Homework 4 (40 pts)

Figure 1 The ground state cross field on the bunny.

For the end of the quarter, HW4 only has the implementation part. The implementation is the best way to understand the complex line bundle field design in geometry processing. The goal is to

- **(40 pts)** Compute the “smoothest cross field” as shown in the figure above by the smallest eigenvalue problem of a complex Laplacian.
- **(Bonus +10 pts)** Find and visualize all the singularities of the cross field as shown in the figure.

**Representation of the cross field**

In the template file hw4.hipnc, the final cross field is generated by copying the cross geometry (or any quarter-turn symmetric geometry) to all the points of the mesh with a certain orientation defined pointwise. This orientation has been setup in the pointwrangle node “setup_orientation.”

Indeed the cross orientation on each 2D tangent plane has only one rotation-about-normal degree of freedom. As shown there, a reference direction “vector basis” is given by the direction of a half-edge i@hedge sourced from this point. (The point attribute i@hedge has been set up for you; it is a pointer from point to vertex as an arbitrary half-edge sourced from this point.) With respect to this base direction “basis”, the cross field’s orientation is the rotation by angle @phase/4 about the normal from this direction. The division by 4 in @phase/4 makes it so that as @phase traverses from 0 to 2π, the actual rotation traverses from 0 to π/2 (quarter turn). Indeed the cross field returns to the original state after only a quarter turn.

Now go to the pointwrangle node “select_eigenfunction” where

\[ @\text{phase} = \text{atan2}(\text{Im}(\psi), \text{Re}(\psi)) = \text{arg}(\psi), \quad i.e. \psi = e^{i(\text{@phase})} \]  

(1)
for some $\mathbb{C}$ value $\psi$ (with magnitude $r$). That is, the unknown @phase is now encoded in the complex-valued function $\psi$ defined on points. Again, as @phase traverses from 0 to $2\pi$, the cross field rotates by a quarter turn and returns to the original state, and simultaneously the complex variable $\psi$ also traverses through all complex phases and returns to the original state.

**Smallest eigenvalue problem of the connection Laplacian**

To measure how the neighboring cross fields are parallel to each other, we take their difference: let $ij$ be a half-edge

$$(d^\nabla \psi)_{ij} = e^{-i\alpha_{ij}}\psi_j - \psi_i$$

where the angle-valued 1-form $\alpha$ (the vertex attribute $\mathbf{f@A}$ in the Houdini file) is necessary to account for the fact that the reference directions ($\mathbf{i@hedge}$) are also chosen incoherently on each point.

The crucial part is to implement a correct $\alpha$ otherwise the resulting cross field would just be a random mess.

Suppose a correct $\alpha$ is given, then minimizing

$$\sum_{(ij)\in\text{hedges}} (\text{hedge weight})_{ij}|(d^\nabla \psi)_{ij}|^2$$

is accomplished by finding the eigenfunction corresponding to the smallest eigenvalue of the eigenvalue problem

$$\langle d^\nabla \star_1 d^\nabla \rangle L \psi = \lambda \star_0 \psi.$$  (4)

Here the conjugate-transpose $\langle \cdot \rangle^\dagger$ is called the Hermitian transpose. (In `scipy.sparse`, while `transpose()` gives the transpose of the matrix, `getH()` gives the Hermitian transpose of the matrix.)

Suppose you have $\alpha$, you can modify the familiar Laplacian into this connection Laplacian (by modifying $d$ to $d^\nabla$).

**The connection $\alpha$ ($\mathbf{f@A}$)**

In the context of vector field (rather than cross field) the standard connection is the **Levi-Civita connection** given by the following.

Let $i,j$ be neighboring points connected by the half-edge $ij$ as well as $ji$. Using the reference direction (labeled as $\mathbf{1}_i$ and $\mathbf{1}_j$) we can assign the angle $\phi_{ij}$ and $\phi_{ji}$ of the shared half edges $ij$ and
respectively. This angle $\phi_{ij}, \phi_{ji}$ is the partial sum of the \textit{scaled} interior angles (labeled as $\beta'_k$ and $\gamma'_l$) where $\beta'_k = \frac{2\pi}{2\ell} \beta_k$ and similarly for all other vertices.

Now insisting that the shared half-edge vectors $e^{i\phi_{ij}}$ (in reference to $\mathbf{1}_i$) and $e^{i\phi_{ji}}$ (in reference to $\mathbf{1}_j$) are parallel vectors but with opposite signs, we know that the Levi-Civita connection $\alpha_{ij}^{LC}$ must satisfy

$$e^{\alpha_{ij}^{LC}} e^{i\phi_{ij}} = -e^{i\phi_{ji}}$$

(5)

Since $-1 = e^{i\pi}$ the above equation simply states

$$\alpha_{ij}^{LC} = \phi_{ji} - \phi_{ij} - \pi \mod 2\pi.$$  

(6)

How about the connection $\alpha$ for the cross field? Since a rotation $e^{i\alpha} \psi$ by angle $\alpha$ to the complex number $\psi$ amounts to only rotating $\alpha/4$ of the actual physical cross field in the tangent plane, we would need to set

$$\alpha_{ij} = 4\alpha_{ij}^{LC}.$$  

(7)

\textbf{Input}: A closed triangle mesh. A reference half edge per point.

1: For each point $i$, assign $\phi_{ij}$ as the angle of the half-edge $ij$ relative to the reference half edge at $i$.

2: Compute the Levi-Civita connection $\alpha_{ij}^{LC}$ for the tangent bundle.

3: Compute the connection for cross fields $\alpha_{ij} = 4\alpha_{ij}^{LC}$.

4: Build the covariant derivative matrix $\nabla^V$ so that $(\nabla^V \psi)_{ij} = e^{-i\alpha_{ij}} \psi_j - \psi_i$.

5: Build $\star_0, \star_1$ by the area weights and cotan weights as usual, and build $L = \bar{d}^V \star_1 d^V$.

6: Solve the smallest eigenvalue problem $L \psi = \lambda \star_0 \psi$.

7: Let $\@phase$ be the complex phase of $\psi$. Then set the cross field direction as the one rotated from the reference half edge direction by angle $\@phase/4$.

\textbf{By the way...}

the cross field is a quarter-turn symmetric object. You can replace the “4” by any $n$ in the entire setup then you will obtain an $n$-direction field. An $n$-direction symmetric object will return to its original state after a $2\pi/n$ turn. For $n = 1$ we have vector field. For $n = 2$ we have bidirection field. $n = 3, 5, 6$ also gives interesting patterns.

A singularity of an $n$ direction field is that, as we walk around that singularity, the field has turned $m \cdot 2\pi/n$ for some integer $m$ analogous to the Poincaré index for vector fields. The total sum of indices $m$ will be $n \chi(M) = (2 - 2g)n$.

In fact, this hairyball formula does not prevent us from considering $n = \frac{1}{2}$, since the Euler characteristic is always even. The corresponding $n$-direction field would be objects that returns to original state only after $4\pi$ turn! (Quaternions are like that.) The $n$-direction fields with $n = \frac{1}{2}$ are called the \textit{spinor fields}. This is what it means in quantum physics that some elementary particles are spin-$\frac{1}{2}$; though in that context $M$ is a 4D spacetime.