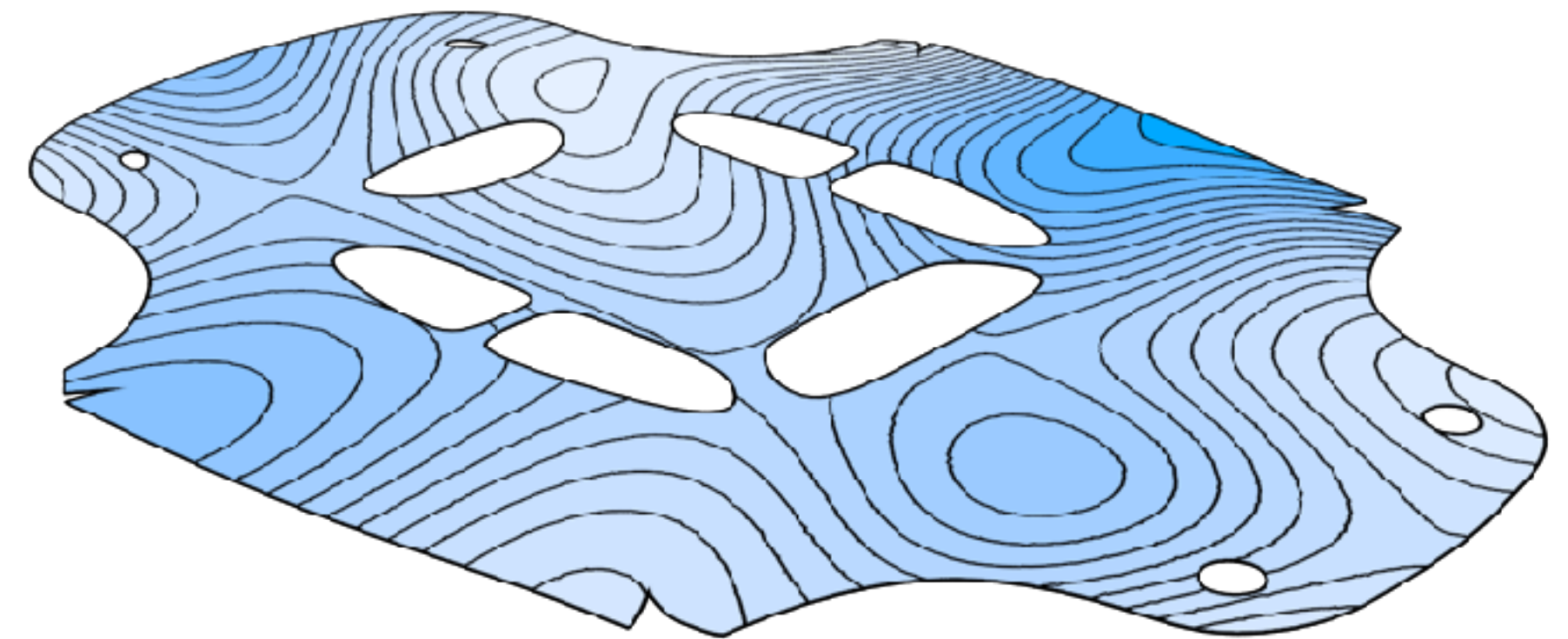
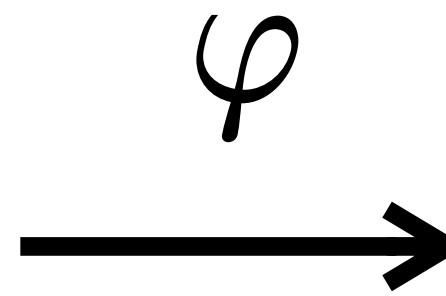
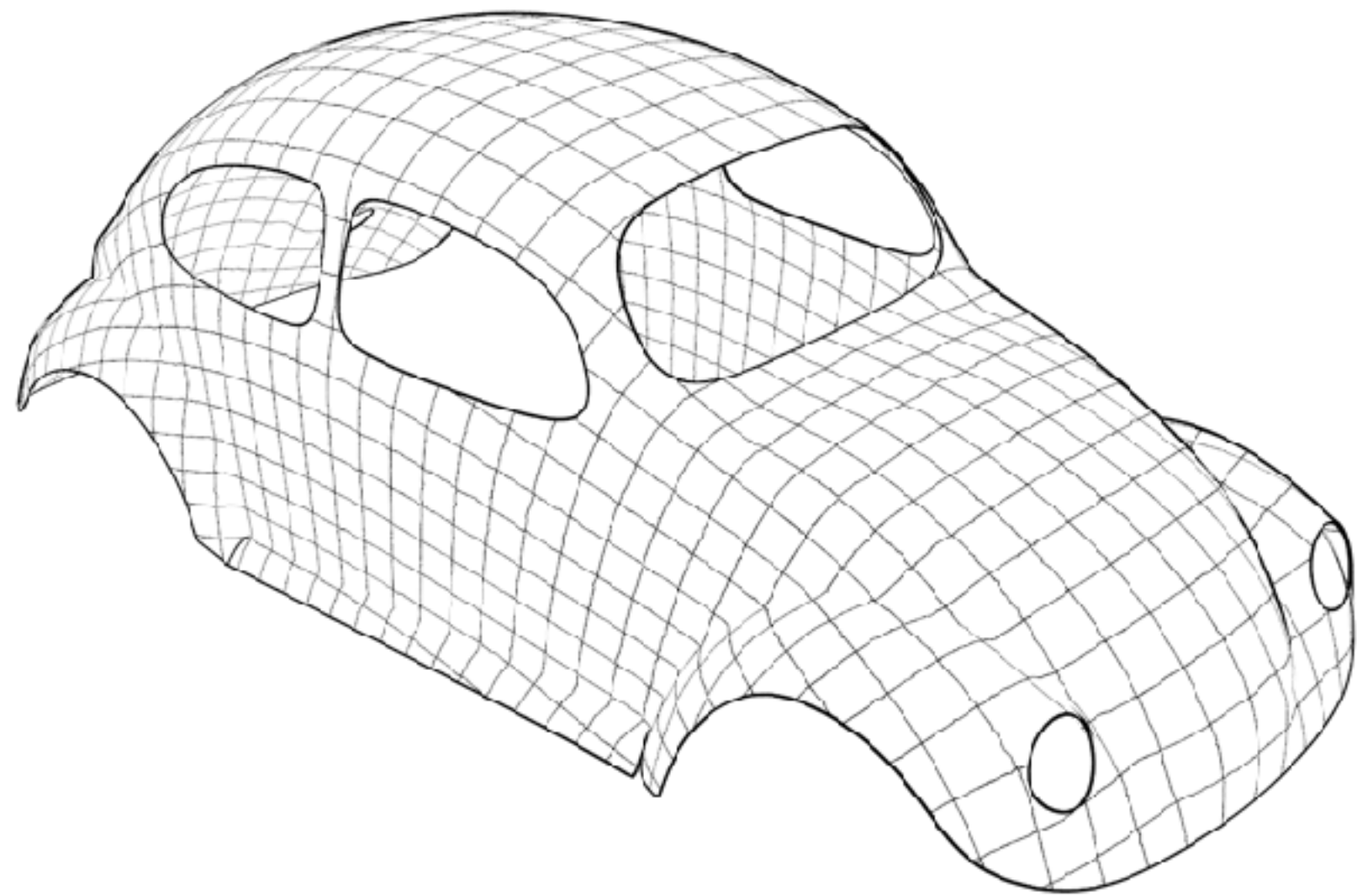


Pullback

$$\varphi : M \rightarrow N$$

$$\varphi^* : \Omega^k(N) \rightarrow \Omega^k(M)$$

$$\omega \in \Omega^k(N)$$



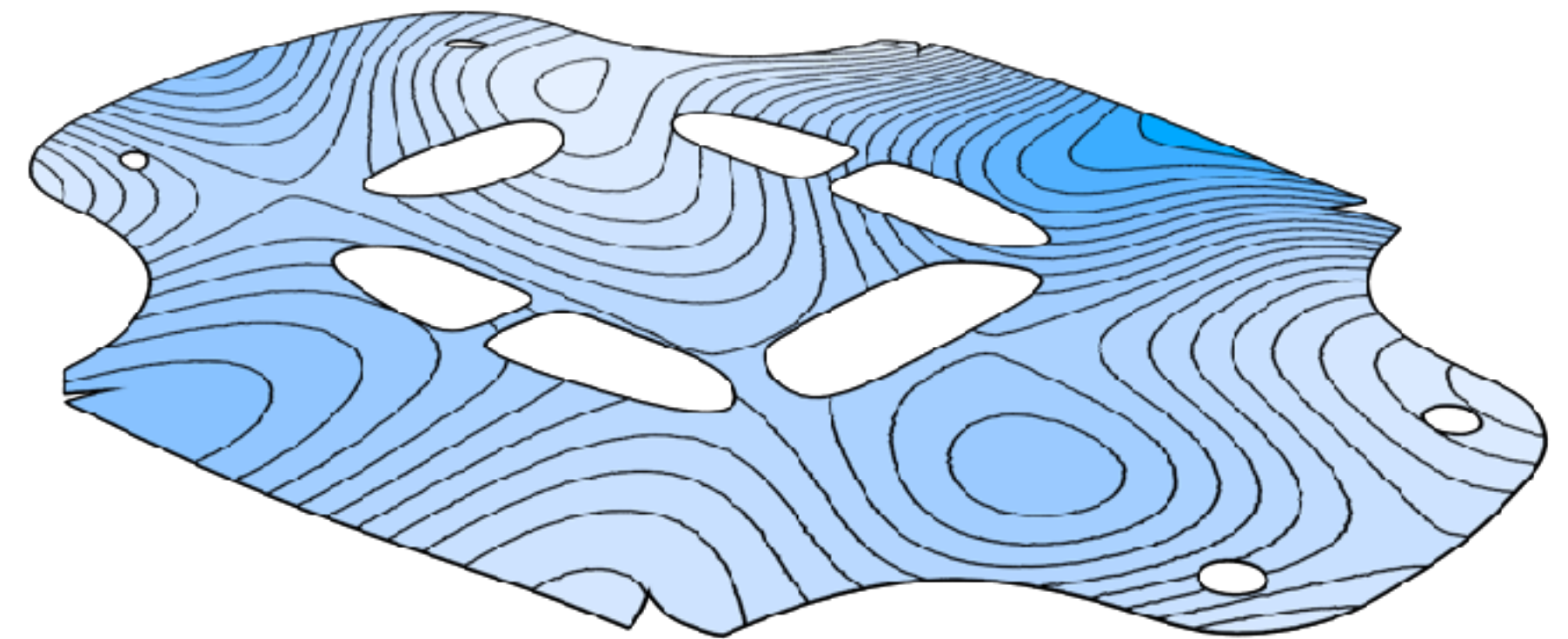
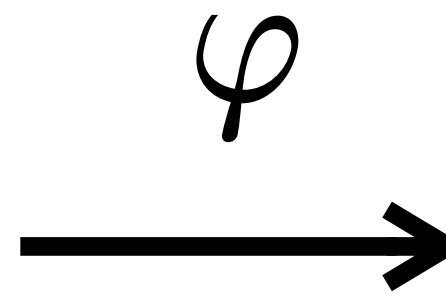
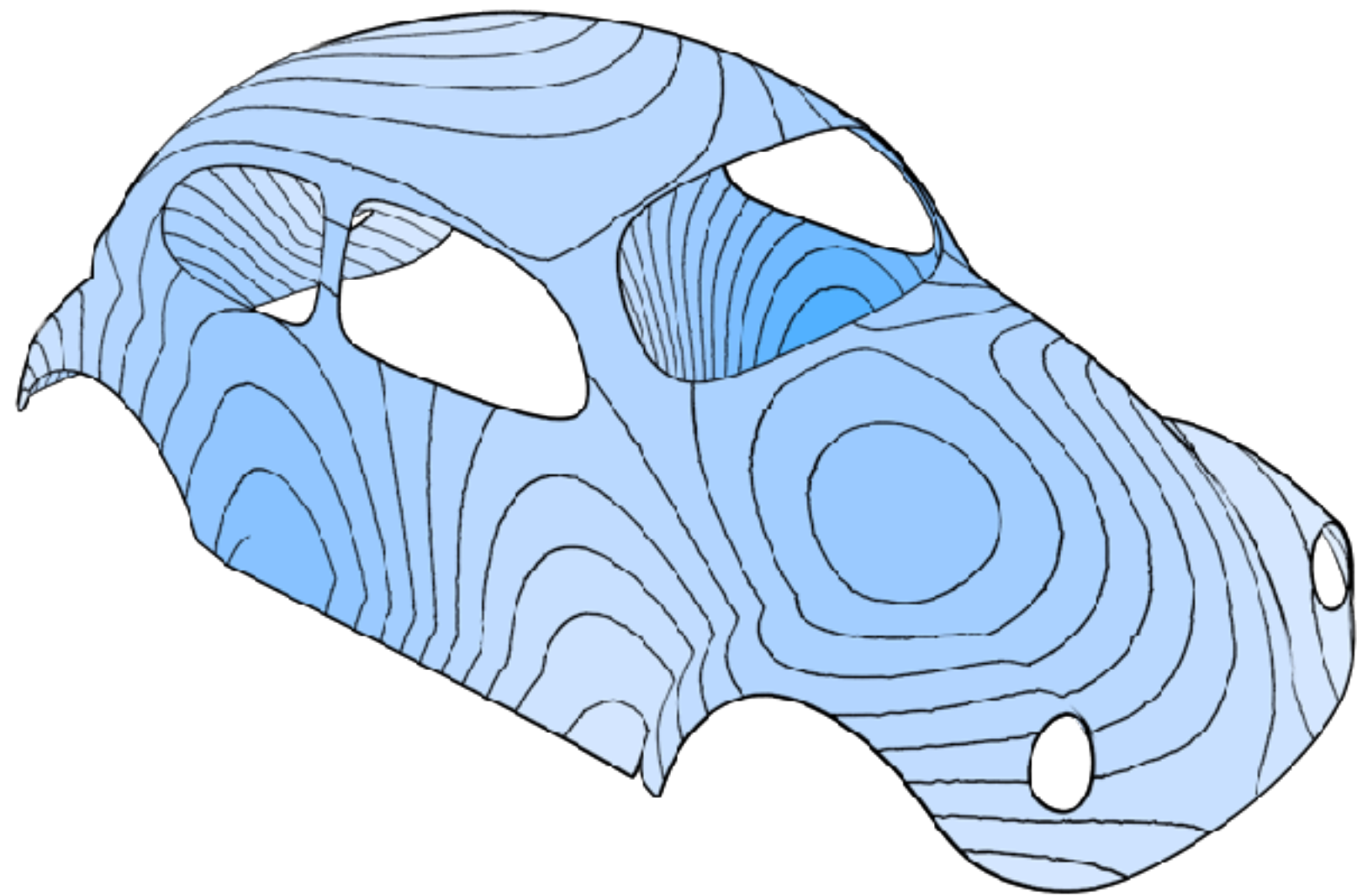
Pullback

$$\varphi : M \rightarrow N$$

$$\varphi^* : \Omega^k(N) \rightarrow \Omega^k(M)$$

$$\varphi^* \omega \in \Omega^k(M)$$

$$\omega \in \Omega^k(N)$$

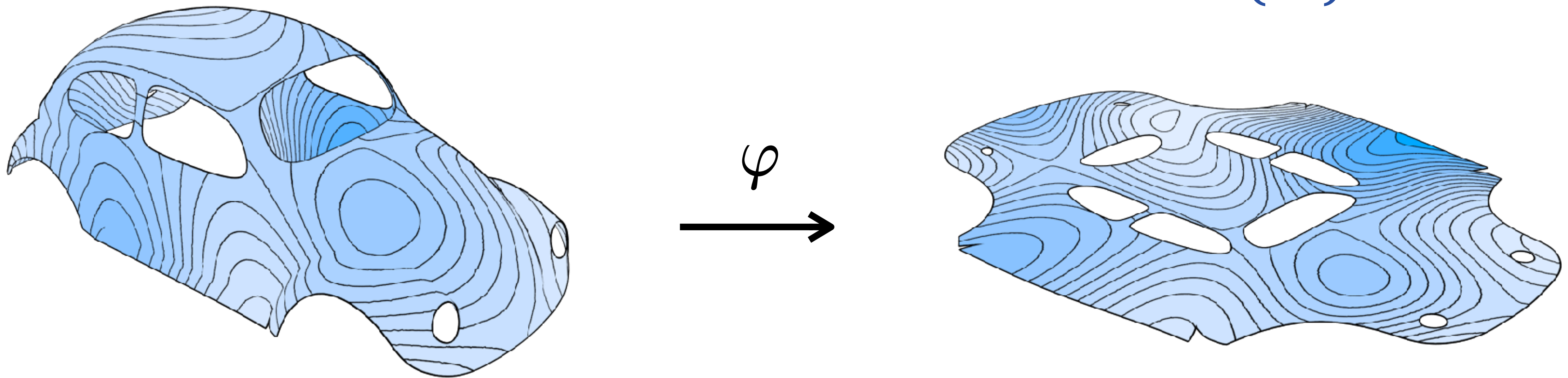


Pullback

$$\varphi : M \rightarrow N \qquad \varphi^* : \Omega^k(N) \rightarrow \Omega^k(M)$$

$$\varphi^* \omega \in \Omega^k(M)$$

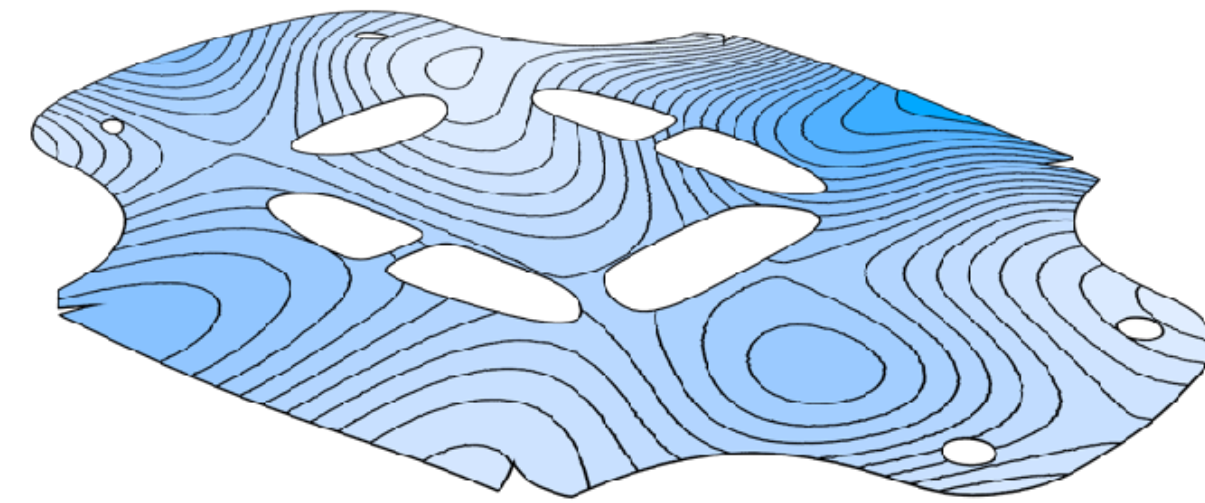
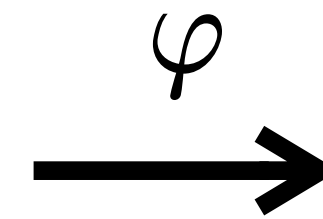
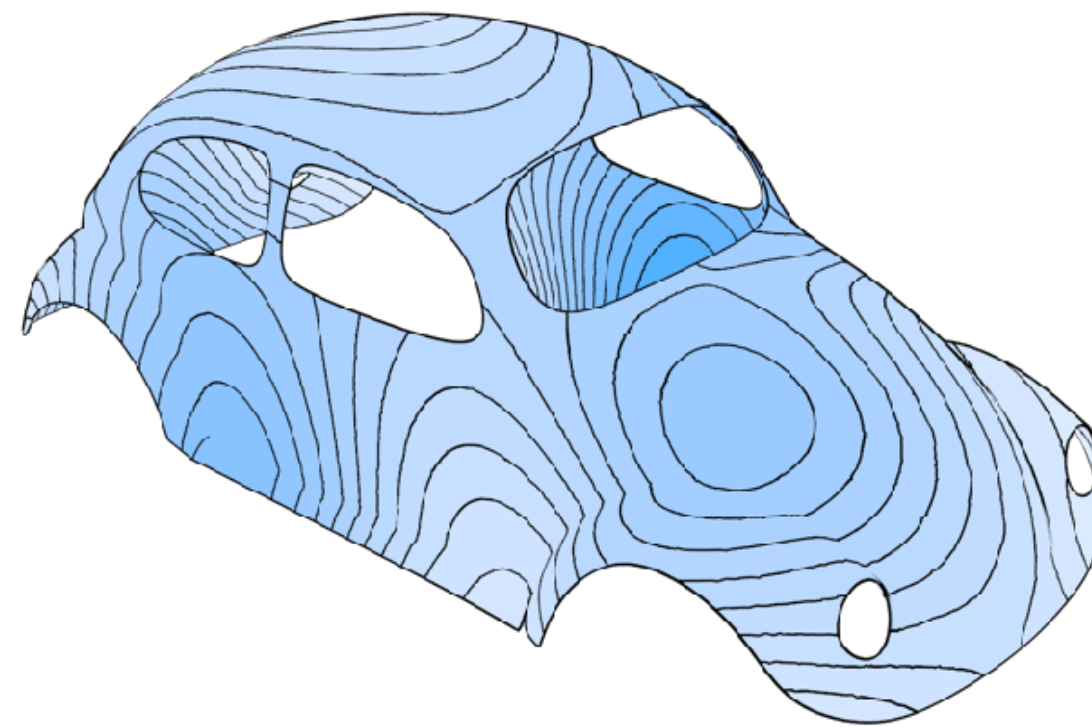
$$\omega \in \Omega^k(N)$$



$$(\varphi^* \omega)_p(\mathbf{v}_1, \dots, \mathbf{v}_k) = \omega|_{\varphi(p)}(d\varphi(\mathbf{v}_1), \dots, d\varphi(\mathbf{v}_k))$$

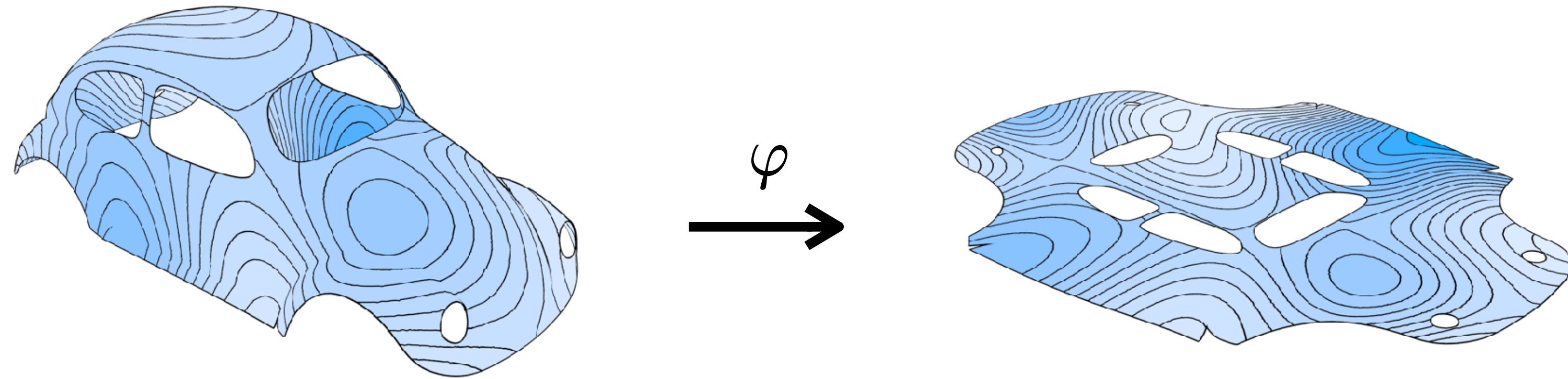
Simple change of variable formula

$$\int_{\varphi(S)} \omega = \int_S \varphi^* \omega$$



Simple change of variable formula

$$\int_{\varphi(S)} \omega = \int_S \varphi^* \omega$$



In vector calculus, if $\mathbf{F} = d\varphi$ is the Jacobian matrix
 $J = \det(\mathbf{F})$ the Jacobian determinant

function $(\varphi^* f)(\mathbf{x}) = f(\varphi(\mathbf{x}))$

circulation $\varphi^* \mathbf{v}^b = (\mathbf{F}^T \mathbf{v})^b$

flux $\varphi^*(\star \mathbf{v}^b) = \star(J \mathbf{F}^{-1} \mathbf{v})^b$

density $\varphi^*(\star f) = J f$

Simple change of variable formula

Conversion formulae [hide]						
	σ	P	S	τ	T	M
$\sigma =$	σ	$J^{-1} P F^T$	$J^{-1} F S F^T$	$J^{-1} \tau$	$J^{-1} R T F^T$	$J^{-1} F^{-T} M F^T$ (non isotropy)
$P =$	$J \sigma F^{-T}$	P	$F S$	τF^{-T}	$R T$	$F^{-T} M$
$S =$	$J F^{-1} \sigma F^{-T}$	$F^{-1} P$	S	$F^{-1} \tau F^{-T}$	$U^{-1} T$	$C^{-1} M$
$\tau =$	$J \sigma$	$P F^T$	$F S F^T$	τ	$R T F^T$	$F^{-T} M F^T$ (non isotropy)
$T =$	$J R^T \sigma F^{-T}$	$R^T P$	$U S$	$R^T \tau F^{-T}$	T	$U^{-1} M$
$M =$	$J F^T \sigma F^{-T}$ (non isotropy)	$F^T P$	$C S$	$F^T \tau R^{-T}$ (non isotropy)	$U T$	M
$J = \det(F), \quad C = F^T F = U^2, \quad F = R U, \quad R^T = R^{-1}$						
$P = J \sigma F^{-T}, \quad \tau = J \sigma, \quad S = J F^{-1} \sigma F^{-T}, \quad T = R^T P, \quad M = C S$						

https://en.wikipedia.org/wiki/Stress_measures

function

$(\varphi^* f)(\mathbf{x}) = f(\varphi(\mathbf{x}))$

circulation

$\varphi^* \mathbf{v}^b = (F^T \mathbf{v})^b$

flux

$\varphi^*(\star \mathbf{v}^b) = \star (J F^{-1} \mathbf{v})^b$

density

$\varphi^*(\star f) = J f$

Exterior derivative

$$d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$$

- For 0-form $f \in \Omega^0(M)$ (i.e. ordinary function $f : M \rightarrow \mathbb{R}$)

$\langle df | \mathbf{v} \rangle$ is the directional derivative

- Leibniz rule

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge d\eta$$

where ω is a k -form

- $dd\alpha = 0$

Exterior derivative (in 3D)

On 0-form: gradient

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

On 1-form: curl

$$\begin{aligned} d(v_1 dx + v_2 dy + v_3 dz) \\ &= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) dy \wedge dz \\ &\quad + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right) dz \wedge dx \\ &\quad + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) dx \wedge dy \end{aligned}$$

On 2-form: divergence

$$\begin{aligned} d(v_1 dy \wedge dz + v_2 dz \wedge dx + v_3 dx \wedge dy) \\ &= \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \right) dx \wedge dy \wedge dz \end{aligned}$$

Exterior derivative

$$ddf = 0$$

$$\nabla \times (\nabla f) = 0$$

$$dd\mathbf{v}^\flat = 0$$

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0$$

Exterior derivative

How does d interact with other operators?

- **Integral:** Stokes Theorem $\int_S d\omega = \int_{\partial S} \omega$
- **Wedge:** Leibniz rule $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge (d\eta)$
- **Hodge star:** Laplacian $(-1)^k \Delta\omega = \star^{-1} d \star d\omega - d \star^{-1} d \star \omega$
- **Pullback:** Just commute $d\varphi^* \omega = \varphi^* d\omega$
- **Interior product:** Lie derivative/Reynolds transport theorem
$$\mathcal{L}_V \omega = (di_V + i_V d)\omega$$

Stokes theorem

$$\int_S d\omega = \int_{\partial S} \omega$$

Fundamental Theorem of Calculus

$$\int_a^b \nabla f \cdot d\mathbf{l} = f(b) - f(a)$$

Kelvin–Stokes Curl Theorem

$$\iint_S (\nabla \times \mathbf{v}) \cdot \mathbf{n} dA = \oint_{\partial S} \mathbf{v} \cdot d\mathbf{l}$$

Divergence Theorem

$$\iiint_V (\nabla \cdot \mathbf{v}) dV = \oiint_{\partial V} \mathbf{v} \cdot \mathbf{n} dA$$

Leibniz rule

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge (d\eta)$$

1st Green's Identity

$$\iiint_V \mathbf{v} \cdot \nabla f \, dV = - \iiint_V f \nabla \cdot \mathbf{v} \, dV + \oint_{\partial V} f \mathbf{v} \cdot \mathbf{n} \, dA$$

Laplacian (scalar)

$$\Delta f = \star^{-1} d \star df$$

Vector calculus $\Delta f = \nabla \cdot \nabla f$

Under orthonormal coordinate

$$\Delta f = \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f + \frac{\partial^2}{\partial z^2} f$$

Laplacian (scalar)

$$\Delta f = \star^{-1} d \star df$$

Under spherical coordinate

$$\star d\theta = \sin \theta d\phi$$

$$\star d\phi = -\frac{1}{\sin \theta} d\theta$$

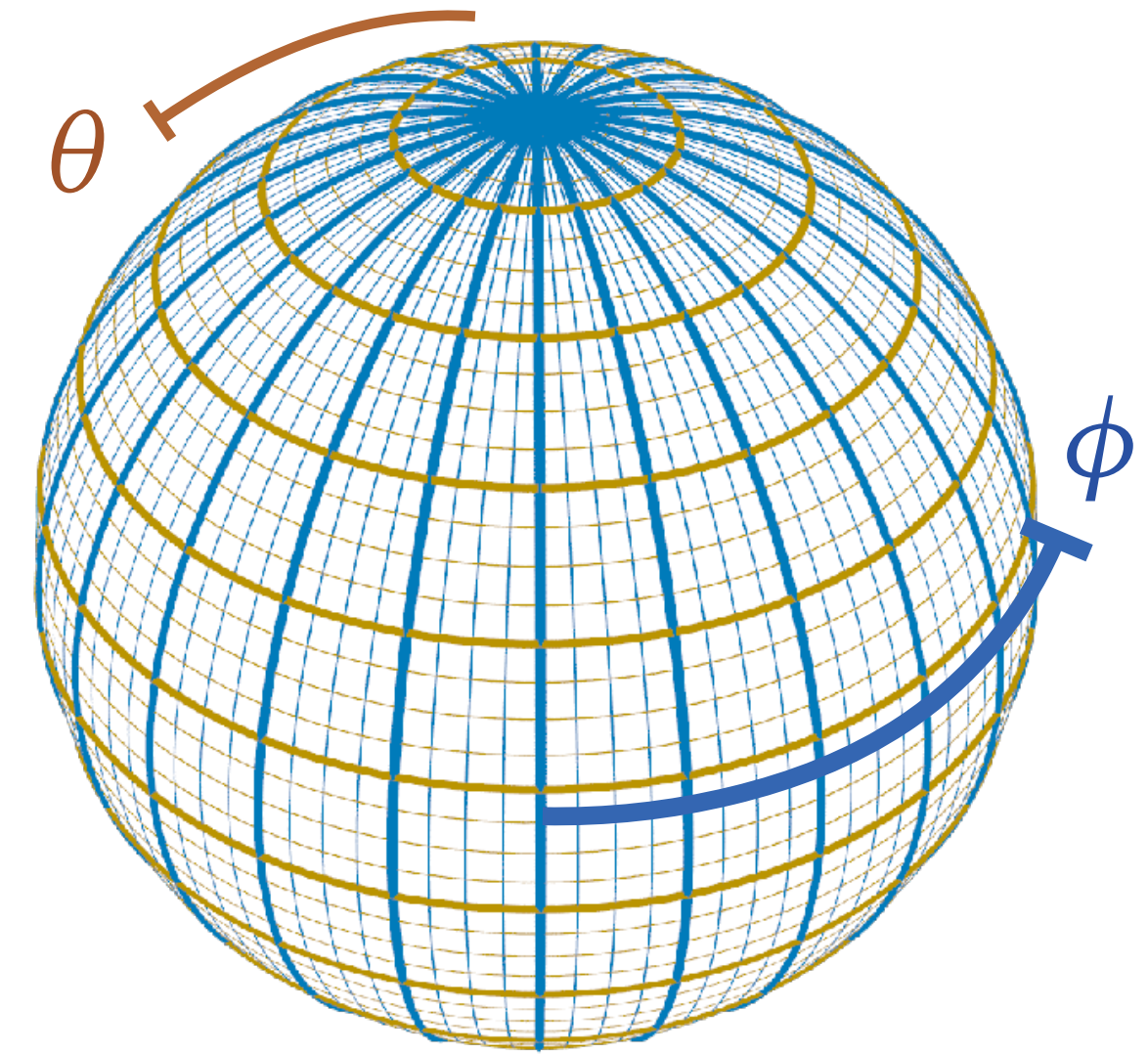
$$\star 1 = \sin \theta d\theta \wedge d\phi$$

$$df = \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial \phi} d\phi$$

$$\star df = \frac{\partial f}{\partial \theta} \star d\theta + \frac{\partial f}{\partial \phi} \star d\phi = \sin \theta \frac{\partial f}{\partial \theta} d\phi - \frac{1}{\sin \theta} \frac{\partial f}{\partial \phi} d\theta$$

$$d \star df = \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) d\theta \wedge d\phi + \frac{1}{\sin \theta} \frac{\partial^2 f}{\partial \phi^2} d\theta \wedge d\phi$$

$$\Delta f = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}$$



Exterior derivative and pullback

$$d\varphi^*\omega = \varphi^*d\omega$$

$$\int_S d(\varphi^*\omega) = \int_{\varphi(\partial S)} \omega = \int_{\partial(\varphi(S))} \omega = \int_S \varphi^*(d\omega)$$

Exterior derivative and pullback

$$\int_S d(\varphi^* \omega) = \int_{\varphi(\partial S)} \omega = \int_{\partial(\varphi(S))} \omega = \int_S \varphi^*(d\omega)$$

In vector calculus, if $\mathbf{F} = d\varphi$ is the Jacobian matrix

$J = \det(\mathbf{F})$ the Jacobian determinant

$$\mathbf{F}^T \nabla f = \nabla(f(\varphi(\mathbf{x})))$$

$$J\mathbf{F}^{-1}(\nabla \times \mathbf{v}) = \nabla \times (\mathbf{F}^T \mathbf{v})$$

$$J\nabla \cdot \mathbf{v} = \nabla \cdot (J\mathbf{F}^{-1} \mathbf{v})$$

Next Topics

- Lie derivative (how to differentiate integrals)

$$\frac{d}{dt} \int_{S(t)} \omega(t) = \int_{S(t)} \frac{\partial \omega(t)}{\partial t} + \int_{S(t)} i_{\dot{S}} d\omega + \int_{\partial S(t)} i_{\dot{S}} \omega$$

- L^2 -norm of differential forms

$$\langle\langle \alpha, \alpha \rangle\rangle = \int_M \alpha \wedge \star \alpha$$