CSE 270 (WI24)
Vector Field Design
Albert Chern
Hairy Ball Theorem

- Hairy ball theorem
- Optimal sections of a bundle
- Curvature and singularities
- Gauge theory in physics
Hairy ball theorem
**Theorem** (Hairy ball theorem)

Every continuous vector field on a topological sphere must have zeros (a.k.a. singularities).

(singularity in orientation) & (continuity) forces the vector field dipping to zero
Hairy ball theorem

**Theorem** (Hairy ball theorem)
Every continuous vector field on a topological sphere must have zeros (a.k.a. singularities).

Keenan Crane 2005
**Theorem** (General hairy ball theorem)

Every continuous vector field on a closed surface must have total index equal to the Euler characteristic.

**Definition** The index of a singularity is the turning number of the vector field around the singularity. That is, it is the number of full (counterclockwise) turns the vector makes when we walk counterclockwise around a small circle around the singularity.
Recall:

\[(K \, dA)_i = 2\pi - \left( \text{turning angle of a loop around the point } i \right)\]
Poincaré index in angle defect

Recall:

$$(K \, dA)_i = 2\pi - \left( \text{turning angle of a loop around the point } i \right)$$

Total turning angle of $\mathbf{v} = 2\pi \sum \text{index} - \int \text{region enclosed } K \, dA$

*Index observation from a distance*
Theorem (General hairy ball theorem)
Every continuous vector field on a closed surface must have total index equal to the Euler characteristic.

Total turning angle of $v = 2\pi \sum \text{index} - \int \text{region enclosed} \ K \ dA$

Let the enclosed region be the entire surface. Apply $\int_M K \ dA = 2\pi \chi$
Revising the notion of function

**Theorem** (General hairy ball theorem)

Every continuous vector field on a closed surface must have total index equal to the Euler characteristic.

- If vector field is like a function over a surface, what prevents us from assigning nonzero value everywhere?

- Unlike a traditional function \( f : S_1 \rightarrow S_2 \)
  a vector field \( \mathbf{u} \) takes value in different spaces \( \mathbf{u}|_x \in T_x M \)

- In fact a more unifying picture to talk about “function” is to look at the *function graph*. 
Revising the notion of function

Plotting $f : S_1 \rightarrow S_2$
Revising the notion of function

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Revising the notion of function

Plotting a tangent vector field $\mathbf{u}$ on $M$
Revising the notion of function

Plotting a tangent vector field $\mathbf{u}$ on $M$
Revising the notion of function

The “canvas” as a (higher-dimensional) manifold is called a **bundle**. Here we have the tangent bundle $T M$. $u \in \Gamma(T M)$ is called a section of the tangent bundle.
**Theorem** (Hairy ball theorem)

The tangent bundle of a closed surface is twisted in such a way that every pair of sections must have the total signed intersection equals to the Euler characteristic.
Optimal Sections of a Bundle

- Hairy ball theorem
- Optimal sections of a bundle
- Curvature and singularities
- Gauge theory in physics
Optimal section

**Goal** Find the optimal section of a bundle.

- Smoothest vector field
- Patterns best align with prescribed directions
- Automatically finds the optimal singularity locations.
Optimal section

Smoothest tangent vector fields

[Knöppel, Crane, Pinkall & Schröder 2013]
Globally optimal direction field
Optimal section

Impainting n-direction field

[Brandt, Scandolo, Eisemann & Hildebrandt 2018]
Modeling n-symmetric vector fields using higher-order energies
Non-orthogonal cross fields

[Diamanti, Vaxman, Panozzo & Sorkine-Hornung 2014]
Designing N-PolyVector fields with complex polynomials
Non-orthogonal cross fields with physical constraints

[Sageman-Furnas, Chern, Ben-Chen & Vaxman 2019]
Chebyshev nets from commuting polyvector field
Optimal section

N-direction field with physical constraints

[Vekhter, Zhuo, Fandino, Huang & Vouga 2019]
Weaving geodesic foliations
Optimal section

Complex phase field

[Knöppel, Crane, Pinkall & Schröder 2015]
Stripe patterns on surfaces
Optimal section

Complex phase field for vortex detection

[Weißmann, Pinkall & Schröder 2014]
Smoke rings from smoke

[Chern, Knöppel, Pinkall & Schröder 2017]
Inside fluid: Clebsch maps for visualization and processing
Optimal section

$C^2$ or $H$ phase field for quad meshing

[Fang, Bao, Tong, Desbrun & Huang 2018]
Quadrangulation through Morse-parametrization hybridization
Optimal section

Phase field for nearly conformal deformations

[Chern, Pinkall & Schröder 2015]
Close-to-conformal deformation of volumes
Optimal section

$\mathbb{H}$ spinor field for nearly isometric reconstruction

[Chern, Knöppel, Pinkall & Schröder 2018]
Shape from metric
Optimal section

Frame field for hexahedral meshing

[Corman & Crane 2019]
Symmetric moving frame
Optimal section

Octahedral variety phase field for hexahedral meshing

[Palmer, Bommes & Solomon 2020]
Algebraic representations for volumetric frame fields
Common ingredient: gauge theory

- Find an algebraic representation of the value space $E_x$ at each $x \in M$.
- Describe the **connection** (a.k.a. **parallel transport**) between adjacent value spaces.
- The derivatives (finite differences) of the field respect the connections.
- Model some energy to minimize.
Discrete tangent bundle
Discrete tangent bundle

Each value space $E_i$

$\Psi_i \in E_i$
Discrete tangent bundle

Each value space $E_i$ is viewed as a complex plane via an arbitrary basis.

$\Psi_i \leftrightarrow \psi_i = r_i e^{i\theta_i} \in \mathbb{C}$

vector field $\leftrightarrow \mathbb{C}$-valued function
Discrete connection

For each adjacent value spaces

\[ 1_i \in E_i \]

\[ \Psi_i \leftrightarrow \psi_i = r_i e^{i\theta_i} \in \mathbb{C} \]

vector field \( \leftrightarrow \) \( \mathbb{C} \)-valued function
Discrete connection

For each adjacent value spaces

\[ \Psi_i \leftrightarrow \psi_i = r_i e^{i \theta_i} \in \mathbb{C} \]

\[ \Psi_j \leftrightarrow \psi_j = r_j e^{i \theta_j} \in \mathbb{C} \]
Discrete connection

For each adjacent value spaces

\[ \Psi_i \in E_i \]

\[ \Psi_j \in E_j \]

\[ \psi_i = r_i e^{i\theta_i} \in \mathbb{C} \]

\[ \psi_j = r_j e^{i\theta_j} \in \mathbb{C} \]

parallel transported basis

\[ \alpha_{ij} \]

\[ 1_i \in E_i \]

\[ 1_j \in E_j \]
Discrete connection

For each adjacent points $i, j$ encode the parallel transport by an angle $\alpha_{ij}$, with

$$\alpha_{ij} = -\alpha_{ji} \mod 2\pi$$

i.e. an angle-valued 1-form. It is called the connection as an additional structure for the bundle.
Covariant derivative

The derivative of the section $\Psi$ is denoted by $d^\nabla \Psi$.

In terms of the basis $\mathbb{1}_i$

$$(d^\nabla \Psi)_{ij} \leftrightarrow e^{-i\alpha_{ij}} \psi_j - \psi_i$$

In terms of the basis $\mathbb{1}_j$

$$(d^\nabla \Psi)_{ij} \leftrightarrow \psi_j - e^{i\alpha_{ij}} \psi_i$$

The magnitude $|(d^\nabla \Psi)_{ij}|$ is basis-choice independent.
Covariant derivative

The derivative of the section $\Psi$ is denoted by $d^\nabla \Psi$.

In terms of the basis $1_i$

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In terms of the basis $1_j$

$$(d^\nabla \Psi)_{ij} \leftrightarrow \psi_j - e^{i\alpha_{ij}} \psi_i$$

The magnitude $|(d^\nabla \Psi)_{ij}|$ is basis-choice independent.
Dirichlet energy

The magnitude $|(d\nabla \psi)_{ij}|$ is basis-choice independent.

$$\mathcal{E}[\psi] = \sum_{ij \in \text{half edges}} w_{ij} |(d\nabla \psi)_{ij}|^2$$

connection derivative

$$d\nabla = \begin{pmatrix} i & j \\ \vdots & \vdots \\ e^{-\phi \alpha_{ij}} & \vdots \end{pmatrix}$$

cotan weights
The magnitude $| (d^\nabla \Psi)_{ij} |$ is basis-choice independent.

$$\mathcal{E}[\Psi] = \sum_{ij \in \text{half edges}} w_{ij} |(d^\nabla \Psi)_{ij}|^2$$

$$= \overline{\psi}^T \left[ \begin{array}{c} d^\nabla \\ \star_1 \\ d^\nabla \end{array} \right] \psi$$

$$L^\nabla \text{ connection Laplacian}$$
The Dirichlet energy $\mathcal{E}[\Psi] = \overline{\psi}^T L^{\nabla} \psi$ measures the smoothness of the section $\Psi$ (the smaller, the smoother with respect to the parallel transport)

**Ground state problem**

$$\min_{\psi} \overline{\psi}^T [L^{\nabla}] \psi$$

s.t. $\sum_i (\text{point}_{\text{area}})_i |\psi_i|^2 = 1$

$$\Leftrightarrow \min_{\psi} \frac{\overline{\psi}^T [L^{\nabla}] \psi}{\overline{\psi}^T \star_0 \psi}$$

smallest eigenvalue problem

$$\Leftrightarrow L^{\nabla} \psi = \lambda \star_0 \psi$$
The Dirichlet energy $\mathcal{E}[\Psi] = \overrightarrow{\psi}^T L \nabla \psi$ measures the smoothness of the section $\Psi$ (the smaller, the smoother with respect to the parallel transport)

**Ground state with pointwise norm=1 constraint**

$$\min_{\psi} \overrightarrow{\psi}^T [L \nabla] \psi$$

s.t. $|\psi_i|^2 = 1$ $\forall i$

**Ginzburg–Landau model**

$$\min_{\psi} \overrightarrow{\psi}^T L \nabla \psi + \frac{1}{\varepsilon^2} \sum_i \left( \text{point weight}_i \right) (|\psi_i|^2 - 1)^2$$
Complex line bundle setup summary

- The $\mathbb{C}$-valued function $\psi : \{\text{points}\} \rightarrow \mathbb{C}$ describes the geometric object called a section $\Psi \in \Gamma(E)$.

- We need to provide an angle-valued 1-form $\alpha$.

- The Laplacian for $\psi$ is

$$L^\nabla = d^\nabla T \star_1 d^\nabla$$

where $d^\nabla = i_{\psi} - 1 - e^{-\delta_{ij}}$.
Effect of connection

If the complex line bundle is the tangent bundle, then there is a canonical $\alpha$, known as the **Levi-Civita connection**, that describes the metric-induced parallel transport.
Effect of connection

\[ \alpha_{\text{LeviCivita}} \]

\[ \alpha_{\text{customized}} = \alpha_{\text{LeviCivita}} + \mathbf{v}^b \]
Levi-Civita connection

- Assign the polar angle $\phi_{ij}$ of each half-edge
  - In the picture $\phi_{ij} = \sum_{k=1}^{4} \beta'_k$ where $\beta'_k = \frac{\beta_k}{\sum_{\ell} \beta_{\ell}} 2\pi$
- The Levi-Civita connection is $\alpha_{ij} = \phi_{ji} - \phi_{ij} - \pi \mod 2\pi$
Gauge transformation

\[ \varphi_i \in \mathbb{R} \text{ (not angle-valued, but actual real-valued)} \]

\[ \tilde{1}_i = e^{-i\varphi_i} 1_i \quad \tilde{\psi}_i = e^{i\varphi_i} \psi_i \quad \tilde{\alpha}_{ij} = \alpha_{ij} + (d\varphi)_{ij} \]

\[ = \alpha_{ij} + \varphi_j - \varphi_i \]
n-direction field
n-direction field
\[ \psi_i^{\text{vec}} = r_i e^{i\theta_i} \]

\[ \psi_i^{\text{vec}} \rightarrow -\psi_i^{\text{vec}} \]

\[ z \rightarrow z^4 \]

\[ \psi_i^{\text{cross}} = (\psi_i^{\text{vec}})^4 \]
\[ \psi_i^{\text{cross}} = (\psi_i^{\text{vec}})^4 \]

parallel transport for \( \psi_i^{\text{vec}} \)

\[ e^{i\alpha_{ij}} \psi_i^{\text{vec}} \]

parallel transport for \( \psi_i^{\text{cross}} \)

\[ e^{i\alpha_{ij}^{\text{cross}}} \psi_i^{\text{cross}} = e^{i4\alpha_{ij}^{\text{vec}}} (\psi_i^{\text{vec}})^4 \]

Therefore \( \alpha_{ij}^{\text{cross}} = 4\alpha_{ij}^{\text{vec}} \)
\[ \alpha_{ij}^{\text{cross}} = 4 \alpha_{ij}^{\text{vec}} \]

Build
\[ L^{\nabla} = d^{\nabla} \star_1 d^{\nabla} \]

where
\[ d^{\nabla} = \begin{bmatrix} \delta_{ij} & -1 & e^{-i \alpha_{ij}^{\text{cross}}} \\ \vdots & \ddots & \vdots \\ \delta_{ij} & -1 & e^{-i \alpha_{ij}^{\text{cross}}} \end{bmatrix} \]

Solve smallest eigenvalue problem
\[ L^{\nabla} \psi^{\text{cross}} = \lambda \star_0 \psi^{\text{cross}} \]
n-direction field

\[ \psi_i^{\text{vec}} = r_i e^{i \theta_i} \]

\[ \vec{\psi}_i \]

\[ \vec{\psi}_i^{\text{vec}} \]

\[ -i \psi_i^{\text{vec}} \]

\[ \psi_i^{\text{cross}} = (\psi_i^{\text{vec}})^4 \]

\[ z \rightarrow z^{1/4} \]
Turning angle, curvature, singularities

- Hairy ball theorem
- Optimal sections of a bundle
- Curvature and singularities
- Gauge theory in physics
Poincaré index in angle defect

Recall:

$$(K \, dA)_i = 2\pi - \left( \text{turning angle of a loop around the point } i \right)$$

Total turning angle of $v = 2\pi \sum_{\text{singularities enclosed}} \text{index} - \int_{\text{region enclosed}} K \, dA$

*Index observation from a distance*
Poincaré index in angle defect

Total turning angle of $v = 2\pi \sum \text{index} - \int \text{region} K dA$

For a general complex line bundle

- Turning angle from point $i$ to point $j$ $\arg \left( \frac{\psi_j}{e^{i\alpha_{ij}} \psi_i} \right)$

- Total turning angle after a round trip

$\arg \left( \frac{\psi_{i_2}}{e^{i\alpha_{i_1,i_2}} \psi_{i_1}} \right) + \cdots + \arg \left( \frac{\psi_{i_n}}{e^{i\alpha_{i_{n-1},i_n}} \psi_{i_1}} \right)$
Poincaré index in angle defect

Total turning angle of $v = 2\pi \sum_{\text{index}}^{\text{singularity}} - \int_{\text{region}}^{\text{enclosed}} K \, dA$

For a general complex line bundle

- Turning angle from point $i$ to point $j$: $\arg \left( \frac{\psi_j}{e^{i\alpha_{ij}} \psi_i} \right)$

- Total turning angle after a round trip

$$\arg \left( \frac{\psi_{i_2}}{e^{i\alpha_{i_1,i_2}} \psi_{i_1}} \right) + \cdots + \arg \left( \frac{\psi_{i_1}}{e^{i\alpha_{i_n,i_1}} \psi_{i_n}} \right) \text{ mod } 2\pi = -i(\alpha_{i_1,i_2} + \cdots + \alpha_{i_n,i_1})$$
Poincaré index in angle defect

Total turning angle of $v = 2\pi$ \[\sum_{\text{singularities enclosed}} \text{index} - \int_{\text{region enclosed}} K\, dA\]

\[\arg\left(\frac{\psi_{i_2}}{e^{i\alpha_{i_1,i_2}} \psi_{i_1}}\right) + \cdots + \arg\left(\frac{\psi_{i_1}}{e^{i\alpha_{i_n,i_1}} \psi_{i_n}}\right) \mod 2\pi = -\mathbb{i}(\alpha_{i_1,i_2} + \cdots + \alpha_{i_n,i_1})\]

The \textbf{curvature} of the bundle enclosed by the loop is

\[\int K\, dA = \beta = \sum_{\text{around the loop}} \alpha_{i_k,i_{k+1}} \mod 2\pi\]

Knowing which 2 pi branch of curvature for every face can give us Poincaré index.
The curvature of the bundle enclosed by the loop is

$$\int K \, dA = \beta = \sum_{\text{around the loop}} \alpha_{i_k, i_{k+1}} \mod 2\pi$$

Knowing which 2 pi branch of curvature for every face can give us Poincaré index.

Curvature is gauge invariant

$$\varphi_i \in \mathbb{R} \quad \text{(not angle-valued, but actual real-valued)}$$

$$\tilde{1}_i = e^{-\varphi_i} 1_i \quad \tilde{\psi}_i = e^{i\varphi_i} \psi_i \quad \tilde{\alpha}_{ij} = \alpha_{ij} + (d\varphi)_{ij} = \alpha_{ij} + \varphi_j - \varphi_i$$
A complex line bundle over a triangle mesh is equipped with

- angle-valued 1-form $\alpha$
- real-valued 2-form $\beta$ so that $d\alpha = \beta \mod 2\pi$

A complex-valued 0-form $\psi$ and the connection $\alpha$ are relative to an arbitrary reference basis. Under basis transformation,

$$\psi \mapsto e^{i\varphi} \psi \quad \alpha \mapsto \alpha + d\varphi$$

The turning angle of $\psi$ along an edge is

$$\arg\left(\frac{\psi_j}{e^{i\alpha_{ij}} \psi_i}\right)$$

The index formula:

$$(\text{Total turning angle}) = 2\pi \sum \text{index} - \int \beta$$
The index formula: \[(\text{Total turning angle}) = 2\pi \sum \text{index} - \int \beta\]

**Hairball theorem for complex line bundle**

Over the entire closed surface

\[2\pi \sum \text{index} = \int_M \beta\]

**Definition.** The **degree** of a complex line bundle is the integer

\[\frac{1}{2\pi} \int_M \beta\]
Hairball theorem for complex line bundle

Over the entire closed surface

\[ 2\pi \sum \text{index} = \int_M \beta \]

Paint some 2-form total integral being 2pi integer, can you build the rest of the ingredients? (i.e 1-form \( \alpha \))
Singularities in optimal n-direction field

\[ n=1 \]
\[ \alpha = \alpha^{LC} \]
\[ \beta = K dA \]

degree = 2
Singularities in optimal n-direction field

n=2
\[ \alpha = 2\alpha^{LC} \]
\[ \beta = 2K \, dA \]
degree = 4
Singularities in optimal $n$-direction field

$n=3$

$\alpha = 3\alpha^{LC}$

$\beta = 3K \, dA$

degree = 6
Singularities in optimal $n$-direction field

$n = 10$

$\alpha = 10\alpha^{\text{LC}}$

$\beta = 10K \, dA$

degree = 20
Singularities in optimal n-direction field

\[ n = 20 \]
\[ \alpha = 20\alpha^{LC} \]
\[ \beta = 20K \, dA \]
degree = 40
Singularities in optimal n-direction field

singularity of optimal field

independent random samples
Gauge theory in physics

- Hairy ball theorem
- Optimal sections of a bundle
- Curvature and singularities
- Gauge theory in physics
Continuous theory

- In the discrete setting
  \[(d^\nabla \psi)_{ij} = e^{-i\alpha_{ij}} \psi_j - \psi_i\]

- Towards continuous
  \[
  \psi_j \approx \psi_i + d\psi(i\hat{j}) + \cdots \\
  e^{-i\alpha_{ij}} \approx 1 - i\alpha(i\hat{j}) + \cdots \\
  (d^\nabla \psi)(i\hat{j}) \approx (1 - i\alpha(i\hat{j})) (\psi_i + d\psi(i\hat{j})) - \psi_i \\
  = d\psi(i\hat{j}) - i\alpha(i\hat{j})\psi_i + O(|i\hat{j}|^2) \\
  d^\nabla \psi = d\psi - i\alpha\psi\]
Continuous theory

- Continuous covariant derivative

\[ d\nabla \psi = d\psi - i\alpha \psi \]

\[ d\nabla = d - i\alpha \wedge : \Omega^k(M; \mathbb{C}) \to \Omega^{k+1}(M; \mathbb{C}) \]

- Double covariant exterior derivative

\[
\begin{aligned}
  d\nabla d\nabla \psi &= (d - i\alpha \wedge)(d - i\alpha)\psi \\
  &= dd\psi - i d(\alpha \psi) - i\alpha \wedge d\psi - \alpha \wedge \alpha \psi \\
  &= -i\beta \psi \quad \text{curvature of the connection}
\end{aligned}
\]
Electrodynamics

charged particle
Electrodynamics

Coulomb electric force

\[ F = q_2 E \]

electric field

\[ E \in \Omega^1(M; \mathbb{R}) \]

charge density

\[ d \star E = \rho \]

n-form
Electrodynamics

\[ F = q_2E \]

magnetic field
\[ \beta \in \Omega^2(M; \mathbb{R}) \]
\[ d\beta = 0 \]
\[ d \star \beta = J \]

electric current flux form
Electrodynamics

Lorentz magnetic force

\[ F = q_2 \mathbf{v}_2 \times \mathbf{B} \]

\[ F = -q_2 i \mathbf{v}_2 \beta \]

\[ F = q_2 E \]

\( \beta \in \Omega^2(M; \mathbb{R}) \)

\[ d\beta = 0 \]

\[ d \star \beta = J \]
This figure seems to violate the conservation of momentum.
The magnetic field $\beta \in \Omega^2(M)$ satisfies $d\beta = 0$.

Kelvin (1851) and Helmholtz (1858): by Hodge decomposition
\[ \beta = d\alpha \]

The potential $\alpha \in \Omega^1(M)$ is not unique
\[ \alpha, \quad \alpha + d\varphi \quad \text{are equality valid representation.} \]

A change of choice $\alpha \mapsto \tilde{\alpha} = \alpha + d\varphi$ is called a gauge transformation.

All physically observable quantities should be gauge-invariant.
This figure seems to violate the conservation of momentum.

Problem solved if we revise the momentum as \( p^\flat = m v^\flat + q \alpha \)

(but now the momentum is not gauge invariant)
Quantum mechanics

$v_1$
Quantum mechanics

\[ \psi_1 = r_1 e^{i\theta_1} \]

\[ \psi_2 = r_2 e^{i\theta_2} \]

\[ p^b = r^2 d\theta \]
Quantum mechanics
Quantum mechanics

\[ \psi = re^{i\theta} \]
\[ \mathbf{p}^b = r^2 d\theta \]
\[ = \overline{\psi}(-i\hbar d)\psi \]

Total kinetic energy is our Dirichlet energy

\[ \int \frac{1}{2m} |\mathbf{p}^b|^2 = \frac{1}{2m} \int \overline{\psi} d \ast d\psi \]
Quantum electrodynamics

\[ \psi = r e^{i \theta} \]

\[ \mathbf{p}^b = r^2 d \theta + r^2 q \alpha \]

\[ = \bar{\psi}(-i d) \psi + q \alpha \bar{\psi} \psi = \bar{\psi}(-i(d + i q \alpha)) \psi = \bar{\psi}(-i d \nabla) \psi \]

Total kinetic energy is our Dirichlet energy

\[ \int \frac{1}{2m} |\mathbf{p}^b|^2 = \frac{1}{2m} \int \bar{\psi} d \star d \psi \]
\[ \psi = r e^{i\theta} \]
\[ \mathbf{p}^\flat = r^2 d\theta + r^2 q\alpha \]
\[ = \overline{\psi}(-id)\psi + q\alpha \overline{\psi}\psi = \overline{\psi}(-i(d + iq\alpha))\psi = \overline{\psi}(-id^\nabla)\psi \]

Total kinetic energy is our Dirichlet energy
\[ \int \frac{1}{2m} |\mathbf{p}^\flat|^2 = \frac{1}{2m} \int \overline{\psi}(d^\nabla^\top \star d^\nabla)\psi \]
Quantum electrodynamics

Gauge transform in quantum electrodynamics

\[ \psi \mapsto e^{i\varphi} \psi \]
\[ \alpha \mapsto \alpha + d\varphi \]

Curvature of the bundle

\[ \beta = d\alpha \text{ is gauge invariant} \]

Interference pattern is gauge invariant

Singularity location is gauge invariant
Aharonov–Bohm effect

What determines the complex line bundle?
By Hodge decomposition

\[ \beta = d\alpha \quad \text{and} \quad \int_{\text{global loop}} \alpha \]

allows us to reconstruct \( \alpha \)
up to a factor of \( d\varphi \)
Optimal singularity placement

- Prescribe magnetic field $\beta$ (e.g. as Gaussian curvature) and global period $\int_{\text{global}} \alpha$

- Construct $\alpha$ and thus

$$L^\nabla = \overline{d^\nabla^T} \star_1 d^\nabla$$

- Find the singularities of the ground state.
Optimal singularity placement

[Knöppel, Crane, Pinkall & Schröder 2015]
Stripe patterns on surfaces
Optimal singularity placement

Complex phase field for vortex detection

[Weißmann, Pinkall & Schröder 2014] Smoke rings from smoke

[Chern, Knöppel, Pinkall & Schröder 2017] Inside fluid: Clebsch maps for visualization and processing