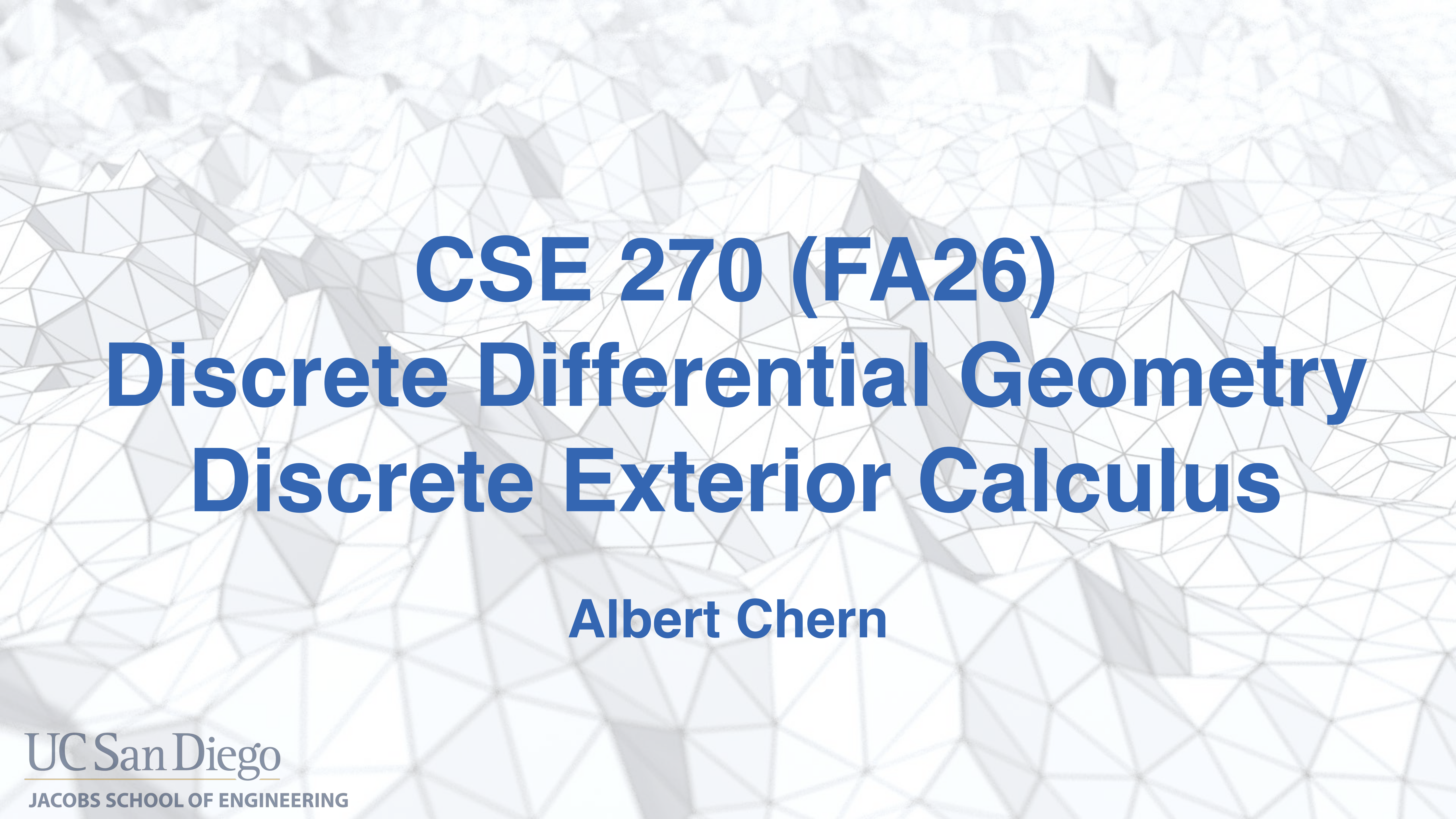




CSE 270 (FA26)

Discrete Differential Geometry

Discrete Exterior Calculus

Albert Chern

Laplacian in Physics

- Laplacian in Physics
- Finite Difference
- Graph Laplacian
- Discrete differential forms
- Discrete Hodge stars
- Example: heat equation

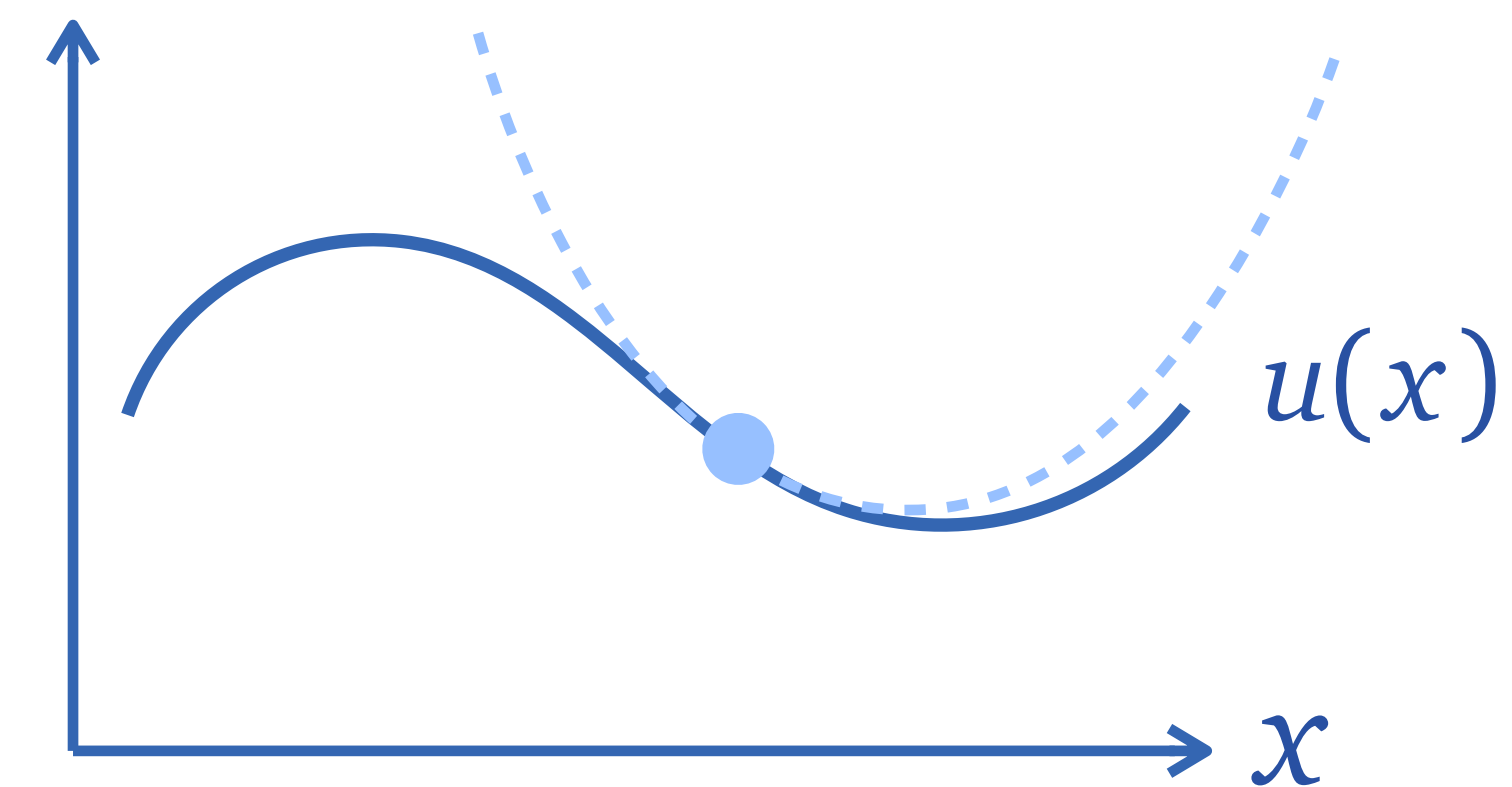
Calculus on Surface

- Given a general (triangulated) domain, consider ***functions*** (***scalar fields***) over it.
- They can represent natural, physical fields, e.g. temperature, displacement, potential,...
- They can also be geometry processing fields like texture coordinates, filters,...
- These field satisfy physical equations or smoothness criterions, involving differential operators on these general domain.

Laplacian

- Laplacian (the Laplace operator) is a 2nd order differential operator that appears in many applications.
- Notation Δ , sometimes written as ∇^2 ; don't confuse it with *hessian*
- For function of one variable $u = u(x)$

$$\Delta u(x) = \frac{d^2 u}{dx^2}(x)$$



- For function of two variables $u = u(x, y)$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Hessian is

$$\text{Hess } u = \begin{bmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial y^2} \end{bmatrix}$$

Laplacian

- For function of one variable $u = u(x)$

$$\Delta u(x) = \frac{d^2 u}{dx^2}(x)$$

- For function of two variables $u = u(x, y)$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

- For function of three variables $u = u(x, y, z)$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Heat equation

- Let $u(x, t)$ represents the (unknown) temperature
- Temperature gradient induces heat flux

$$q(x, t) := -k \frac{\partial u(x, t)}{\partial x} \quad k : \text{conductivity}$$

- Total in-flow of heat flux leads to rise of temperature

$$c \frac{\partial u(x, t)}{\partial t} = - \frac{\partial q(x, t)}{\partial x} \quad c : \text{heat capacity}$$

- Together: $\frac{\partial u}{\partial t} = \frac{k}{c} \frac{\partial^2 u}{\partial x^2} \quad k/c : \text{diffusivity}$

Heat equation

- Let $u(x, y, t)$ represents the (unknown) temperature
- Temperature gradient induces heat flux

$$\begin{bmatrix} q_1(x, y, t) \\ q_2(x, y, t) \end{bmatrix} := -k \begin{bmatrix} \frac{\partial u(x, y, t)}{\partial x} \\ \frac{\partial u(x, y, t)}{\partial y} \end{bmatrix} \quad k : \text{conductivity}$$

- Total in-flow of heat flux leads to rise of temperature

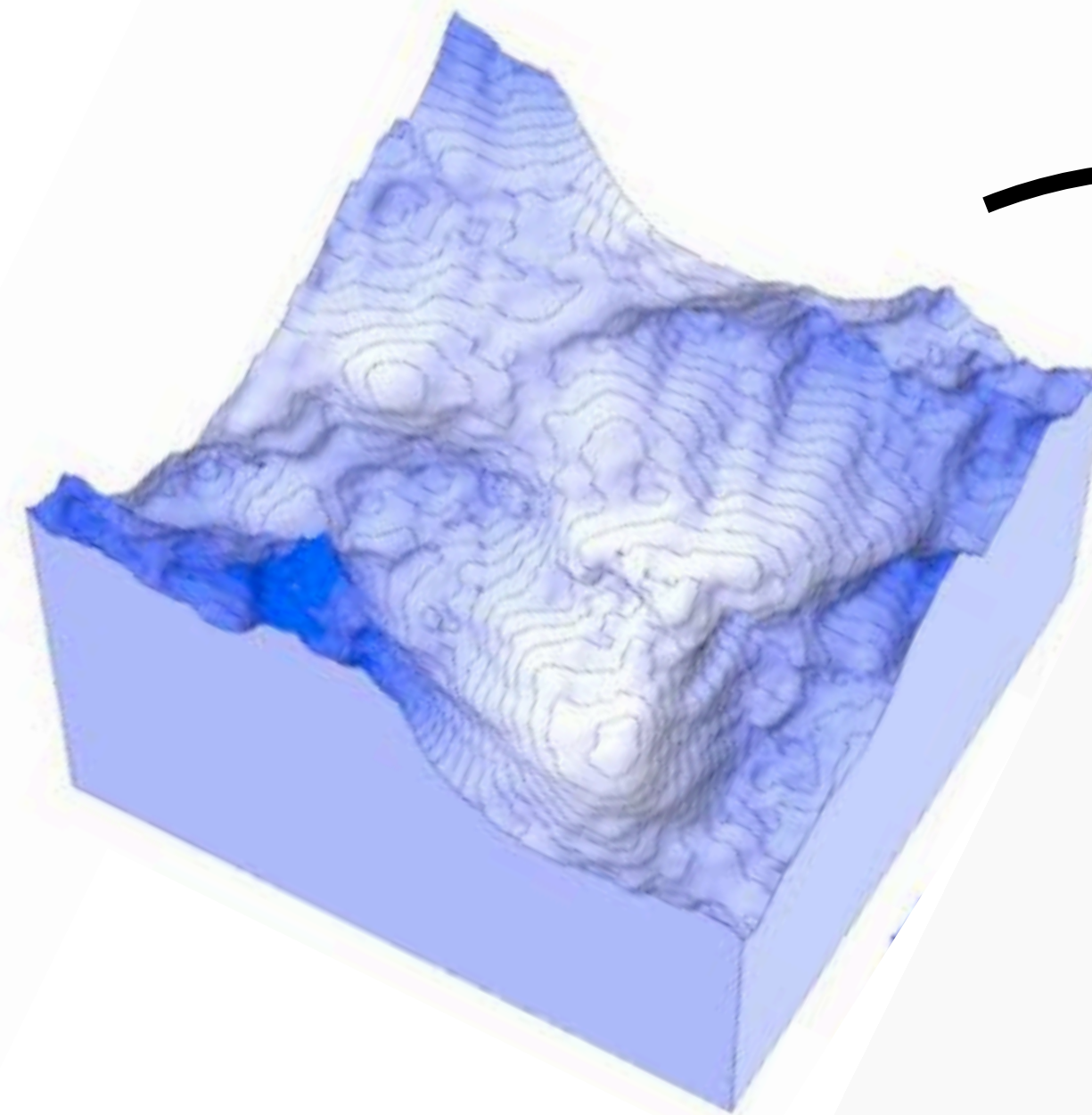
$$c \frac{\partial u(x, y, t)}{\partial t} = - \frac{\partial q_1(x, y, t)}{\partial x} - \frac{\partial q_2(x, y, t)}{\partial y} \quad c : \text{heat capacity}$$

- Together: $\frac{\partial u}{\partial t} = \frac{k}{c} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad k/c : \text{diffusivity}$

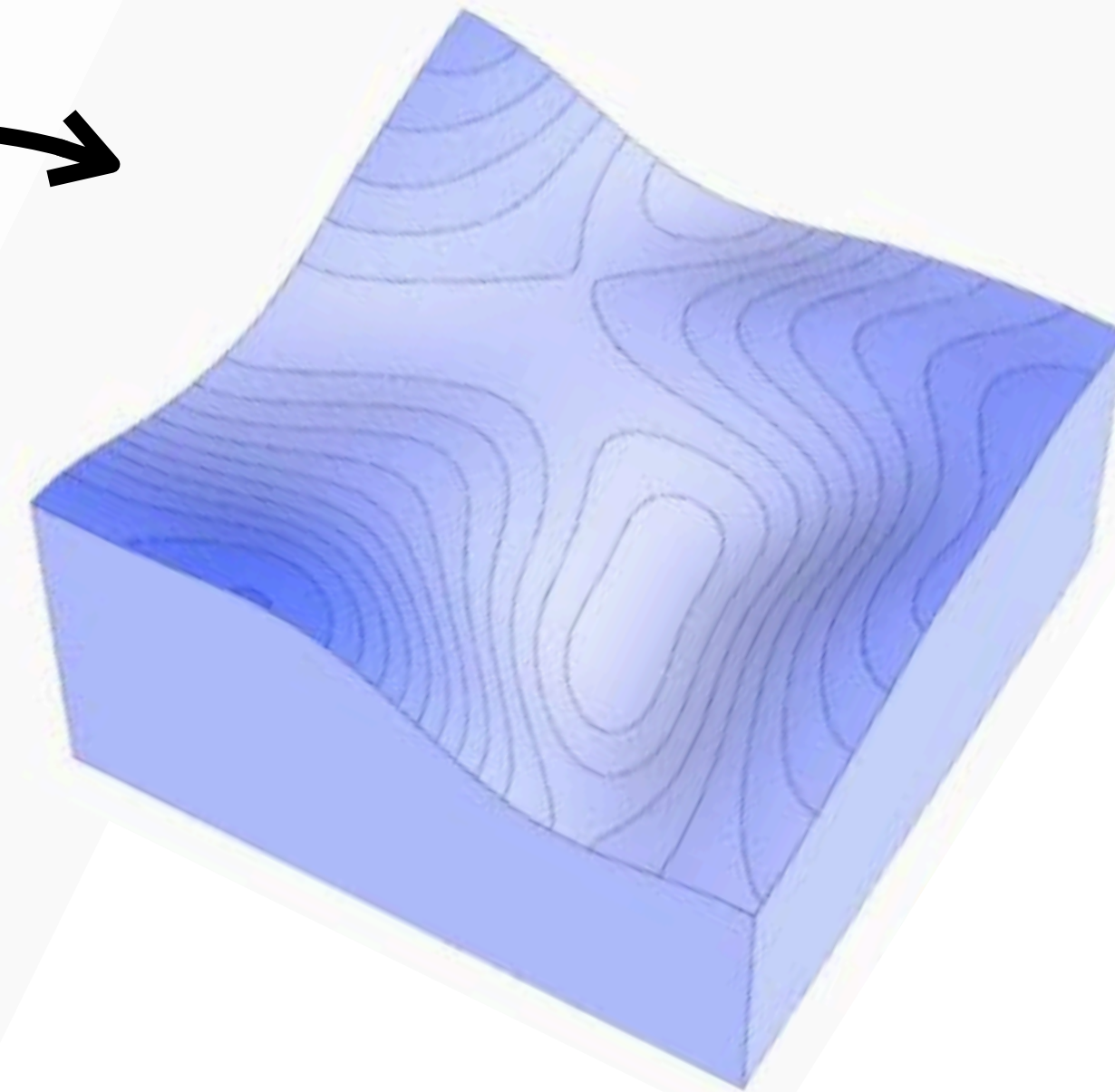
Heat equation

$$\frac{\partial u}{\partial t} = \Delta u$$

initial condition



over time



- What happens at steady/equilibrium state?

$$\Delta u = 0 \quad \text{harmonic!}$$

Harmonic functions

- Function u is called *harmonic* if

$$\Delta u = 0$$

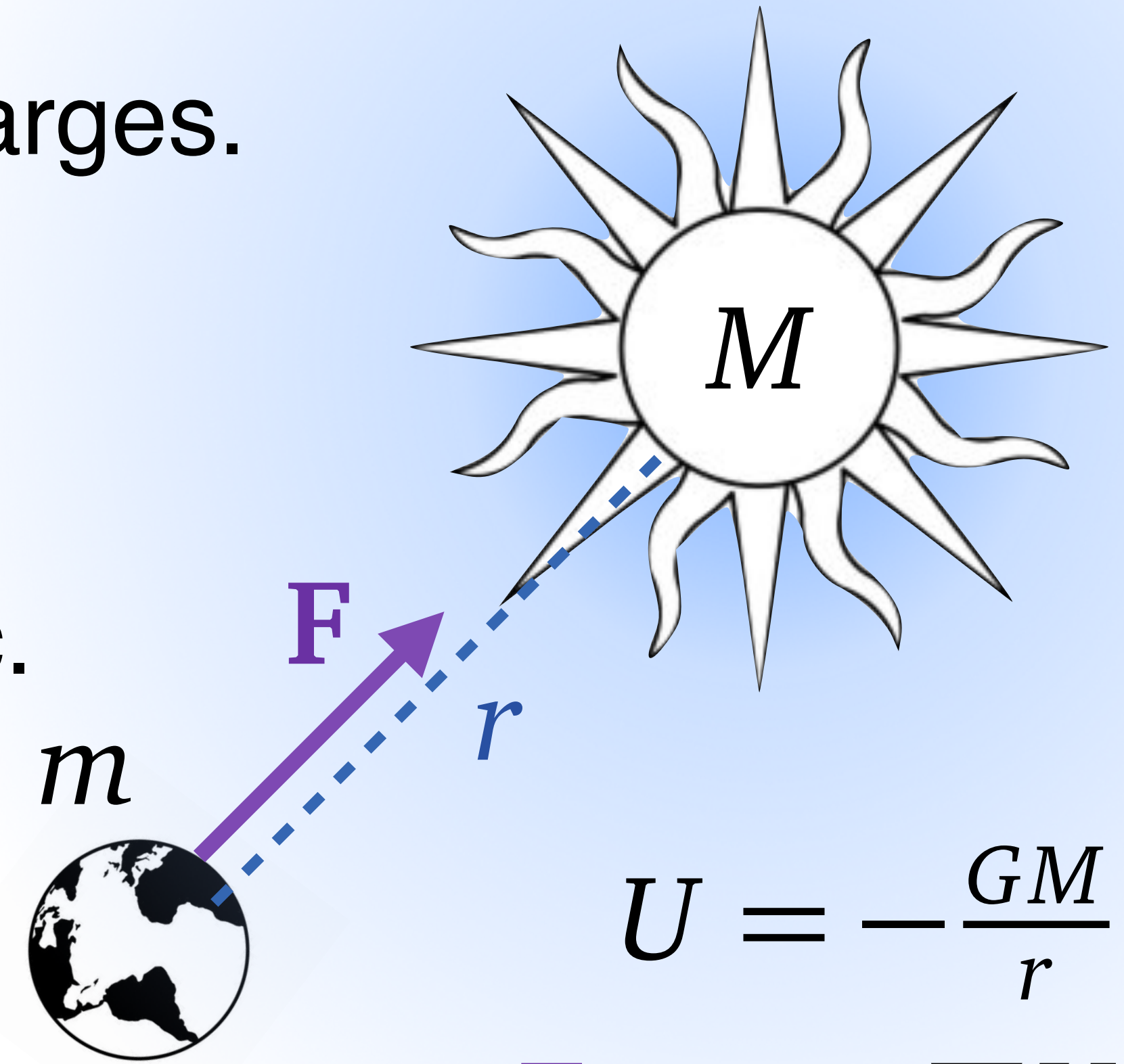
- Geometrically, it satisfies the *mean value property*

$$u(\mathbf{x}) = \frac{1}{|B_{\mathbf{x}}(r)|} \int_{\mathbf{y} \in B_{\mathbf{x}}(r)} u(\mathbf{y}) dA \quad (\text{on flat spaces})$$

where $B_{\mathbf{x}}(r)$ denotes the ball of radius r centered at $\mathbf{x}$.

Harmonic functions

- Gravitational potential is harmonic away from masses.
- Electric potential is harmonic away from charges.
- Each of the u, v coordinates of an angle preserving map is harmonic.
- Temperature field at equilibrium is harmonic.



$$U = -\frac{GM}{r}$$

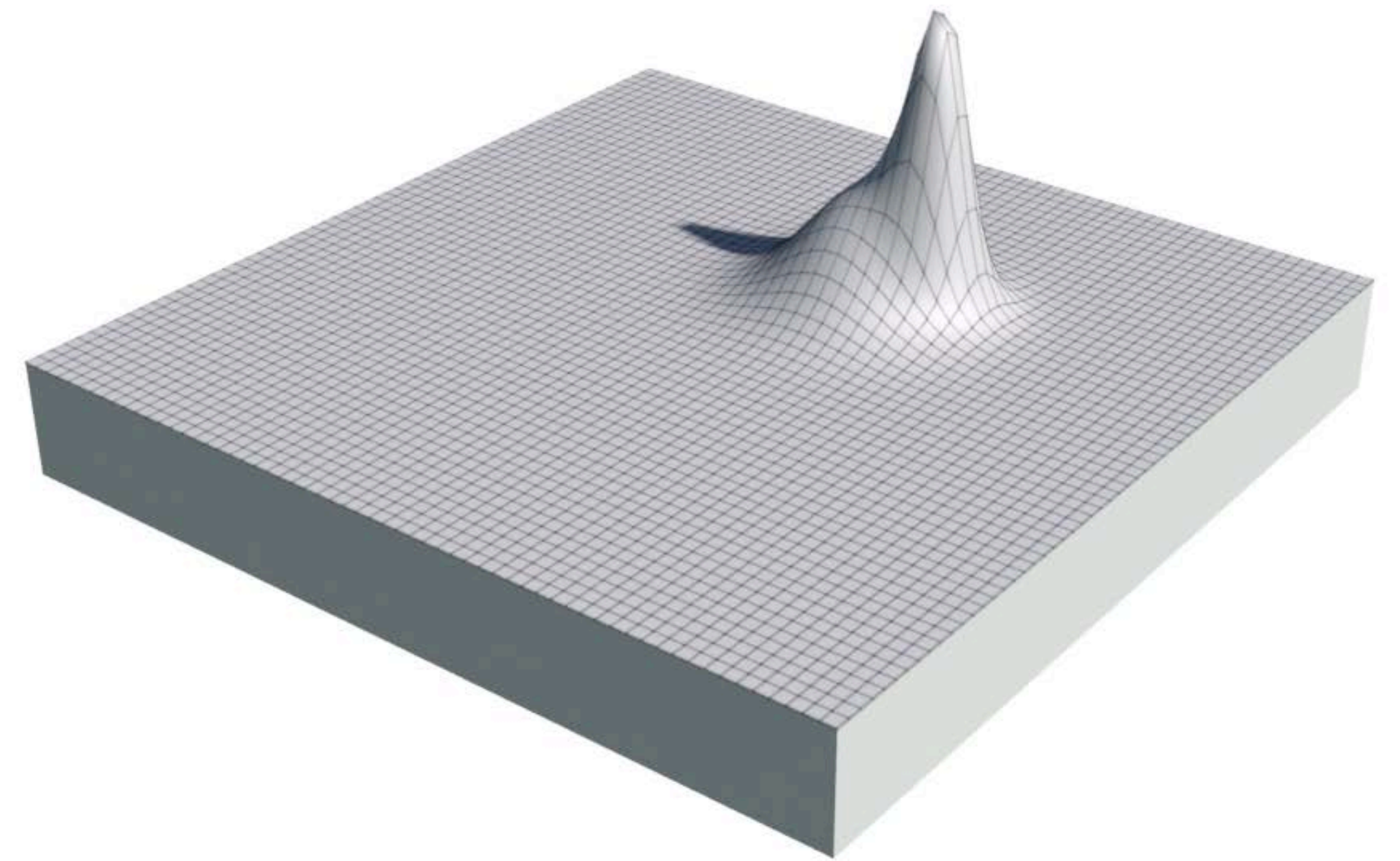
$$\mathbf{F} = -m\nabla U$$

Wave equation

- Wave equation

$u(t; x, y)$ pressure or displacement
at time t and position
(x, y)

$$\frac{\partial^2 u}{\partial t^2} = \Delta u$$

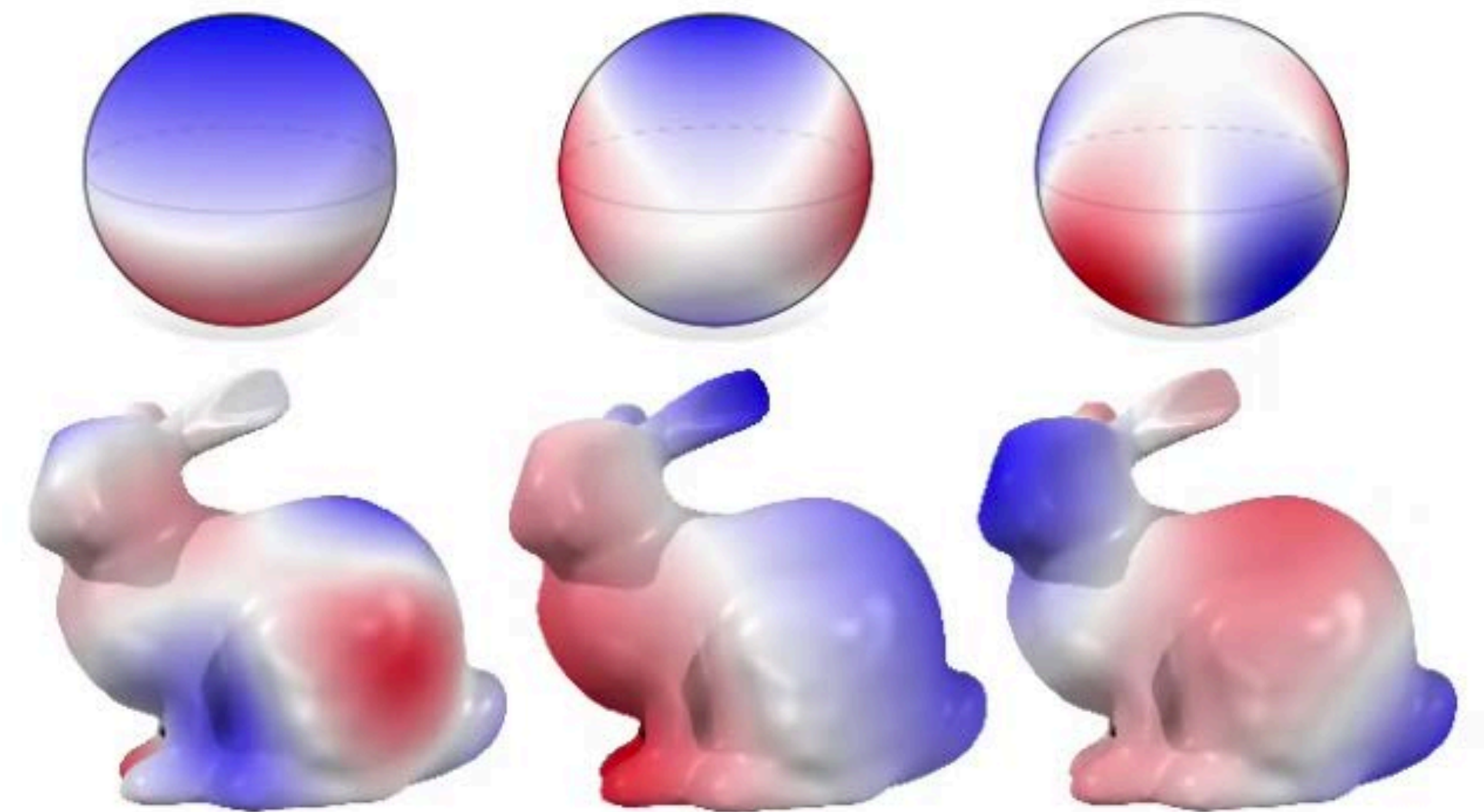
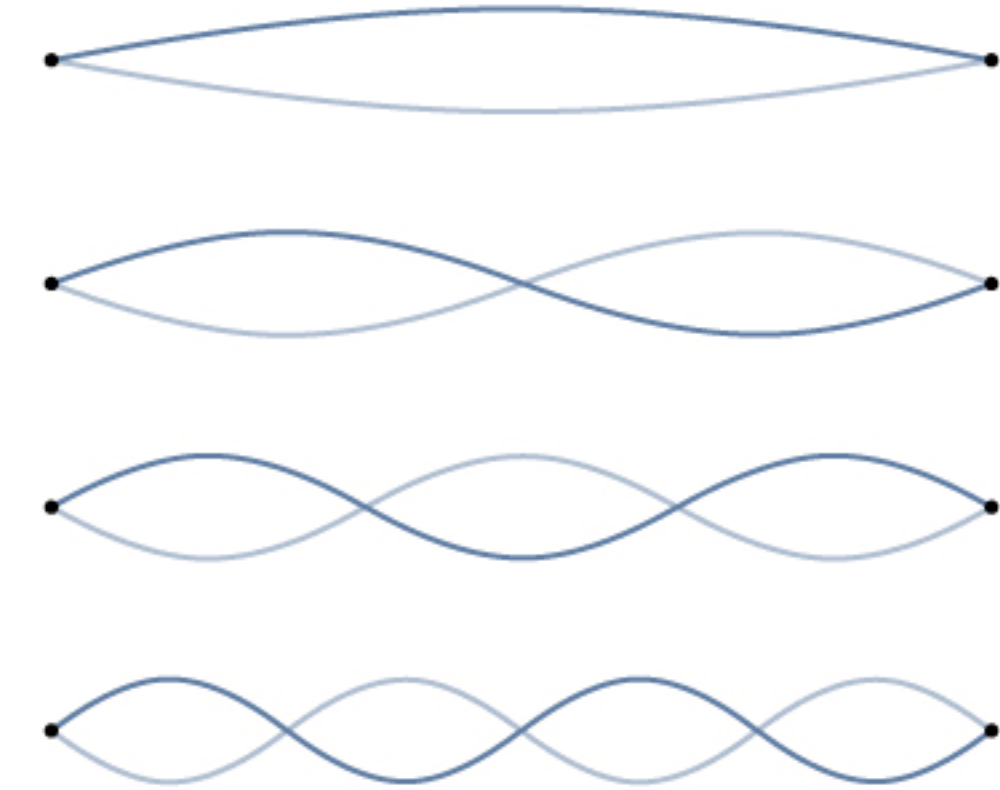


Laplacian eigenfunctions

- There are functions $\varphi_1, \varphi_2, \dots$ satisfying

$$\Delta \varphi_k = \lambda_k \varphi_k$$

- The eigenvalue λ_k is the frequency
- These are like the *Fourier basis* for functions on the surface.
- They are the standing waves solution of the wave equation.



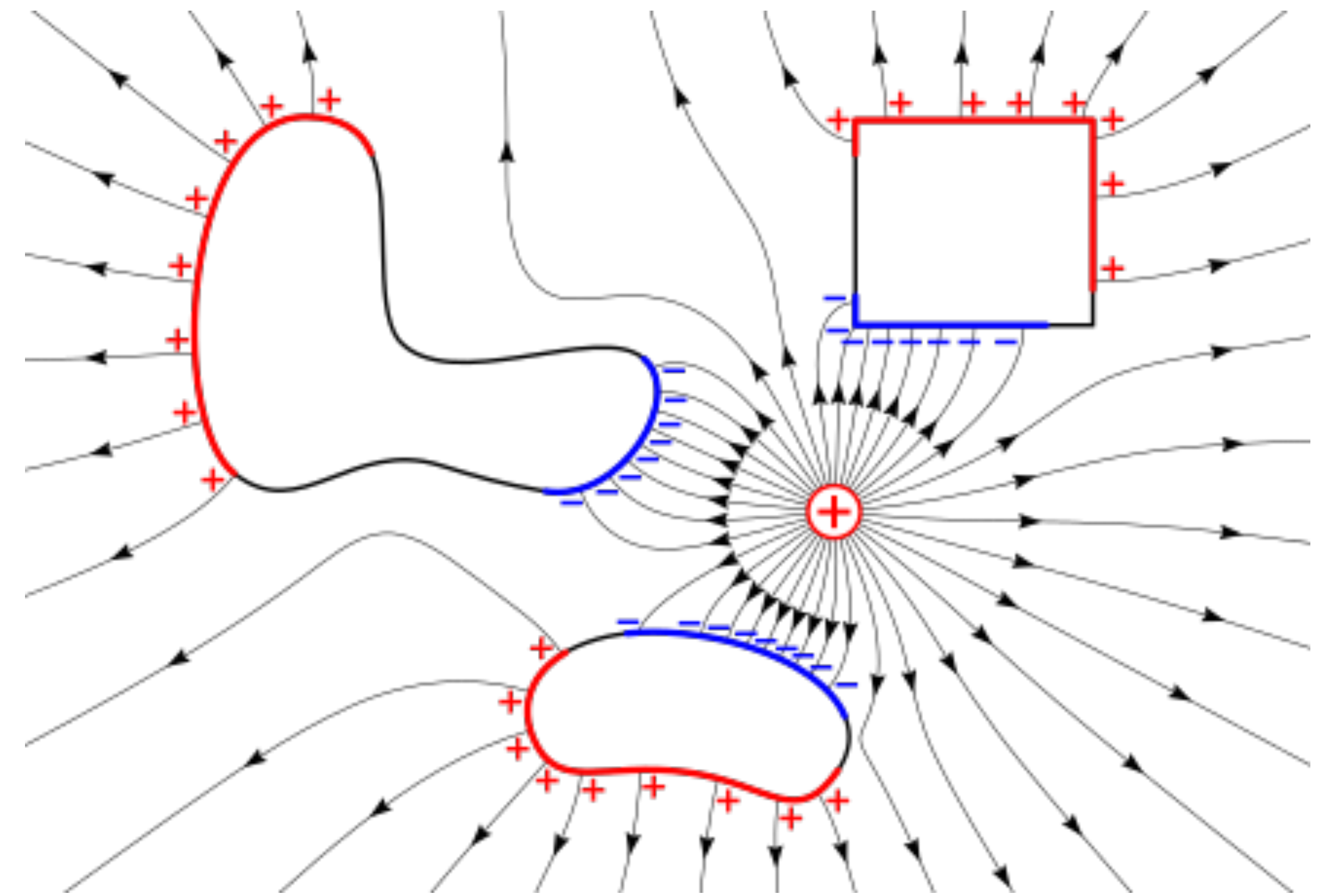
Electrostatics

- Electrostatics

$u(x, y)$ electric potential

$\rho(x, y)$ electric charge

$$\Delta u = \rho \quad \text{Poisson equation}$$



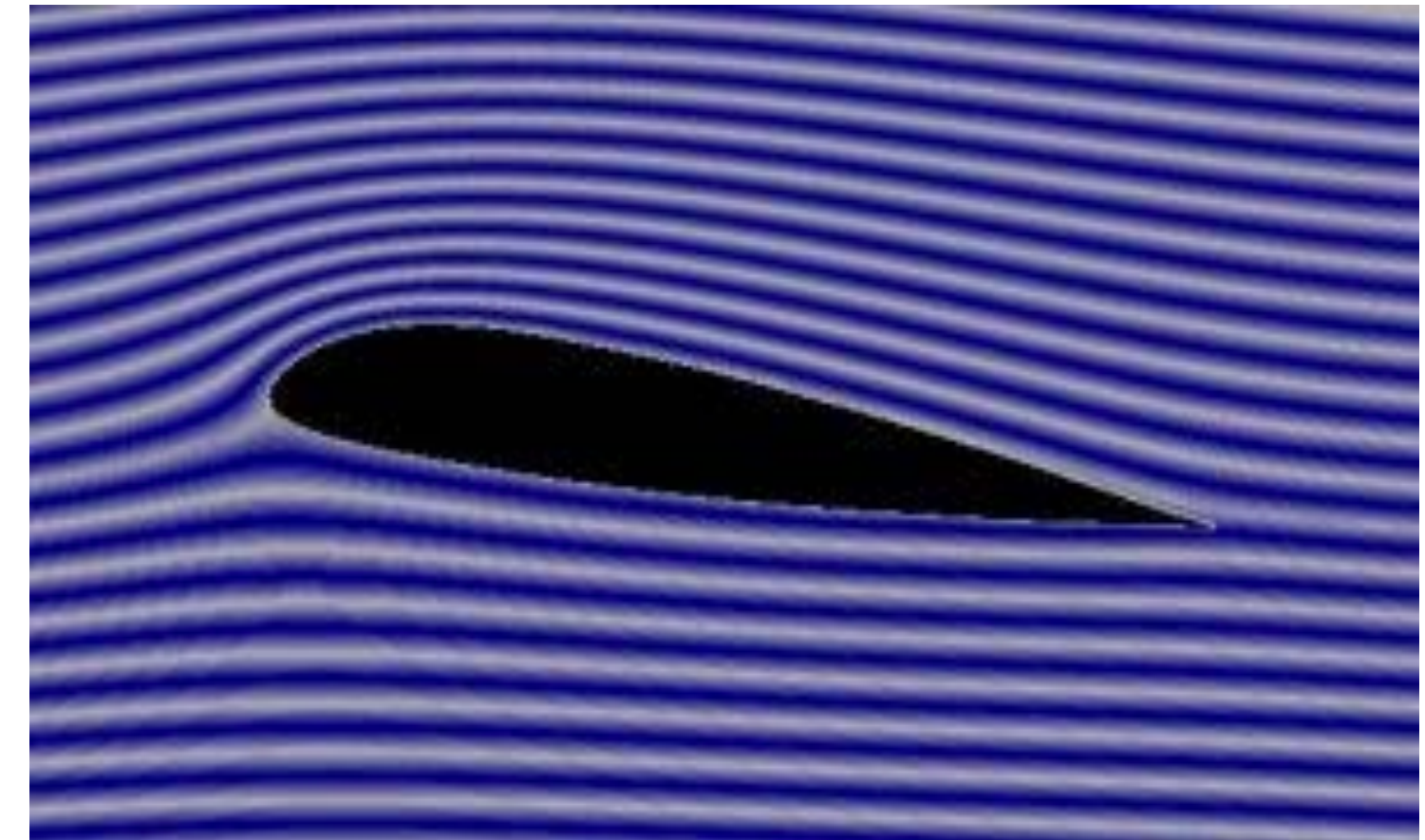
Vorticity and streamlines

- Fluid dynamics

$u(x, y)$ function whose
contour lines are
streamlines

$\rho(x, y)$ vorticity

$$\Delta u = \rho \quad \text{Poisson equation}$$



Viscosity

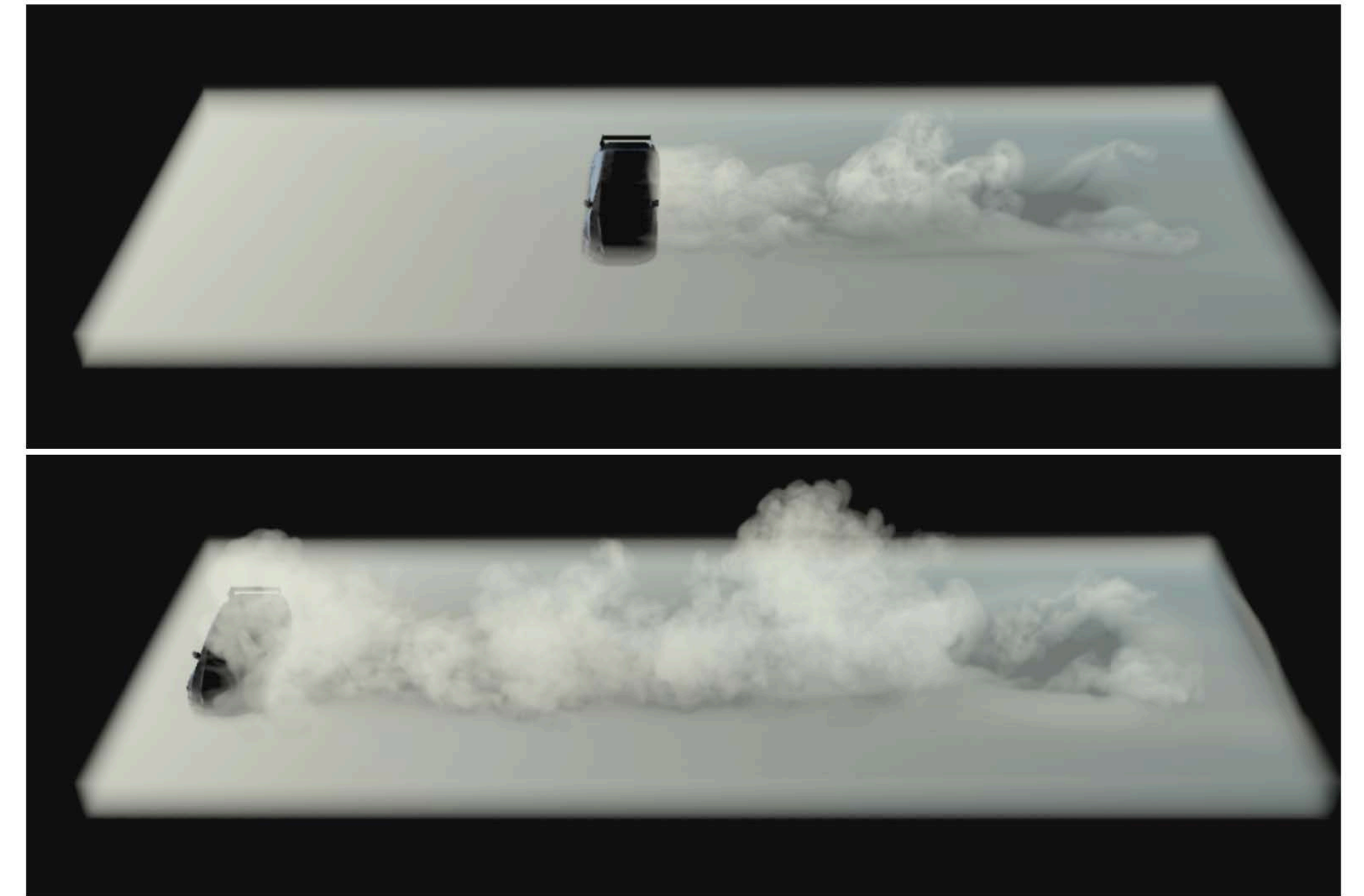
- Fluid dynamics

$\vec{u}(t; x, y, z)$ velocity field

viscosity
is a Laplacian

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} + \nabla p = \epsilon \Delta \vec{u}$$

Navier–Stokes equation



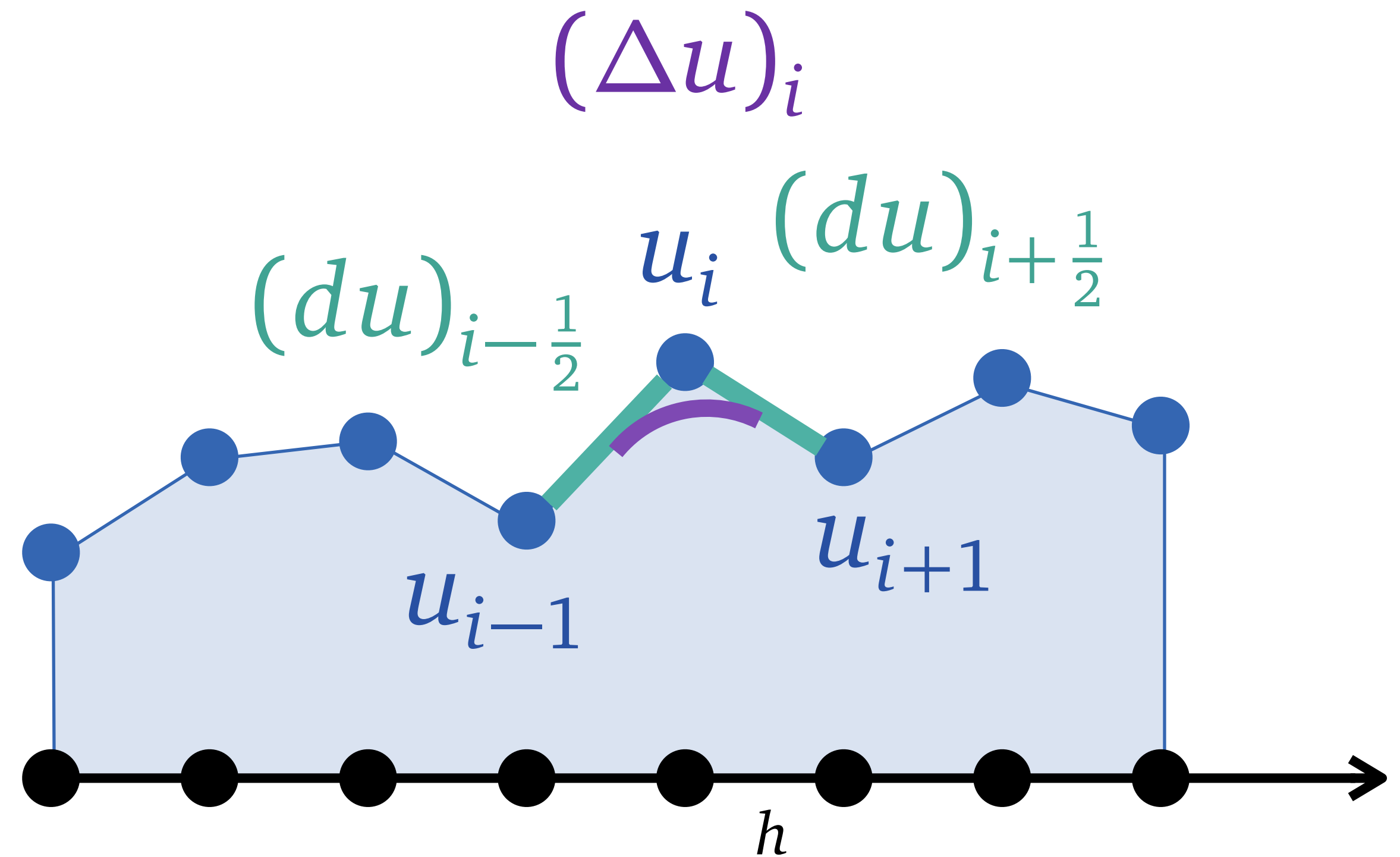
[Qu et al 2019] BiMocq solver

Discrete Laplacian: Finite Difference

- Laplacian in Physics
- Finite Difference
- Graph Laplacian
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- Discrete Hodge stars
- Example: heat equation

A simple discrete Laplacian

- In 1D $\Delta u(x) = u''(x)$
- Discretize 1D space into uniform grid
- grid size h



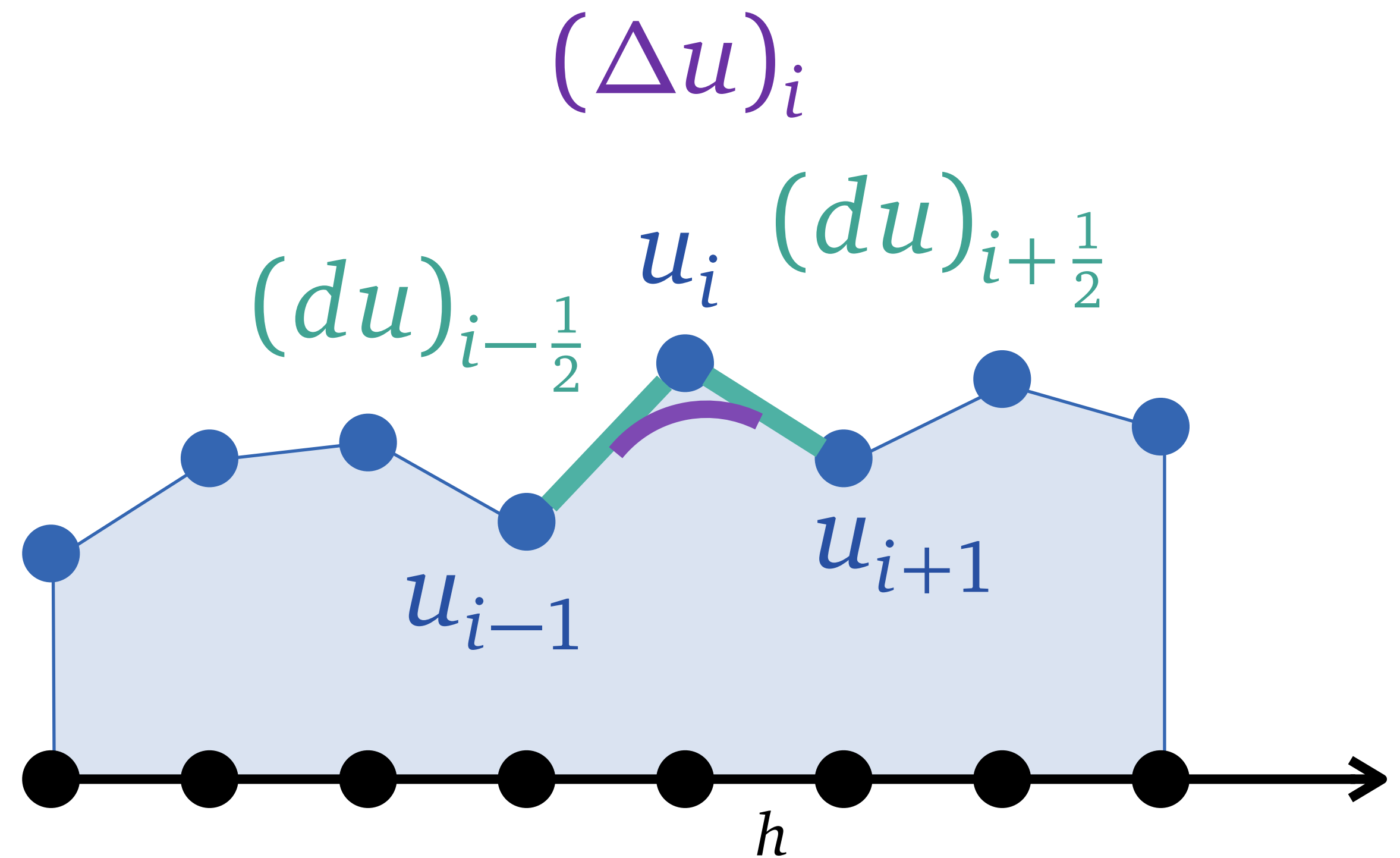
A simple discrete Laplacian

- Function: Assign values u_i on grid points
- Rate of change: On edges

$$(du)_{i+\frac{1}{2}} = \frac{u_{i+1} - u_i}{h}$$

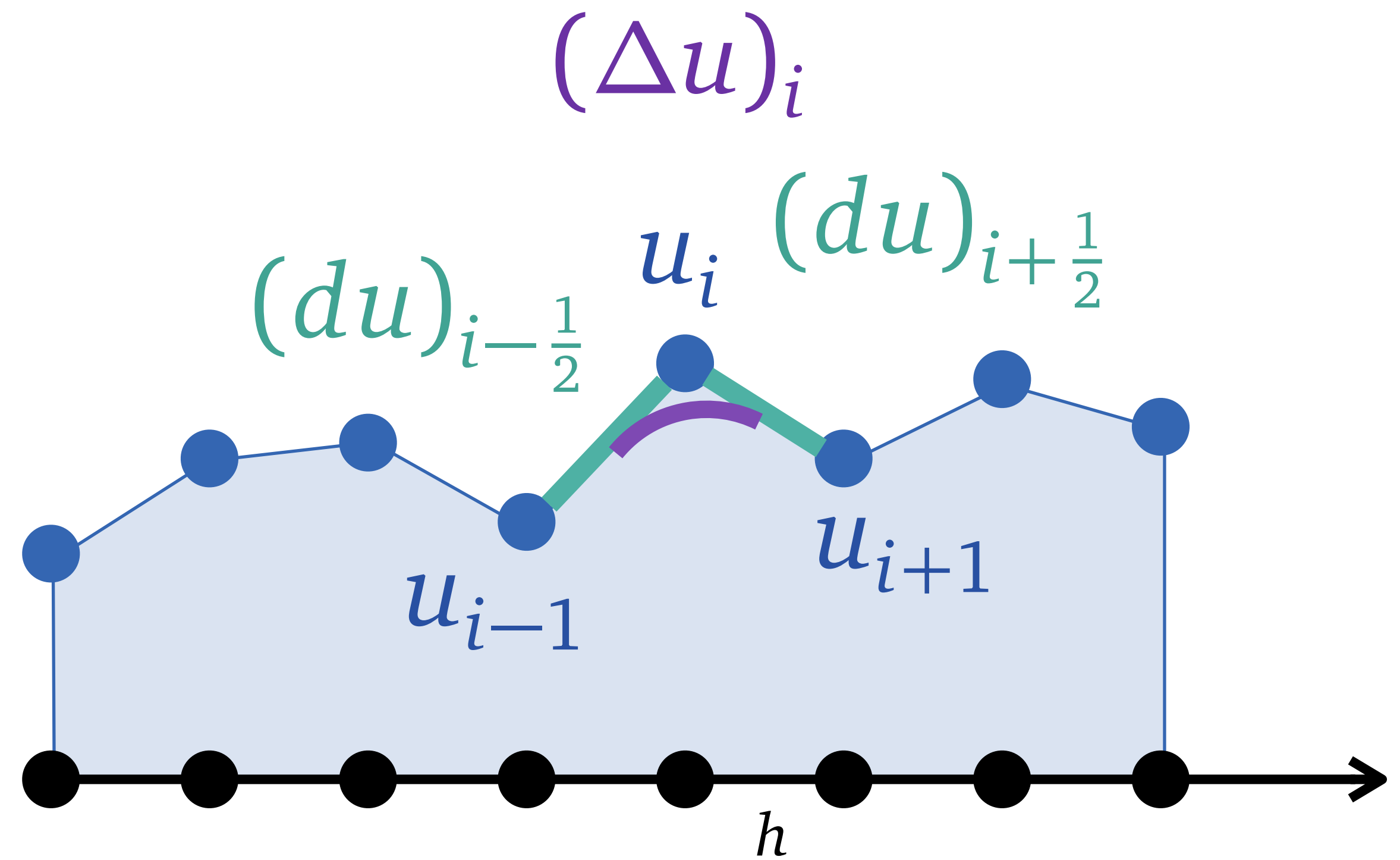
- Laplacian: Back to grid points

$$\begin{aligned}(\Delta u)_i &= \frac{(du)_{i+\frac{1}{2}} - (du)_{i-\frac{1}{2}}}{h} \\&= \frac{(u_{i+1} - u_i) - (u_i - u_{i-1})}{h^2} \\&= \frac{1}{h^2} (u_{i+1} - 2u_i + u_{i-1})\end{aligned}$$



A simple discrete Laplacian

$$(\Delta u)_i = \frac{1}{h^2} (u_{i+1} - 2u_i + u_{i-1})$$



In 2D

- Continuous:

$$\Delta u(x, y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

- Discrete

$$(\Delta u)_{i,j} = \frac{1}{h^2} \left(\begin{array}{c} + u_{i,j+1} \\ u_{i-1,j} - 4u_{i,j} + u_{i+1,j} \\ + u_{i,j-1} \end{array} \right)$$

Remark

- 1D: $(\Delta u)_i = \frac{1}{h^2}(u_{i+1} - 2u_i + u_{i-1})$
- 2D: $(\Delta u)_{i,j} = \frac{1}{h^2} \begin{pmatrix} + u_{i,j+1} \\ u_{i-1,j} - 4u_{i,j} + u_{i+1,j} \\ + u_{i,j-1} \end{pmatrix}$
- In general:
Laplacian measures the difference between the value at a point and its neighbor's average
(up to factors of $1/h^2$ or 2 or 4 or ...)

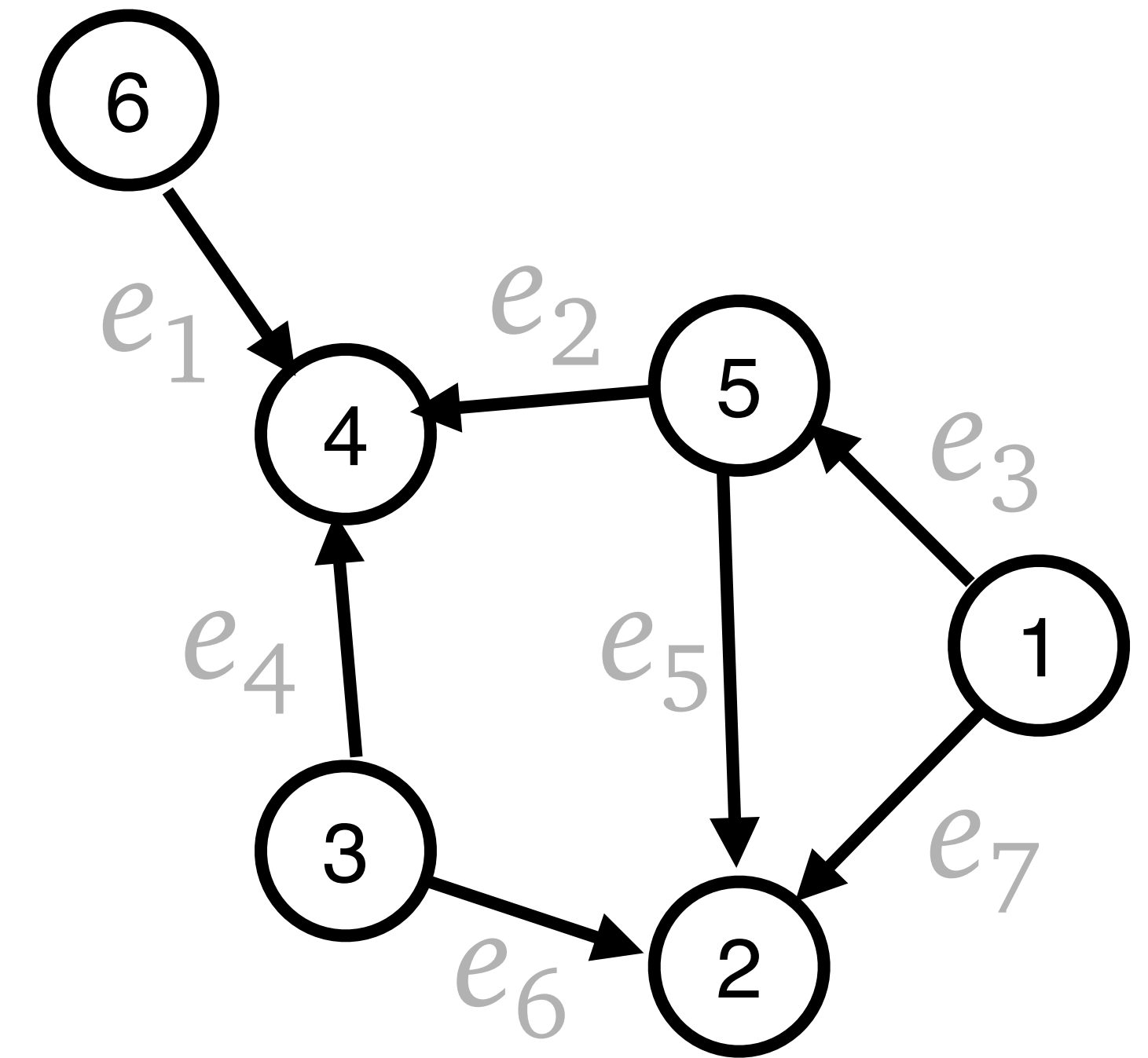
Discrete Laplacian: Graph Laplacian

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Graph Laplacian

- Consider a graph $\mathcal{G} = (\mathcal{P}, \mathcal{E})$ pts oriented edges
- A set of pt data is $\mathbf{u}: \mathcal{P} \rightarrow \mathbb{R}$

$$\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_{|\mathcal{P}|} \end{bmatrix} \in \mathbb{R}^{|\mathcal{P}|} \quad (\mathcal{P} \rightarrow \mathbb{R}) = \mathbb{R}^{|\mathcal{P}|}$$



- The **graph Laplacian** is the linear operator $\mathbf{L}: \mathbb{R}^{|\mathcal{P}|} \rightarrow \mathbb{R}^{|\mathcal{P}|}$

$$(\mathbf{L}\mathbf{u})_i = \boxed{|\text{Nbr}(i)|} u_i - \sum_{j \in \text{Nbr}(i)} u_j = \sum_{j \in \text{Nbr}(i)} (u_i - u_j)$$

valence of i
degree of i

0-forms and 1-forms on Graph

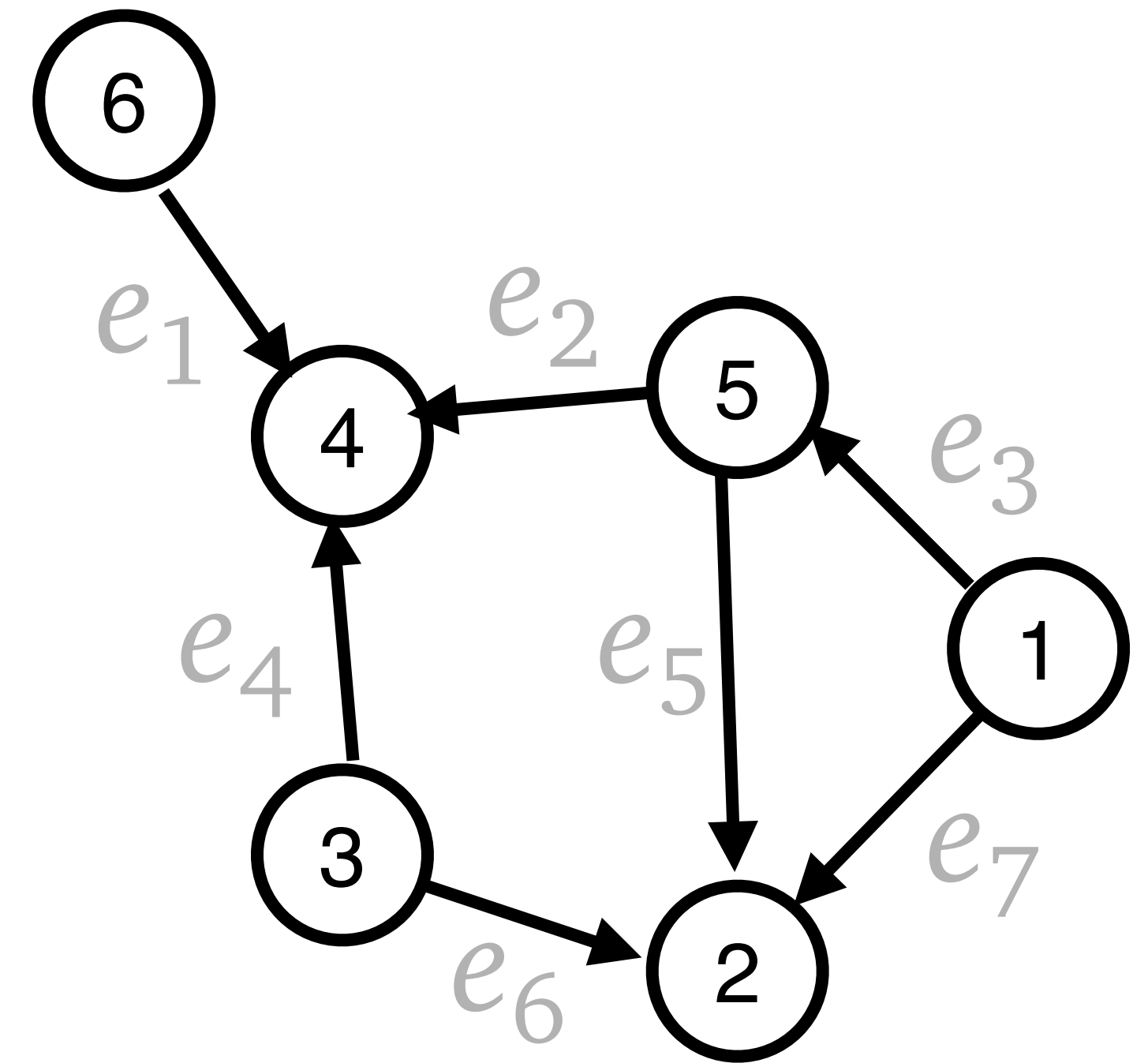
- Call quantities defined on pts **0-forms**
Call those defined on oriented edges **1-forms**

- Space of 0-forms:

$$\mathbb{R}^{|\mathcal{P}|} = \underset{0\text{-cochains}}{C^0(\mathcal{G})} = \underset{\substack{0\text{-forms} \\ \text{(usually for smooth theory)}}}{\Omega^0(\mathcal{G})}$$

Space of 1-forms:

$$\mathbb{R}^{|\mathcal{E}|} = C^1(\mathcal{G}) = \Omega^1(\mathcal{G})$$



Derivative

- Define $\mathbf{d}_0 : C^0(\mathcal{G}) \xrightarrow{\text{linear}} C^1(\mathcal{G})$

$$(\mathbf{d}_0 \mathbf{u})_{[i,j]} := u_j - u_i$$

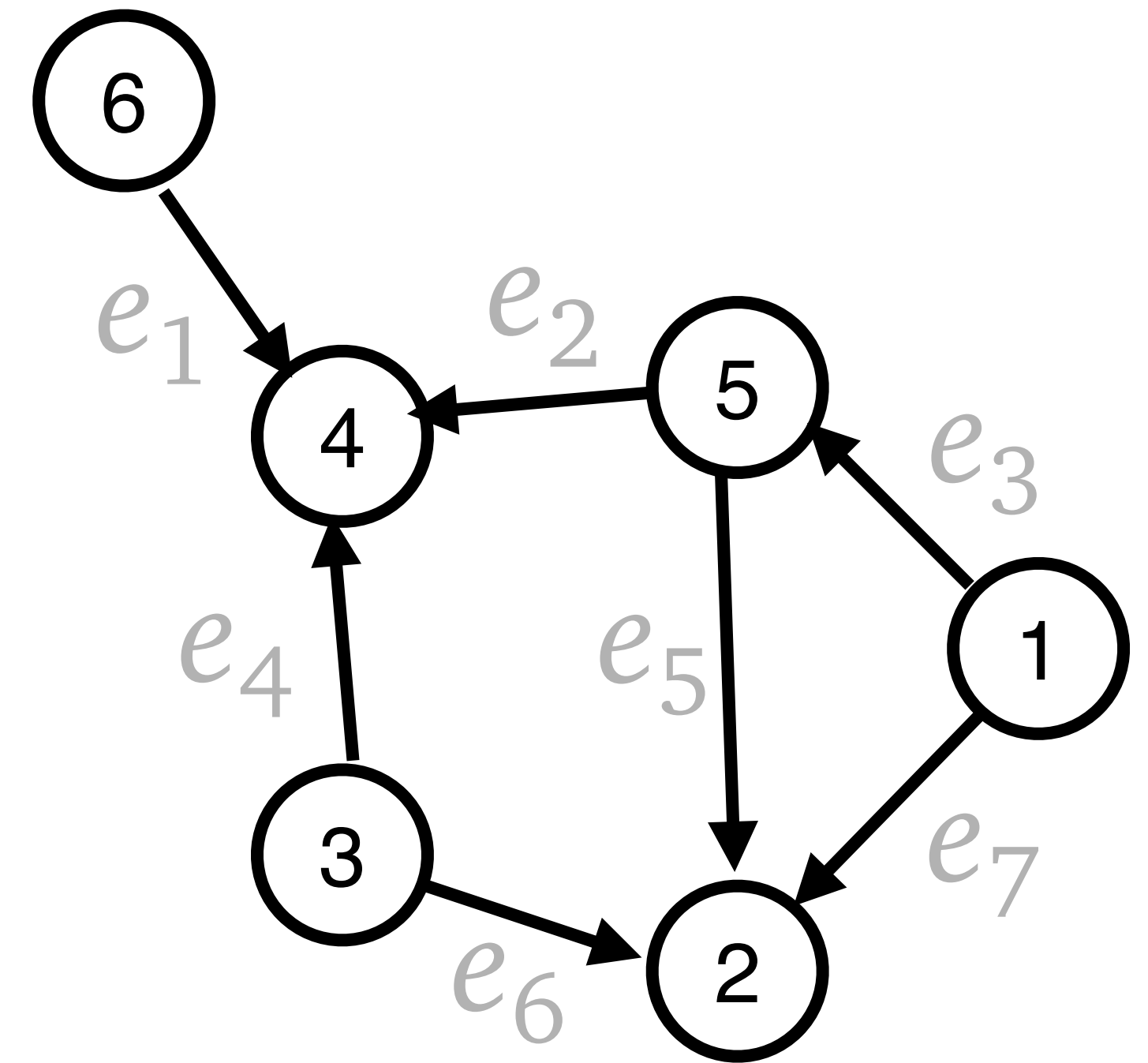
edge from i to j

equivalently:

$$(\mathbf{d}_0 \mathbf{u})_e := u_{\text{dst}(e)} - u_{\text{src}(e)}$$

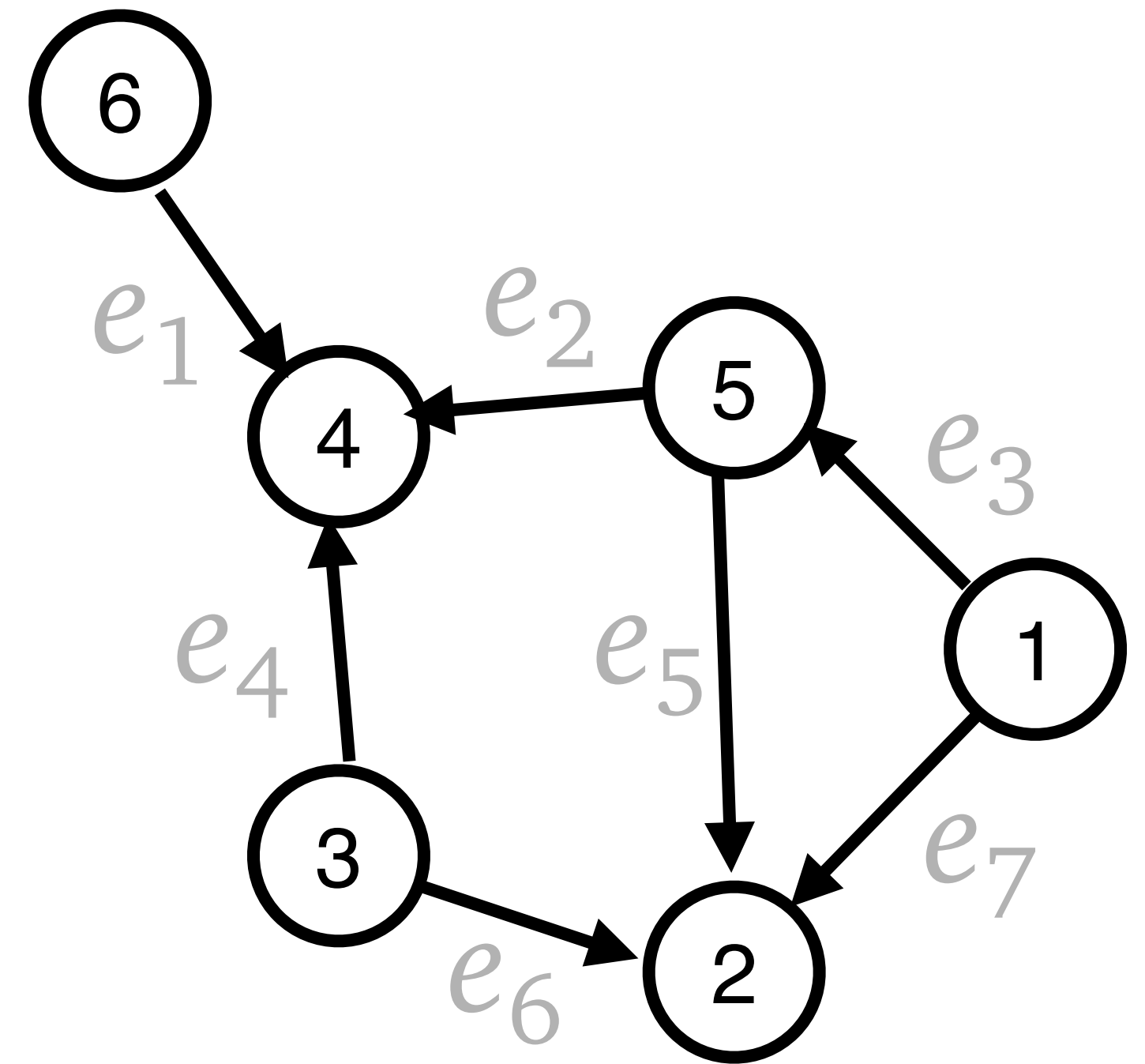
- $\mathbf{d}_0$ is a $|\mathcal{E}| \times |\mathcal{P}|$ sparse matrix

$$\begin{cases} (\mathbf{d}_0)_{e, \text{src}(e)} = -1, & \forall e \in \mathcal{E} \\ (\mathbf{d}_0)_{e, \text{dst}(e)} = 1, & \forall e \in \mathcal{E} \\ (\mathbf{d}_0)_{e,i} = 0, & \text{elsewhere} \end{cases}$$



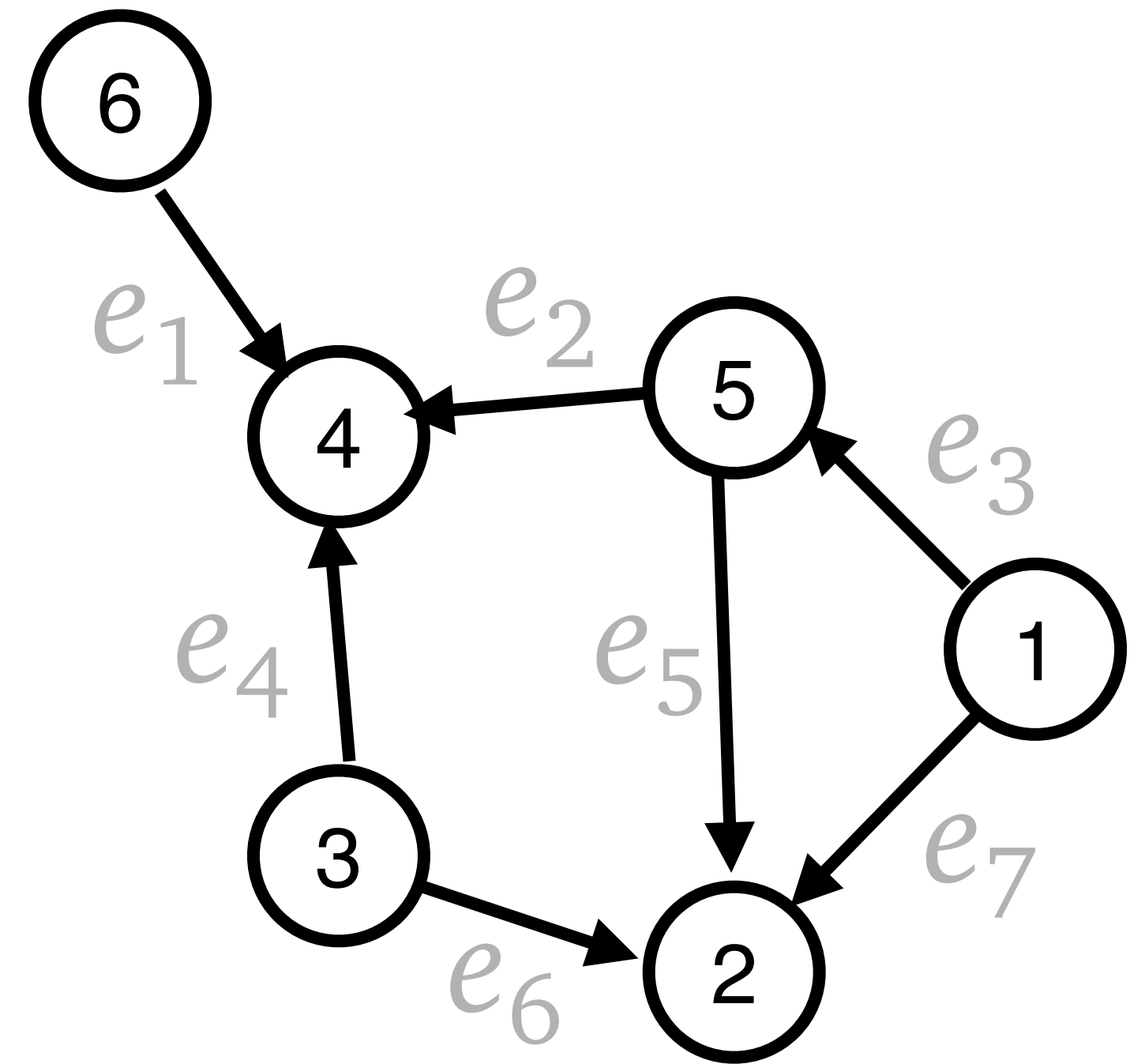
Derivative

$$\mathbf{d}_0 = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{matrix} & \begin{bmatrix} & & & 1 & & -1 \\ & & & 1 & -1 & \\ -1 & & & & 1 & \\ & & -1 & 1 & & \\ & 1 & & & -1 & \\ & 1 & -1 & & & \\ -1 & 1 & & & & \end{bmatrix} \end{matrix}$$



Derivative

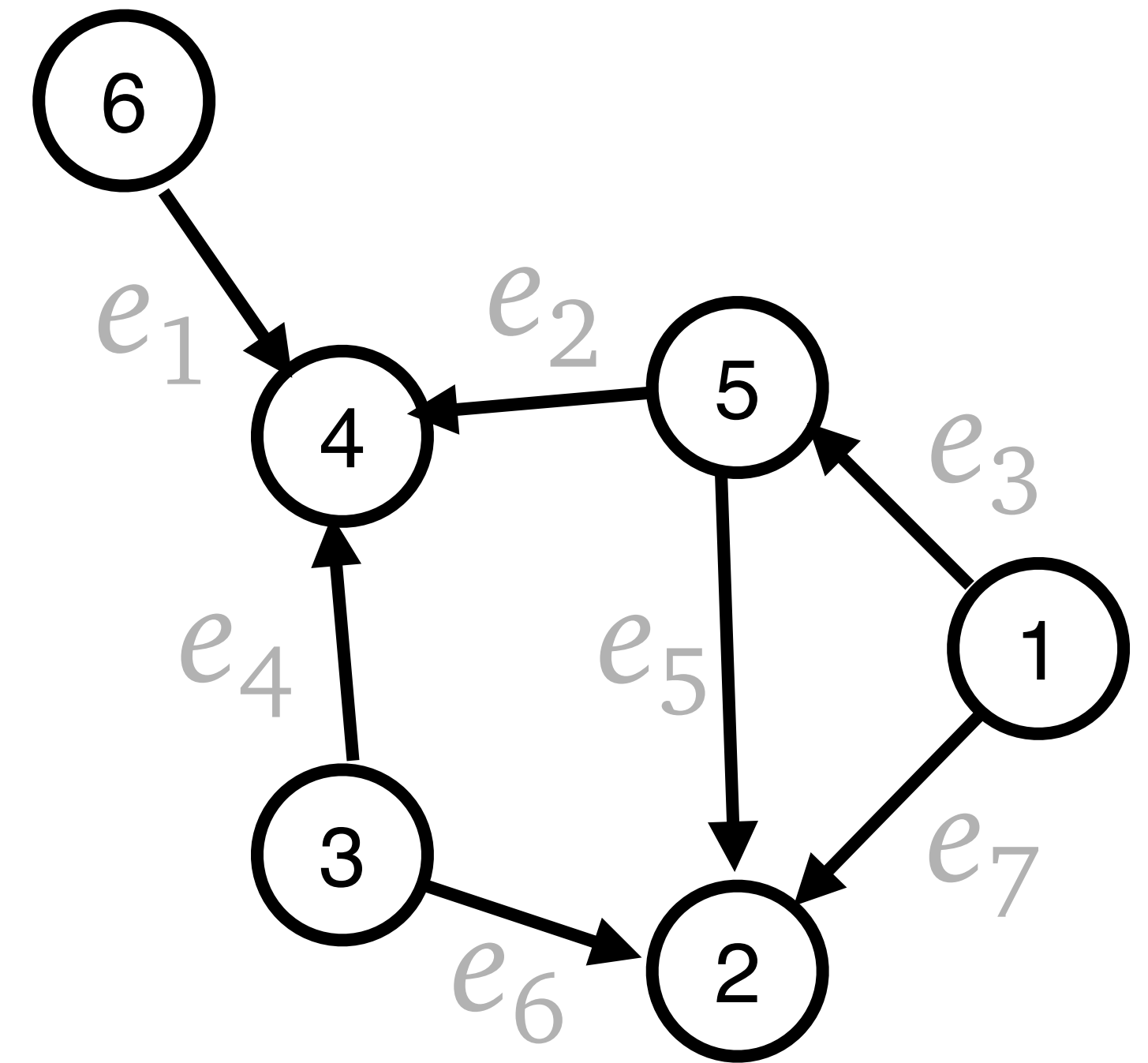
$$\mathbf{d}_0 = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{matrix} & \begin{bmatrix} & & & 1 & & -1 \\ & & & 1 & -1 & \\ -1 & & & & 1 & \\ & & -1 & 1 & & \\ & 1 & & & -1 & \\ & 1 & -1 & & & \\ -1 & 1 & & & & \end{bmatrix} \end{matrix}$$



- In general $\mathbf{d}_k : C^k(M) \rightarrow C^{k+1}(M)$ with $\mathbf{d}_k = \partial_{k+1}^\top$
- They are called **exterior derivatives**.

Derivative

$$\mathbf{d}_0 = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{matrix} & \begin{bmatrix} & & & 1 & & -1 \\ & & & 1 & -1 & \\ -1 & & & & 1 & \\ & & -1 & 1 & & \\ & 1 & & & -1 & \\ & 1 & -1 & & & \\ -1 & 1 & & & & \end{bmatrix} \end{matrix}$$

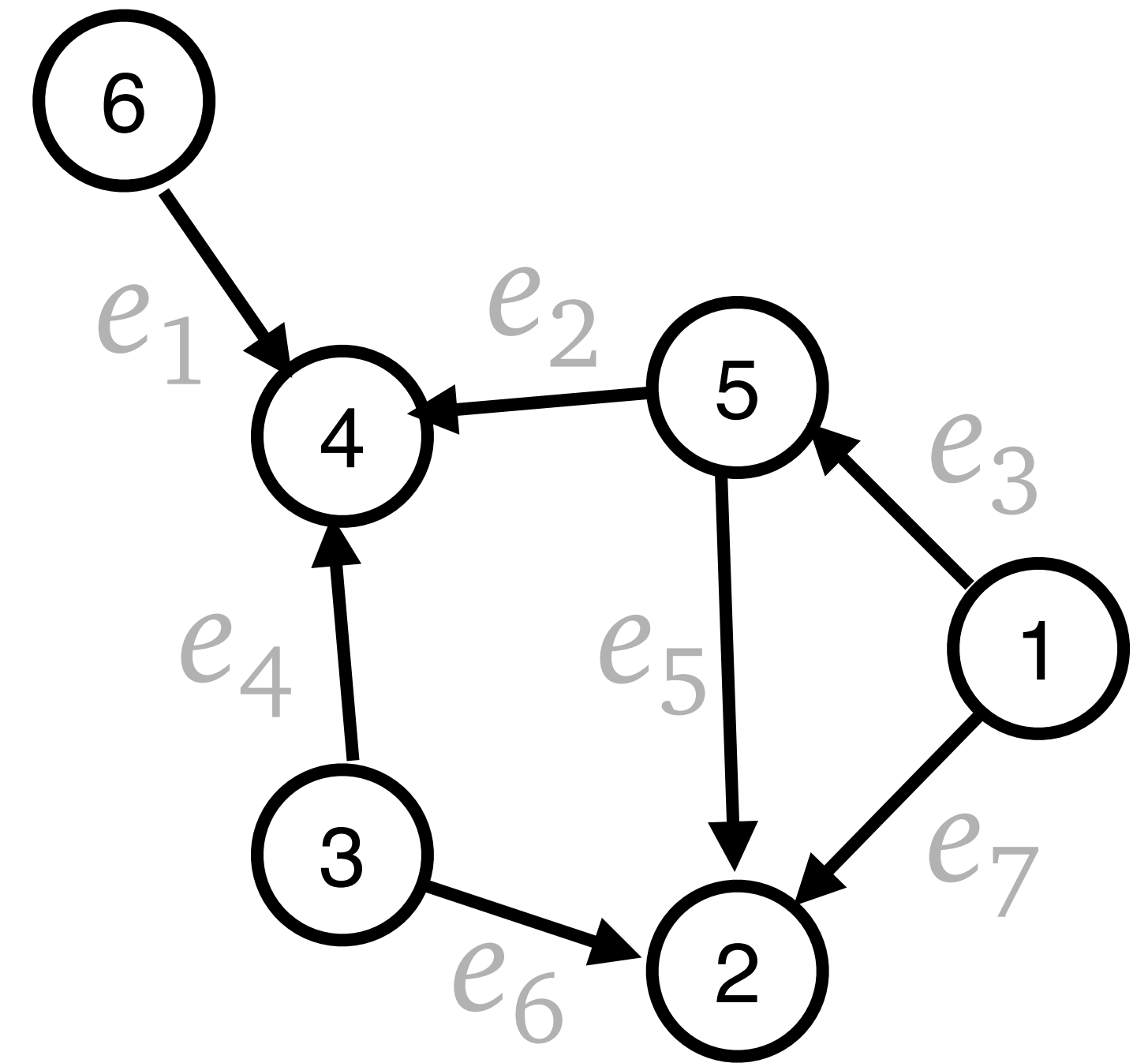


- Look at $\mathbf{d}_0^\top \in \mathbb{R}^{|\mathcal{P}| \times |\mathcal{E}|}$
- For example, for $\alpha \in C^1(\mathcal{G})$, $(\mathbf{d}_0^\top \alpha)_5 = -\alpha_{e_2} + \alpha_{e_3} - \alpha_{e_5}$

Derivative

$$(\mathbf{d}_0^\top \alpha)_i = \sum_{e \succ i} s_{i,e} \alpha_e$$

$$s_{i,e} = \begin{cases} -1 & i = \text{src}(e) \\ 1 & i = \text{dst}(e) \end{cases}$$



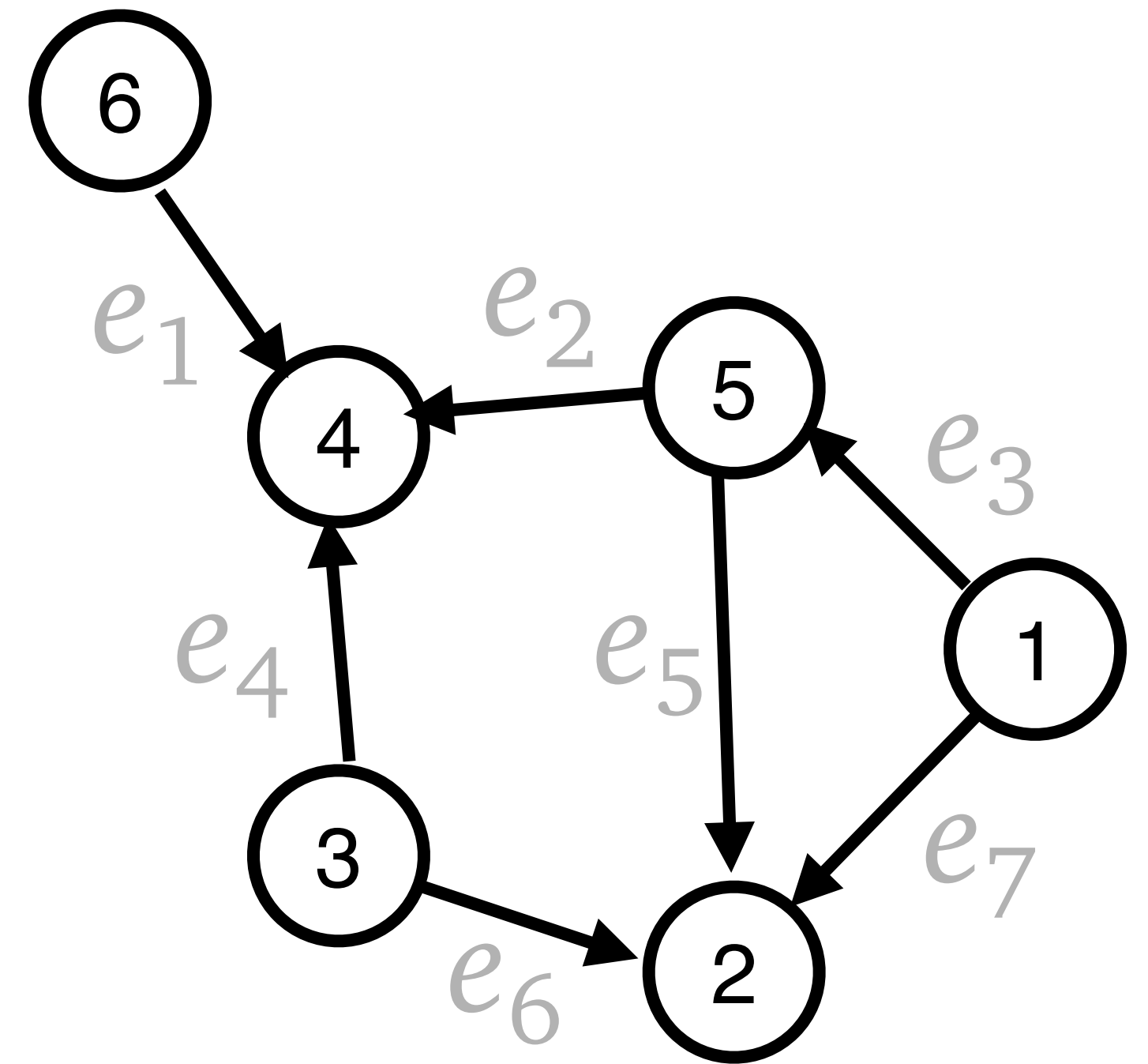
- Interpretation: α is flux for a flow flowing along each edge, and $\mathbf{d}_0^\top \alpha$ is the total in-flow into the cell at each vertex.

Graph Laplacian as 2nd Derivative

Theorem

The graph Laplacian can be written as

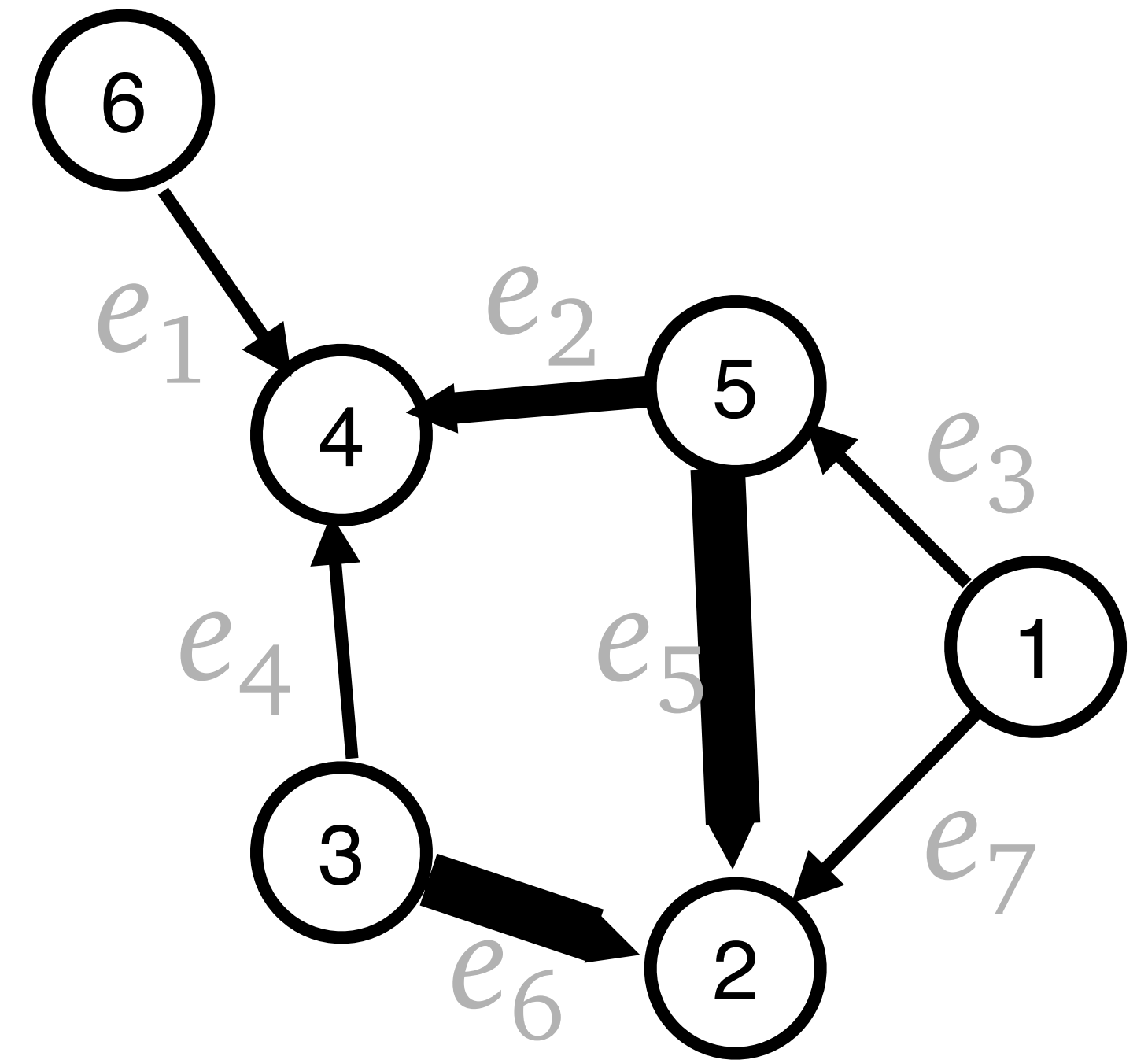
$$\mathbf{L} = \mathbf{d}_0^\top \mathbf{d}_0$$



Graph Laplacian with Edge Weights

- Each edge can have different “amount of connectivity”
- Let $w_e > 0$ represent a given weight for $e \in \mathcal{E}$
- Then the Laplacian is given by

$$(\mathbf{L}\mathbf{u})_i := \sum_{j \in \text{Nbr}(i)} w_{ij} (u_i - u_j)$$



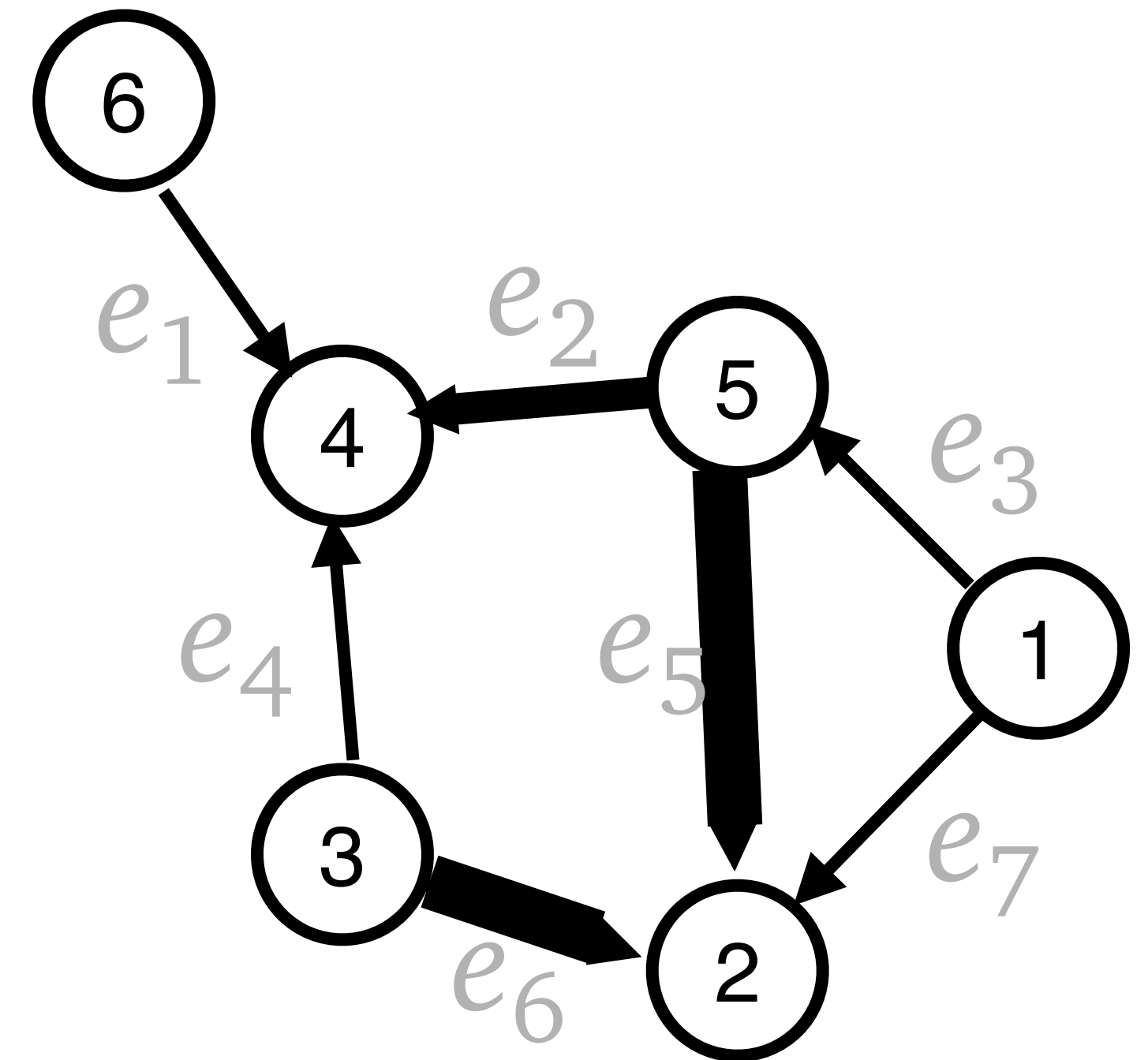
Graph Laplacian with Edge Weights

$$(\mathbf{L}\mathbf{u})_i := \sum_{j \in \text{Nbr}(i)} w_{ij} (u_i - u_j)$$

- Define diagonal matrix $\star_1 \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{E}|}$

$$\star_1 = \begin{bmatrix} w_{e_1} & & & & & \\ & w_{e_2} & & & & \\ & & & & & \\ & & & \ddots & & \\ & & & & & \\ & & & & & w_{e_{|\mathcal{E}|}} \end{bmatrix}$$

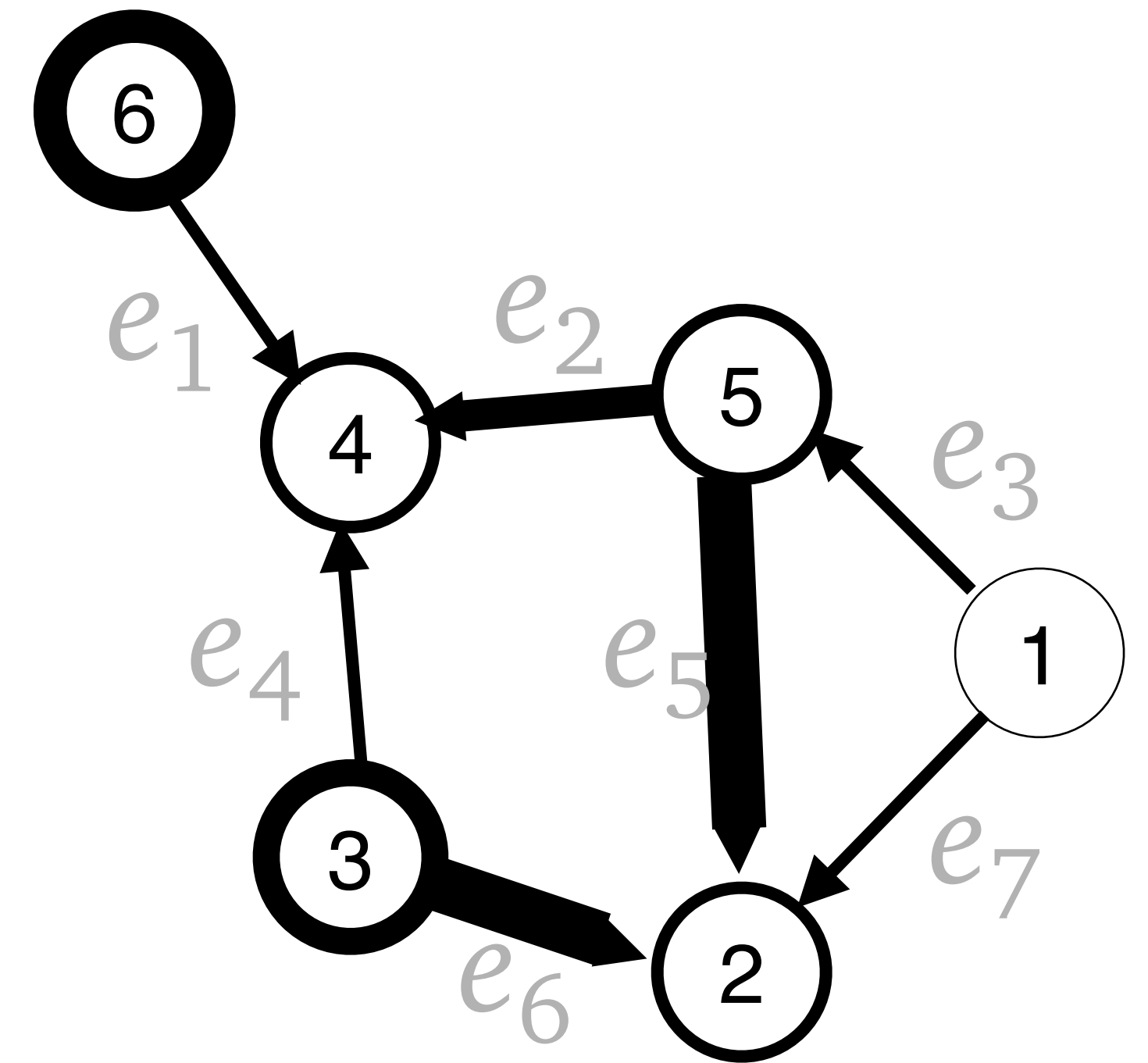
- Then we have: $\mathbf{L} = \mathbf{d}_0^\top \star_1 \mathbf{d}_0$



Graph Laplacian with Point Weights

- Points can also have weights, representing “mass” or “capacity”
- Let $m_i > 0$ represent a given weight for $i \in \mathcal{P}$
- Then the Laplacian normalized by pt weight:

$$(\tilde{\mathbf{L}}\mathbf{u})_i := \frac{1}{m_i} \sum_{j \in \text{Nbr}(i)} w_{ij} (u_i - u_j)$$



Graph Laplacian with Point Weights

$$(\tilde{\mathbf{L}}\mathbf{u})_i := \frac{1}{m_i} \sum_{j \in \text{Nbr}(i)} w_{ij} (\mathbf{u}_i - \mathbf{u}_j)$$

$$= \star_0^{-1} \mathbf{d}_0^\top \star_1 \mathbf{d}_0$$

where $\star_0 = \begin{bmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_{|\mathcal{P}|} \end{bmatrix} \in \mathbb{R}^{|\mathcal{P}| \times |\mathcal{P}|}$

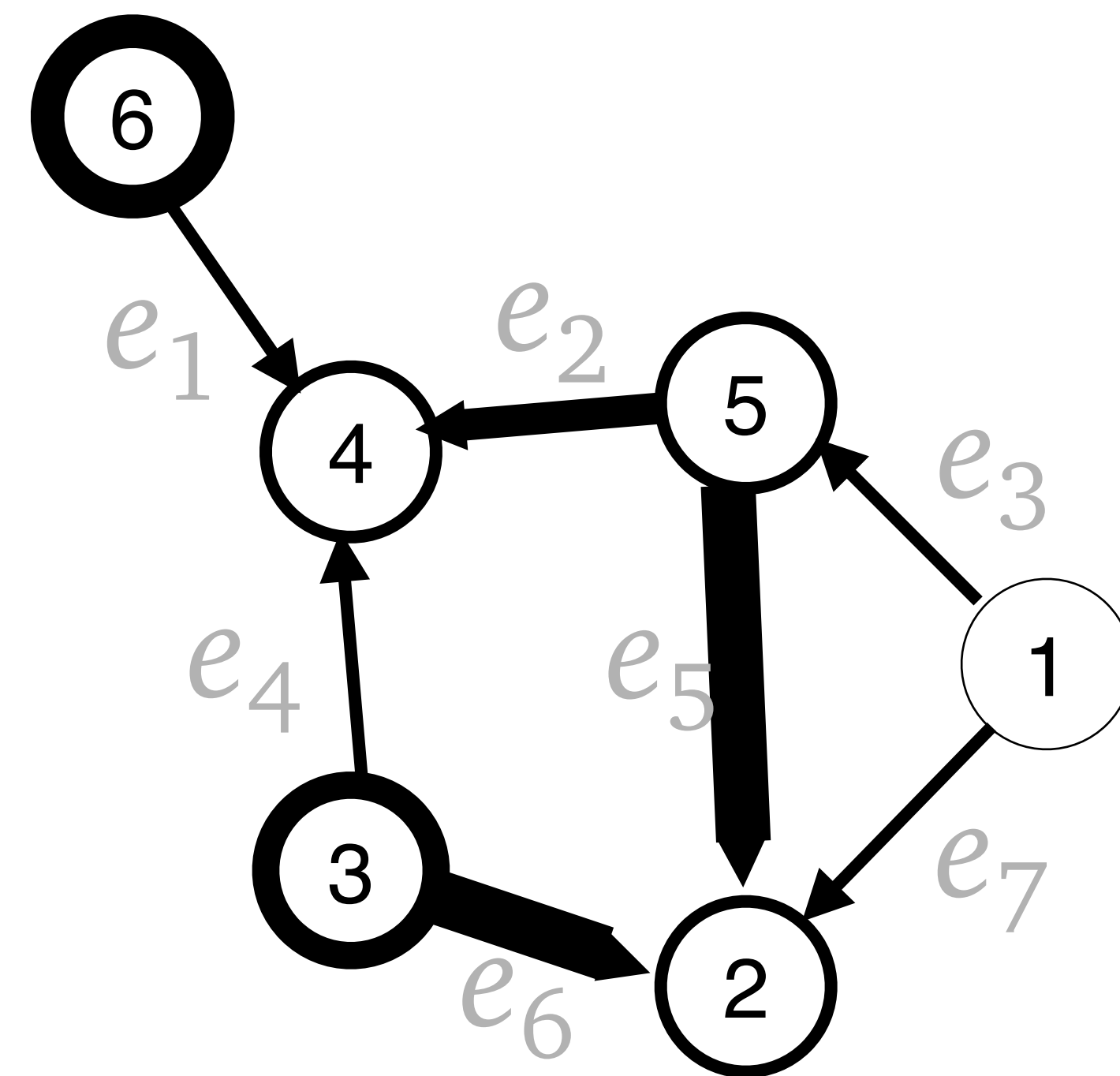


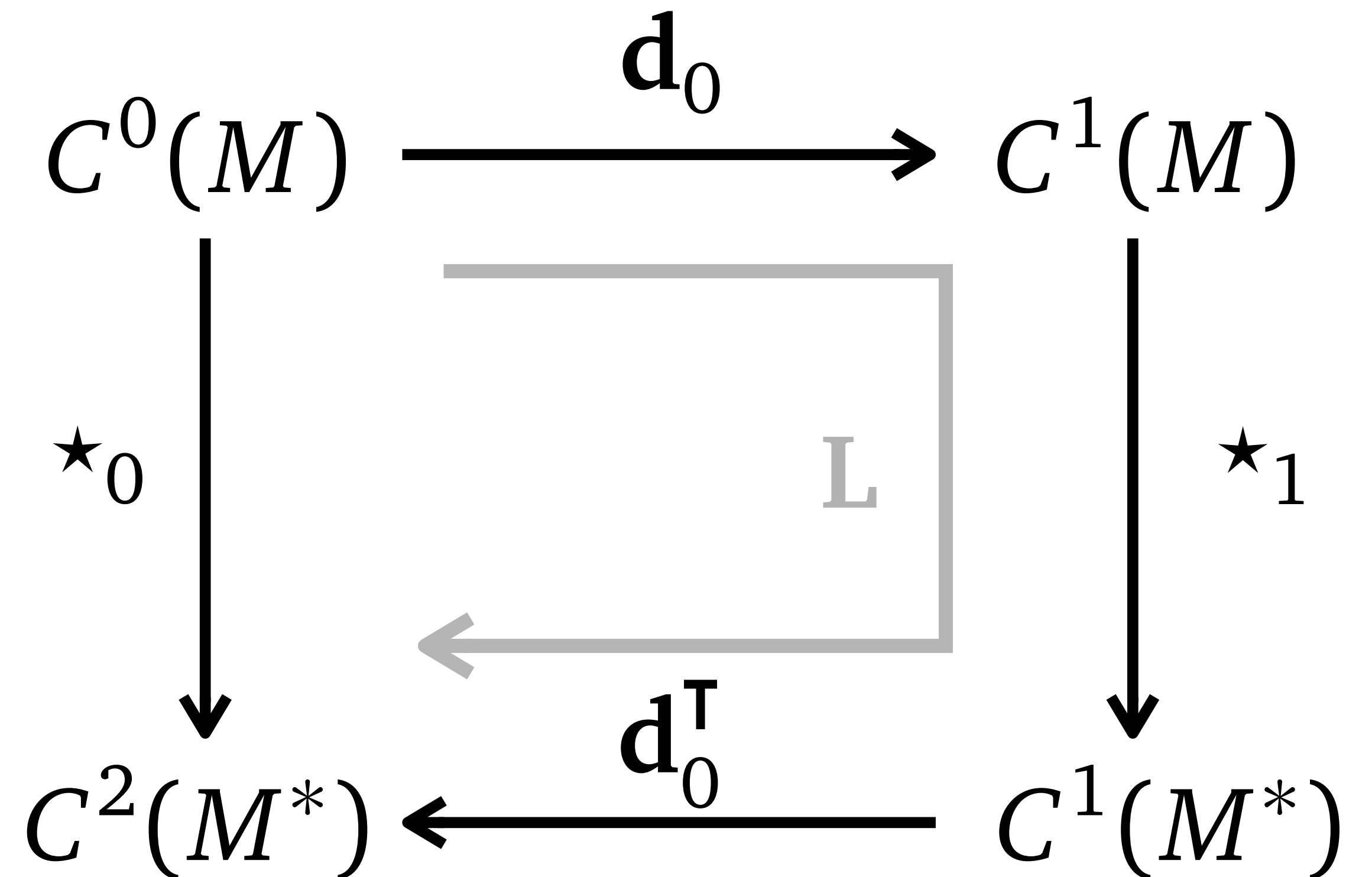
Diagram for what we have so far

$$\tilde{\mathbf{L}} = \star_0^{-1} \mathbf{d}_0^\top \star_1 \mathbf{d}_0$$

$$\mathbf{L} = \mathbf{d}_0^\top \star_1 \mathbf{d}_0$$

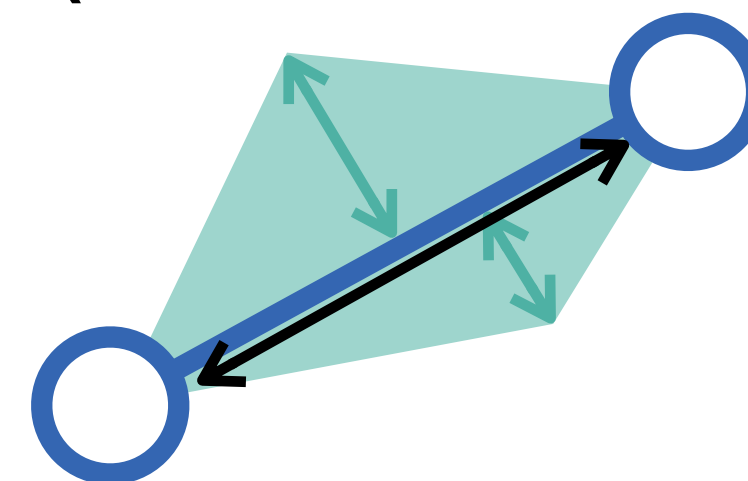
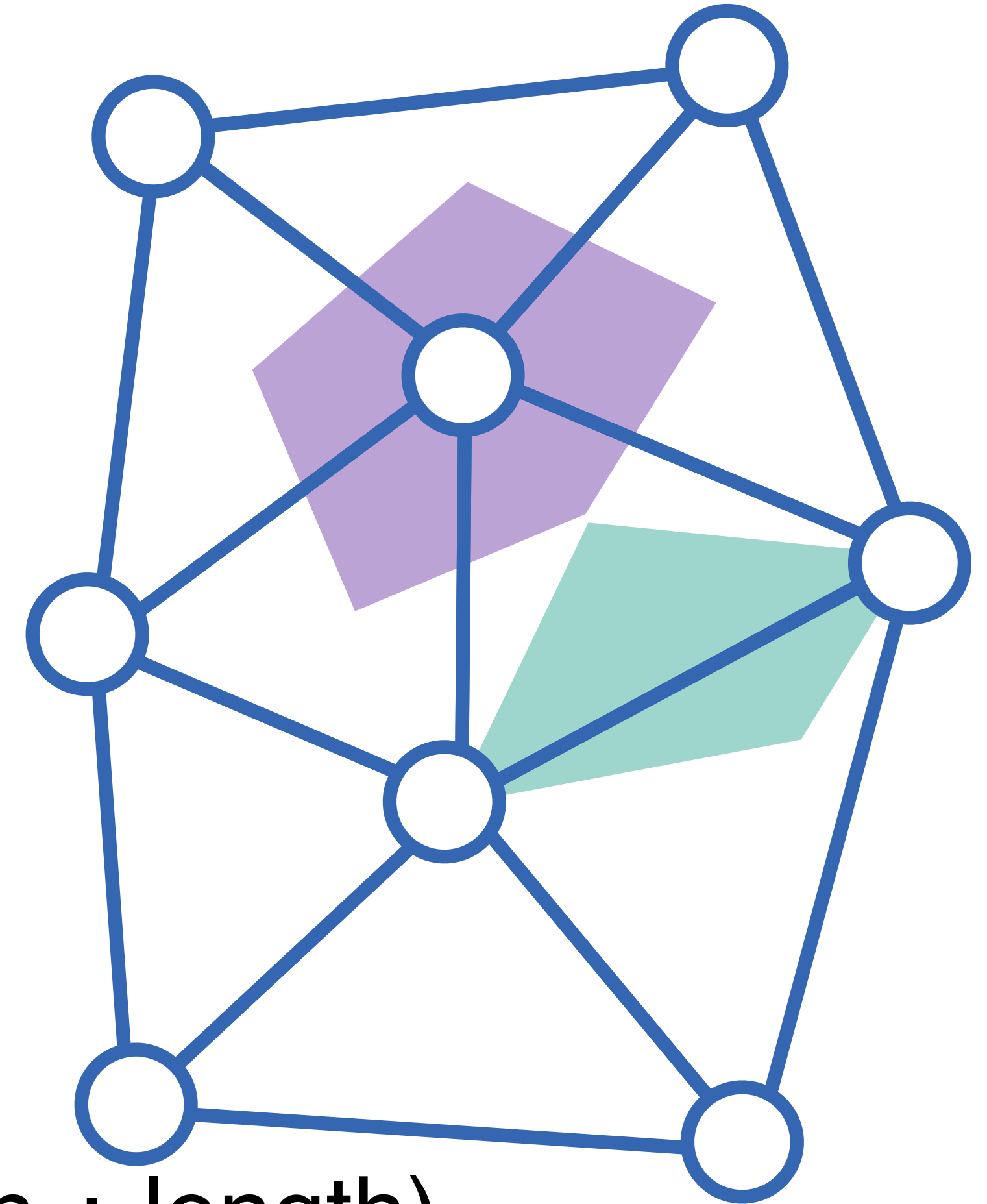
Both $\mathbf{L}$ and $\star_0$ are symmetric positive (semi)definite, but not for $\tilde{\mathbf{L}}$

In practice, we like to rearrange equation so that Laplacian is represented by $\mathbf{L}$ (e.g. moving $\star_0$ to the other side of the equation)



Connectivity and Weights

- When it comes to a triangle mesh representing a surface
- $\mathbf{d}_0$ (as matrix with entries $\{-1,0,1\}$) encodes the topology (connectivity)
- The point weights $\star_0$ and edge weights $\star_1$ encode the geometry/metric
 - ▶ Point weight is the **area** of the point territory
 - ▶ Edge weight is the **conductivity** (crosssection $\div$ length)



Discrete Differential Forms

- Laplacian in Physics
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Differential forms in a nutshell

- **Differential k -forms** are fields to be integrated on k -dim oriented submanifolds.

$$\int_{\sigma} \alpha$$

differential k -form
 k -dimension surface

- ▶ 0-form: functions $\int_p f := f|_p$
- ▶ 1-form: circulation $\int_{\gamma} \vec{u} \cdot d\vec{\ell}$
- ▶ $(n-1)$ -form: flux $\iint_S \vec{u} \cdot \vec{n} dA$
- ▶ n -form: density $\iiint_U f dV$

Differential forms in a nutshell

- The space of differential k -forms on a manifold M is denoted by

$$\Omega^k(M)$$

Derivatives on differential forms

- The **exterior derivative**

$$d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$$

is defined so that

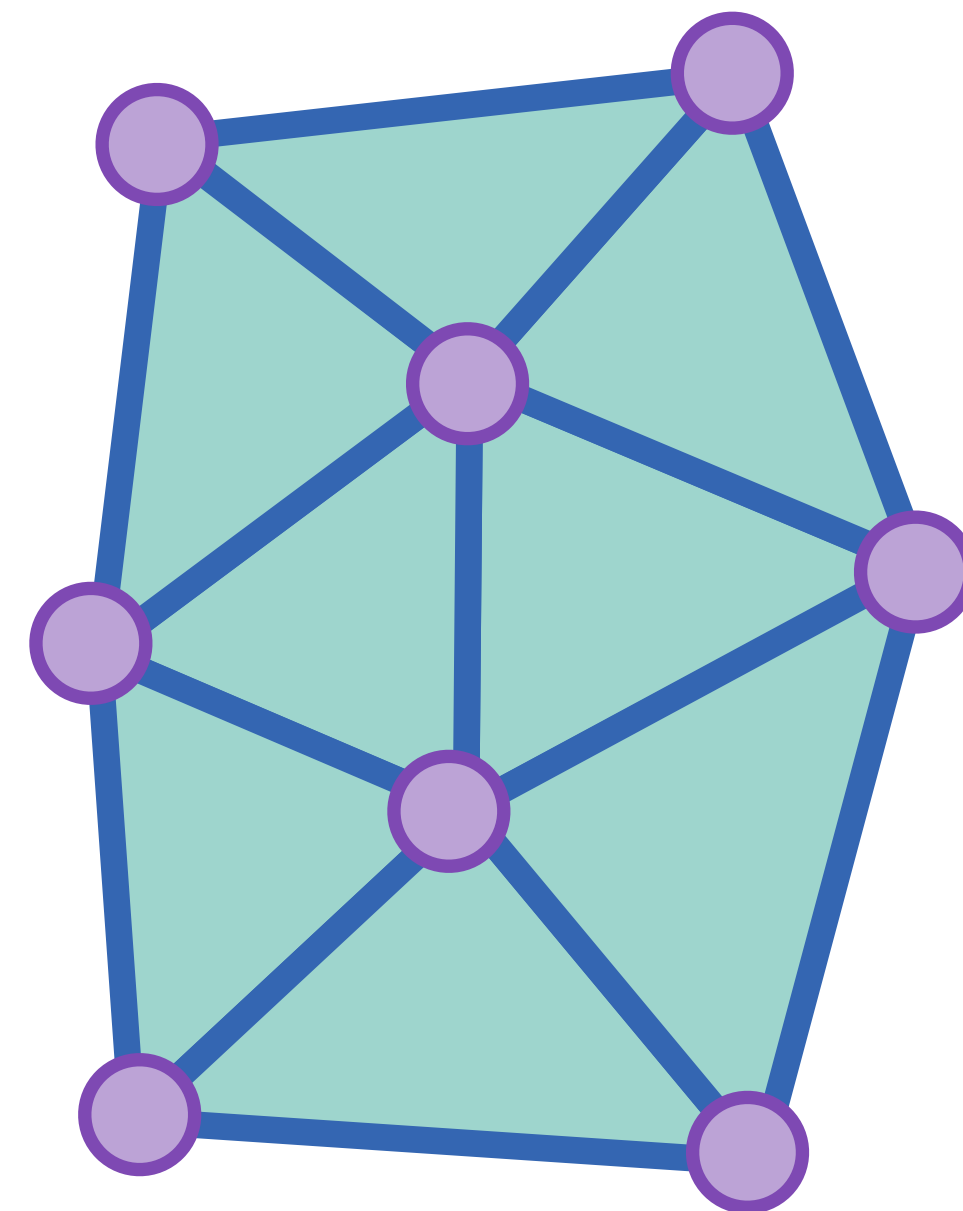
$$\boxed{\int_A d\alpha = \int_{\partial A} \alpha}$$

for all $\alpha \in \Omega^k(M)$
 $A \in C_{k+1}(M)$

- This equation is called **Stokes Theorem** (or definition).
- For now this is just a definition inherited from the boundary operator. We will learn more about how it unifies many calculus identities.

Discrete differential forms

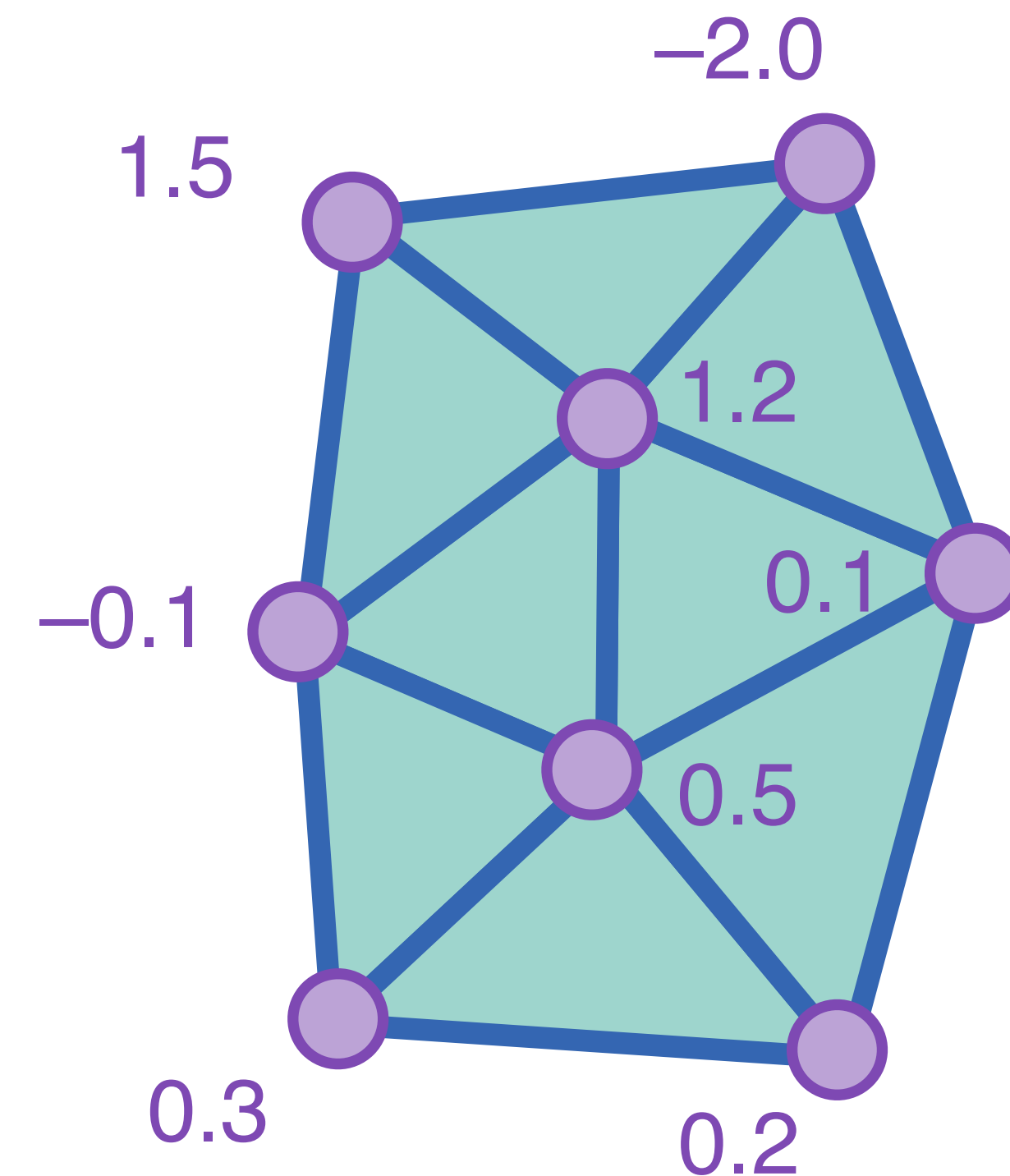
- Consider a discrete triangle mesh $M = (\mathcal{P}, \mathcal{E}, \mathcal{F})$



Discrete differential forms

- Consider a discrete triangle mesh $M = (\mathcal{P}, \mathcal{E}, \mathcal{F})$
- We encode a 0-form $\phi \in \Omega^0(M)$ by its samples on vertices

$$\hat{\phi} : \mathcal{P} \rightarrow \mathbb{R} \quad \hat{\phi}_p = \phi(p) \quad p \in \mathcal{P}$$



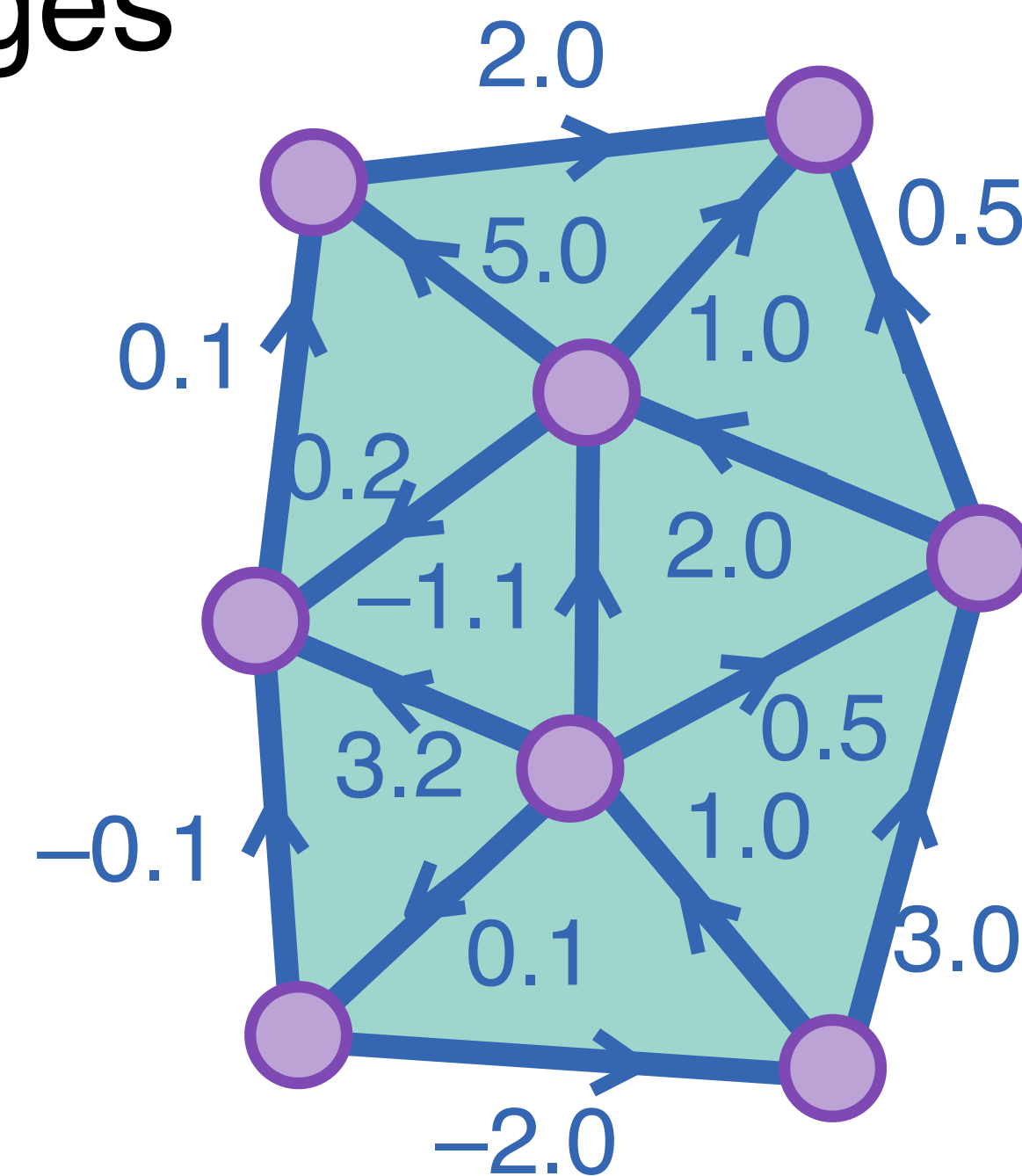
Discrete differential forms

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$$\hat{\phi} : \mathcal{P} \rightarrow \mathbb{R} \quad \hat{\phi}_p = \phi(p) \quad p \in \mathcal{P}$$

- Encode a 1-form $\alpha \in \Omega^1(M)$ by its integrals on edges

$$\hat{\alpha} : \mathcal{E} \rightarrow \mathbb{R} \quad \hat{\alpha}_e = \int_e \alpha \quad e \in \mathcal{E}$$



Discrete differential forms

- Consider a discrete triangle mesh $M = (\mathcal{P}, \mathcal{E}, \mathcal{F})$
- We encode a 0-form $\phi \in \Omega^0(M)$ by its samples on vertices

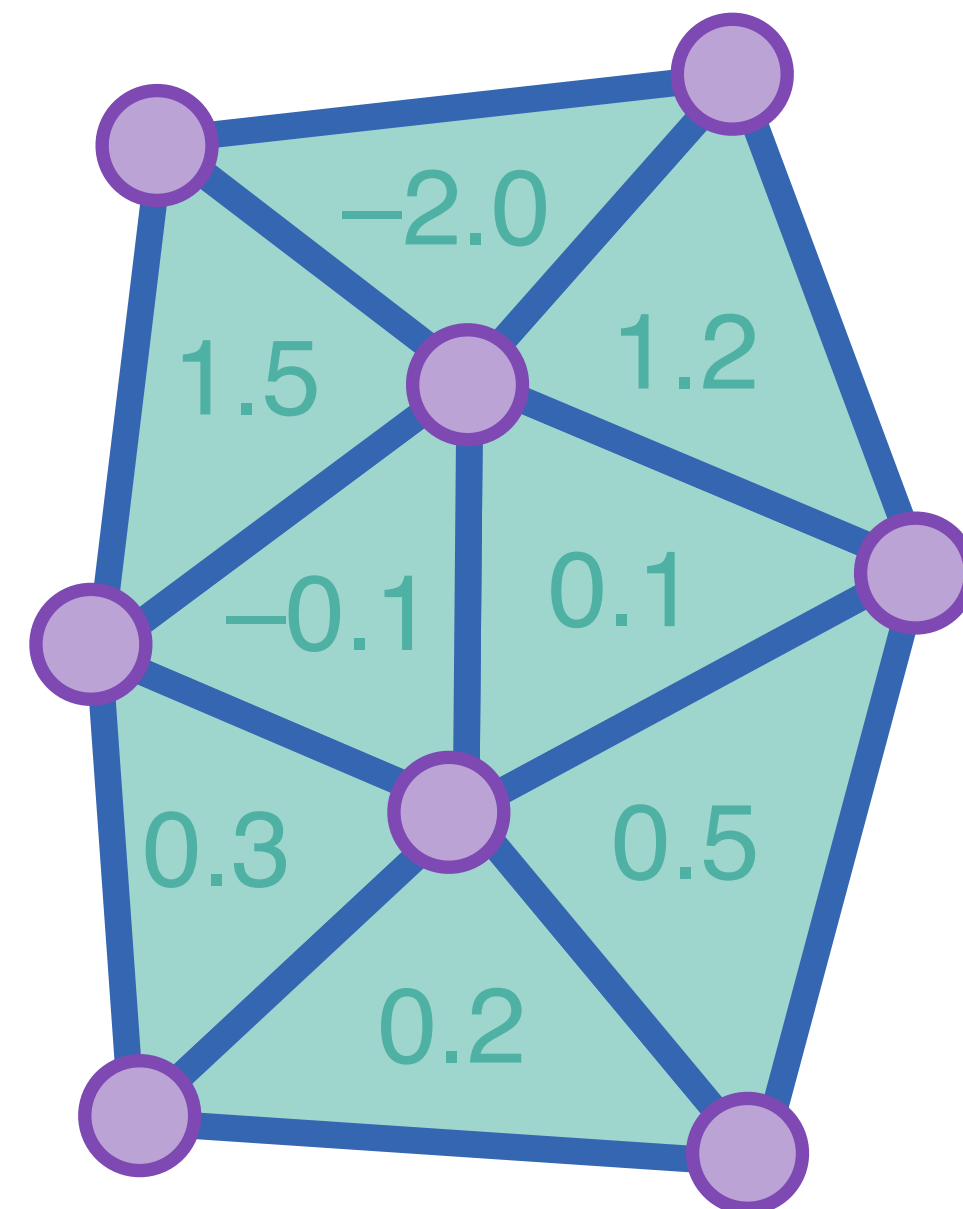
$$\hat{\phi} : \mathcal{P} \rightarrow \mathbb{R} \quad \hat{\phi}_p = \phi(p) \quad p \in \mathcal{P}$$

- Encode a 1-form $\alpha \in \Omega^1(M)$ by its integrals on edges

$$\hat{\alpha} : \mathcal{E} \rightarrow \mathbb{R} \quad \hat{\alpha}_e = \int_e \alpha \quad e \in \mathcal{E}$$

- Encode 2-form $\omega \in \Omega^2(M)$ by its integrals on faces

$$\hat{\omega} : \mathcal{F} \rightarrow \mathbb{R} \quad \hat{\omega}_f = \int_f \omega \quad f \in \mathcal{F}$$



Discrete differential forms

- Each discrete k -form ω is an assignment of numbers per simplex on the k -skeleton

$$\omega = (\omega_{\sigma_1}, \dots, \omega_{\sigma_{|M_k|}})^T$$

- Given any k -chain $\gamma = \sum_{i=1}^{M_k} c_i \sigma_i \in C_k$, represented by the array

$$\mathbf{c} = (c_1, \dots, c_{|M_k|})^T$$

The evaluation of the k -form on the chain is

$$\int_{\gamma} \omega = \sum_{i=1}^{M_k} c_i \omega_{\sigma_i} = \mathbf{c}^T \omega$$

- Discrete k -forms are dual to k -chains.

Formal definition

- Let $M = (M_0, M_1, M_2, \dots, M_n)$ be a simplicial/cell complex
- Recall: space of k -chains $C_k = \text{span}_{\mathbb{R} \text{ or } \mathbb{Z}} M_k$
a k -chain is a linear combination of k -cells
- There are boundary operators making chain spaces a chain complex

$$C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \quad \partial_k \circ \partial_{k+1} = 0$$

Definition The space of k -cochains (discrete k -forms)

$$C^k := \left\{ \alpha \mid \alpha: C_k \xrightarrow{\text{linear}} \mathbb{R} \right\} = C_k^*$$

A k -cochain is an assignment of number on each k -cell.

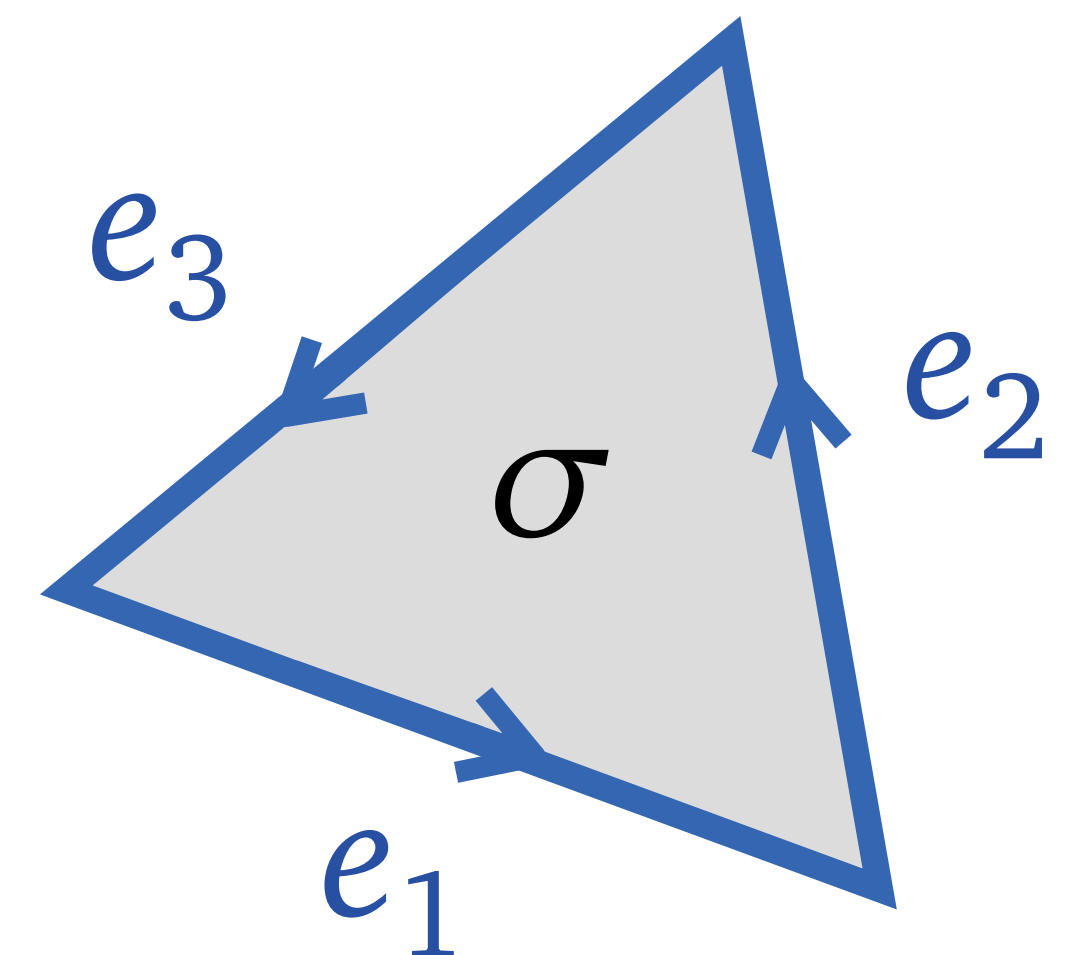
Discrete exterior derivative

- Representing discrete forms as integrated quantity make defining exterior derivatives easy.
- Take advantage of Stokes' Theorem

$$\int_S d\alpha = \int_{\partial S} \alpha$$

- For example,

$$\begin{aligned} \widehat{(d\alpha)}_\sigma &= \int_\sigma d\alpha \stackrel{\text{Stokes' Thm}}{=} \int_{\partial\sigma} \alpha \\ &= \sum_{i=1}^3 \int_{e_i} \alpha = \sum_{i=1}^3 \hat{\alpha}_{e_i} \end{aligned}$$



Discrete exterior derivative

Definition

The ***discrete exterior derivative*** (or **coboundary operator**) on discrete forms is the adjoint of the boundary operator on chains:

$$\mathbf{d}_k : C^k \xrightarrow{\text{linear}} C^{k+1}$$

$$\langle \mathbf{c} | \mathbf{d}_k \boldsymbol{\alpha} \rangle = \langle \partial_{k+1} \mathbf{c} | \boldsymbol{\alpha} \rangle \text{ for all } \mathbf{c} \in C_{k+1}$$

- In matrix form: $\mathbf{d}_k = \partial_{k+1}^\top$

- Immediate consequence

$$\mathbf{d}_{k+1} \circ \mathbf{d}_k = \mathbf{0}$$

*discrete exterior derivatives
form a chain complex*

$$C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \xrightarrow{d_2} \dots$$

Discrete Hodge Stars

- Laplacian in Physics
- Finite Difference
- Graph Laplacian
- Discrete differential forms
- Discrete Hodge stars
- Example: heat equation

Dual mesh

Dual Mesh

Triangle Mesh and its Dual

- Let $M = (M_0, M_1, M_2)$ be a triangle mesh (without boundary for now)

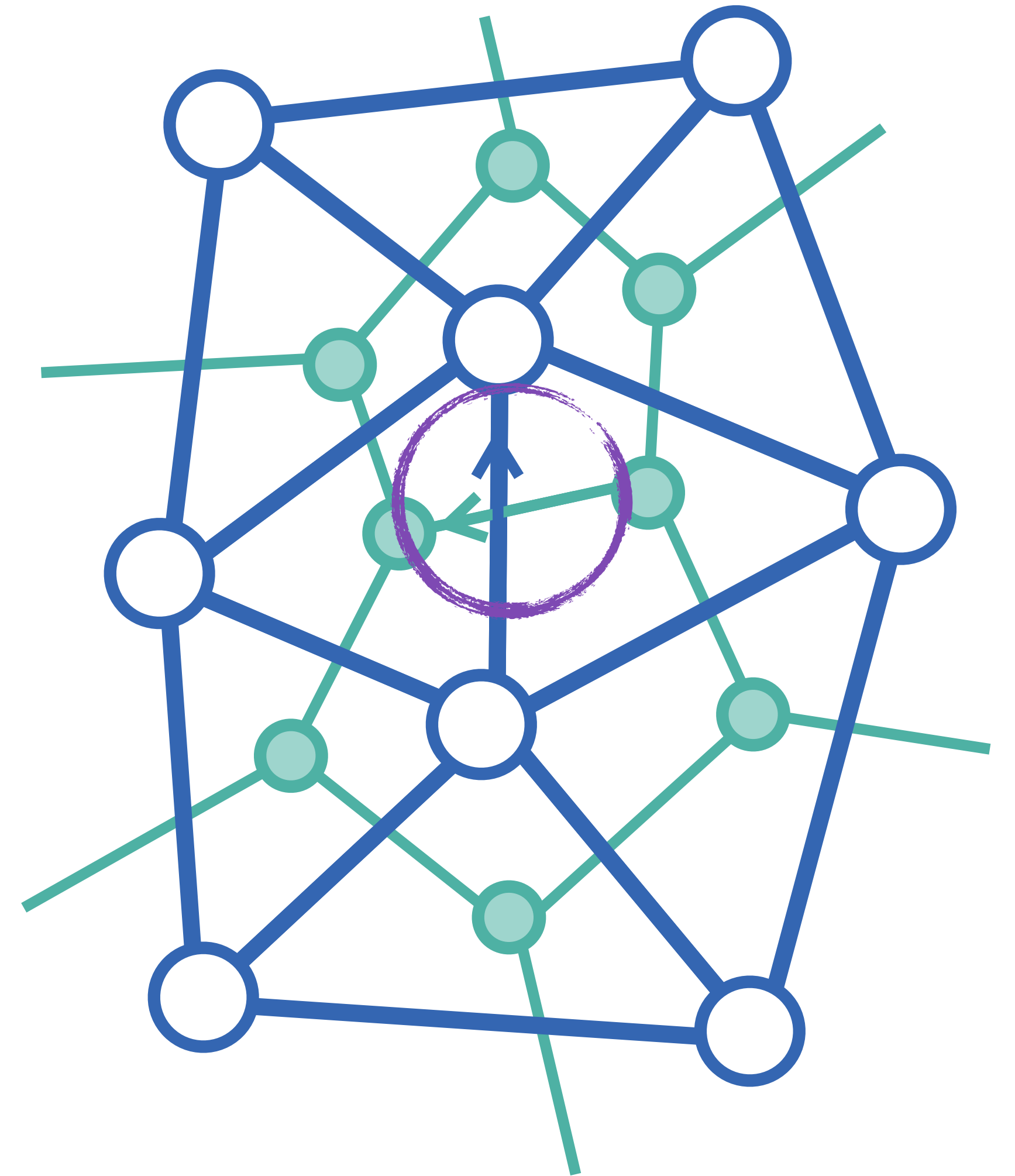
- Consider the *dual mesh*

$$M^* = (M_0^*, M_1^*, M_2^*)$$

dual pts $M_0^* \cong M_2$ primal faces

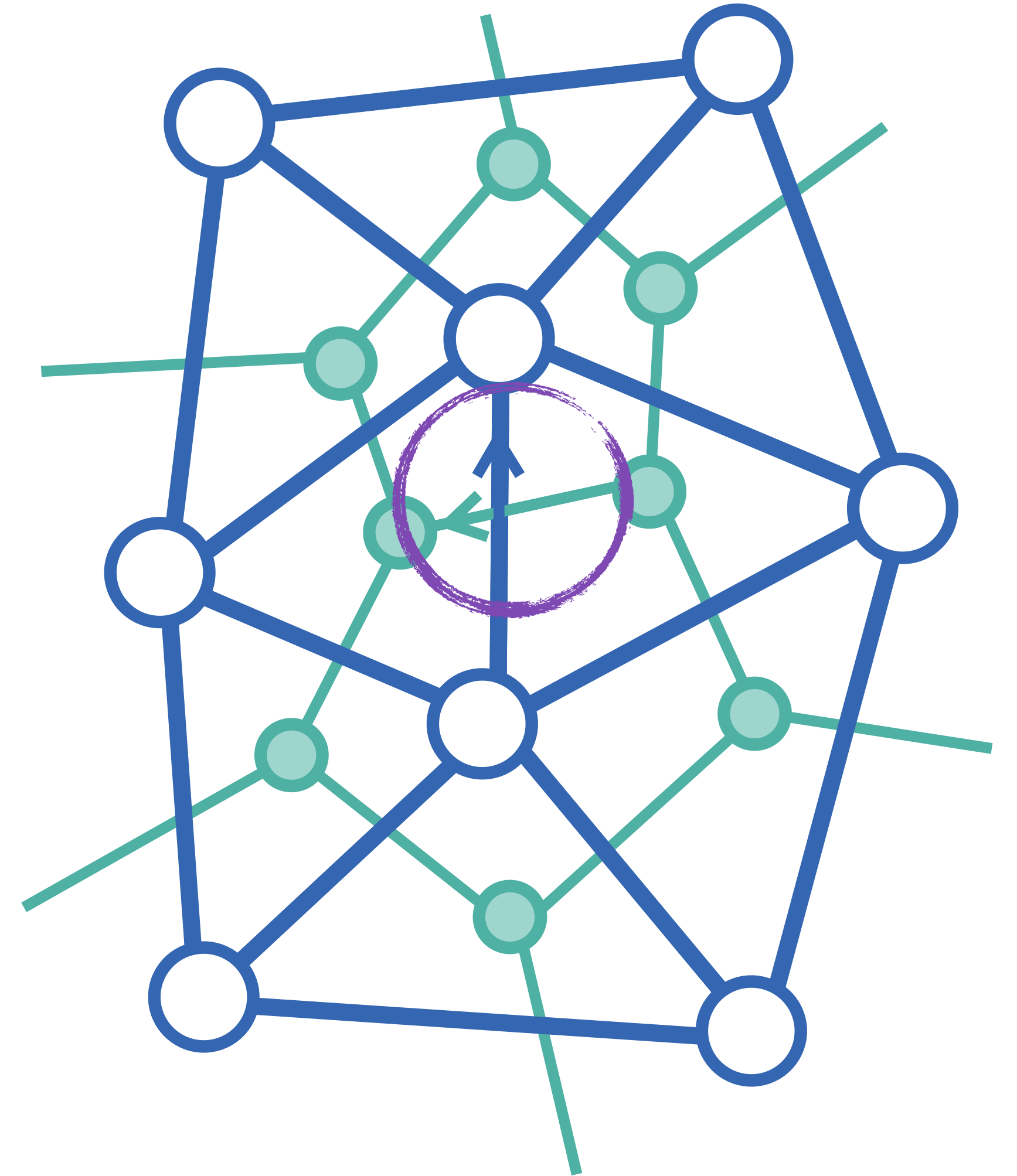
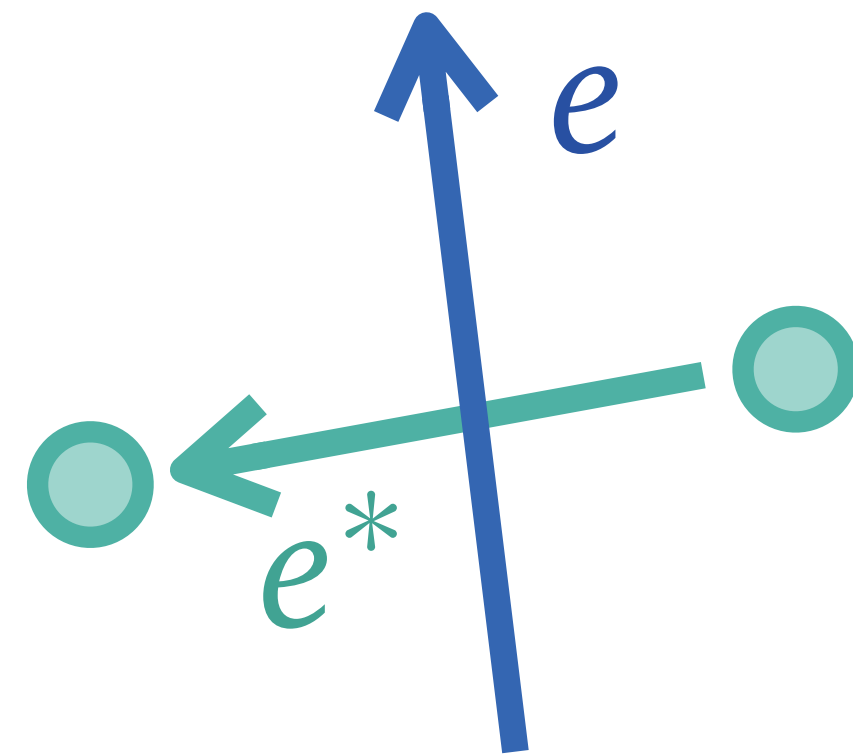
dual edges $M_1^* \cong M_1$ primal edges

dual faces $M_2^* \cong M_0$ primal pts



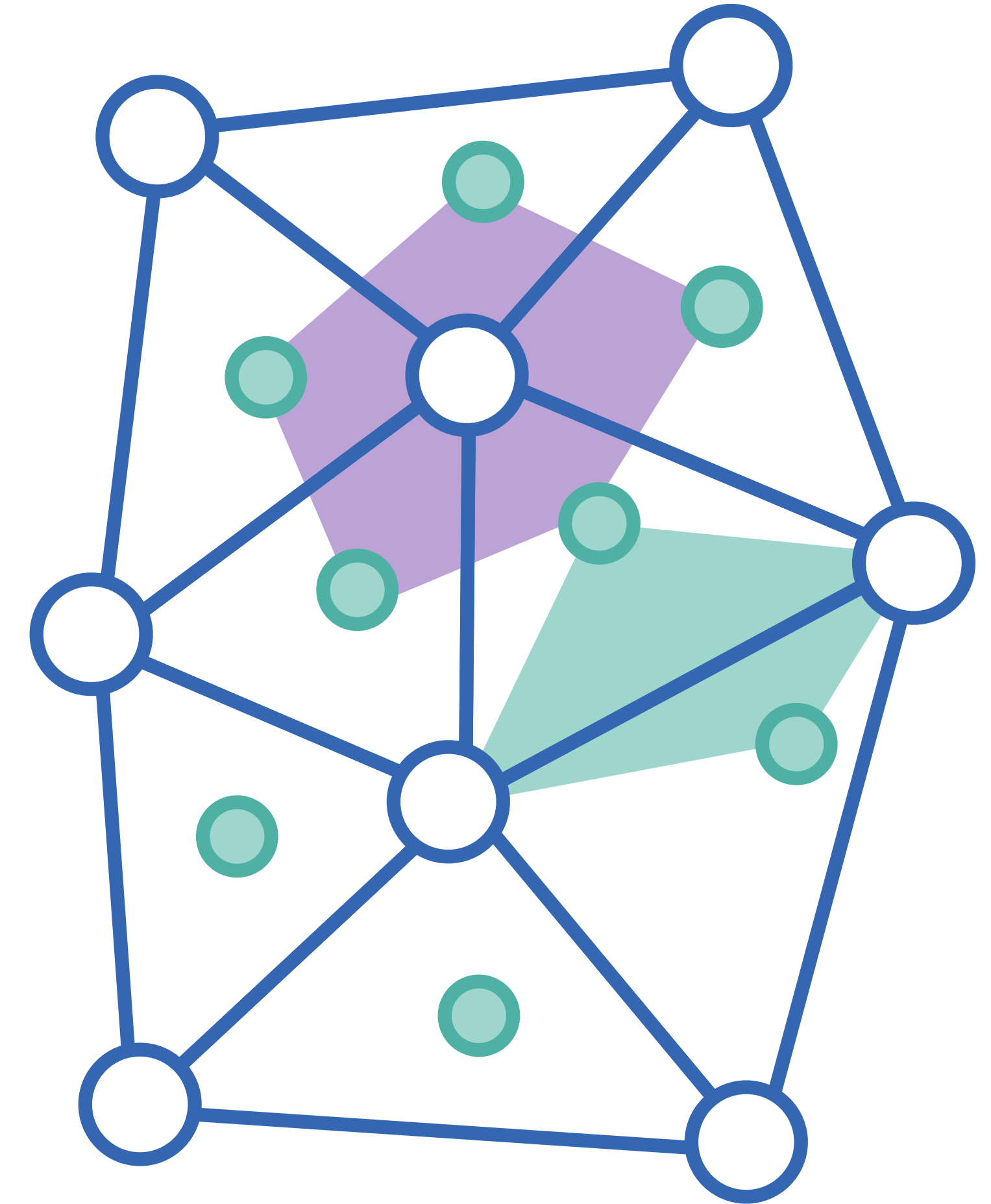
Triangle Mesh and its Dual

- Orientation of the dual edge e^* is chosen as pointing from the RHS triangle of e to the LHS triangle of e



Dual mesh

- By connecting dual mesh vertices to primal mesh vertices
 - ▶ we have notions of “territory” for primal vertex and primal edge.
 - ▶ The entire surface can be partitioned into territories of vertices (dual cell) or into territories of edges.



Dual mesh

- Edge cell has an area of

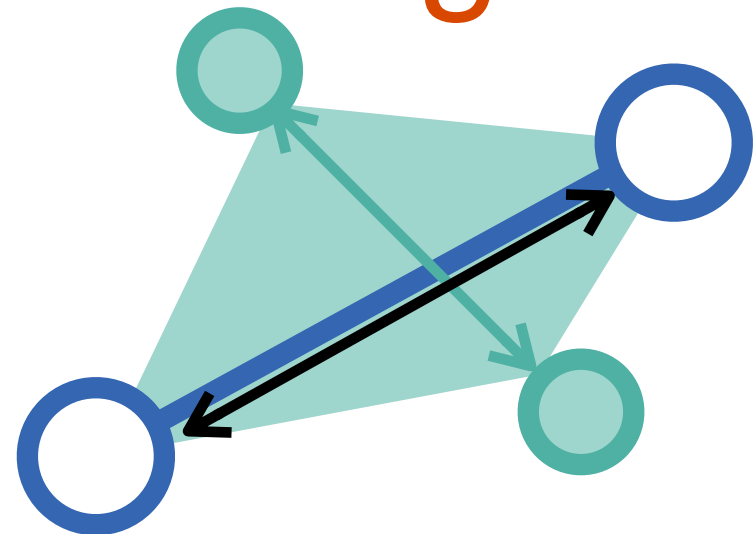
projection

$$\frac{1}{2} |\mathbf{P}_{e^\perp} \mathbf{e}^*| |e|$$

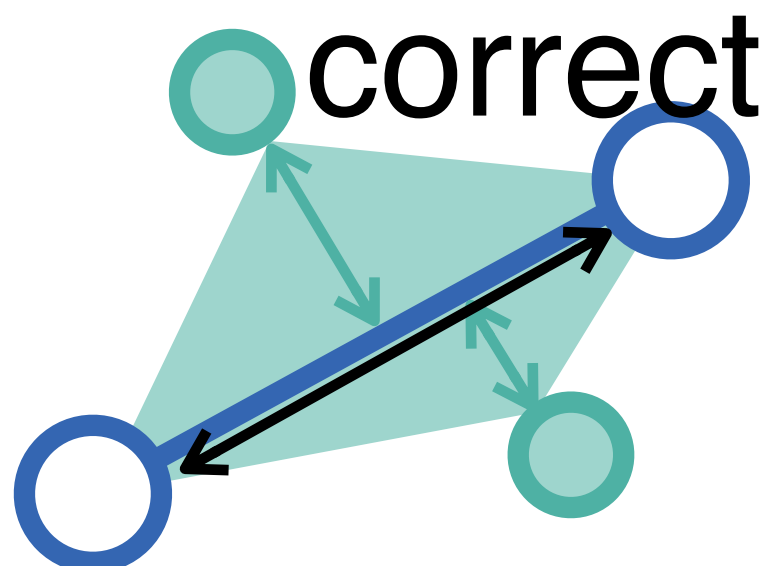
edge's
cross
section
width

edge
length

wrong



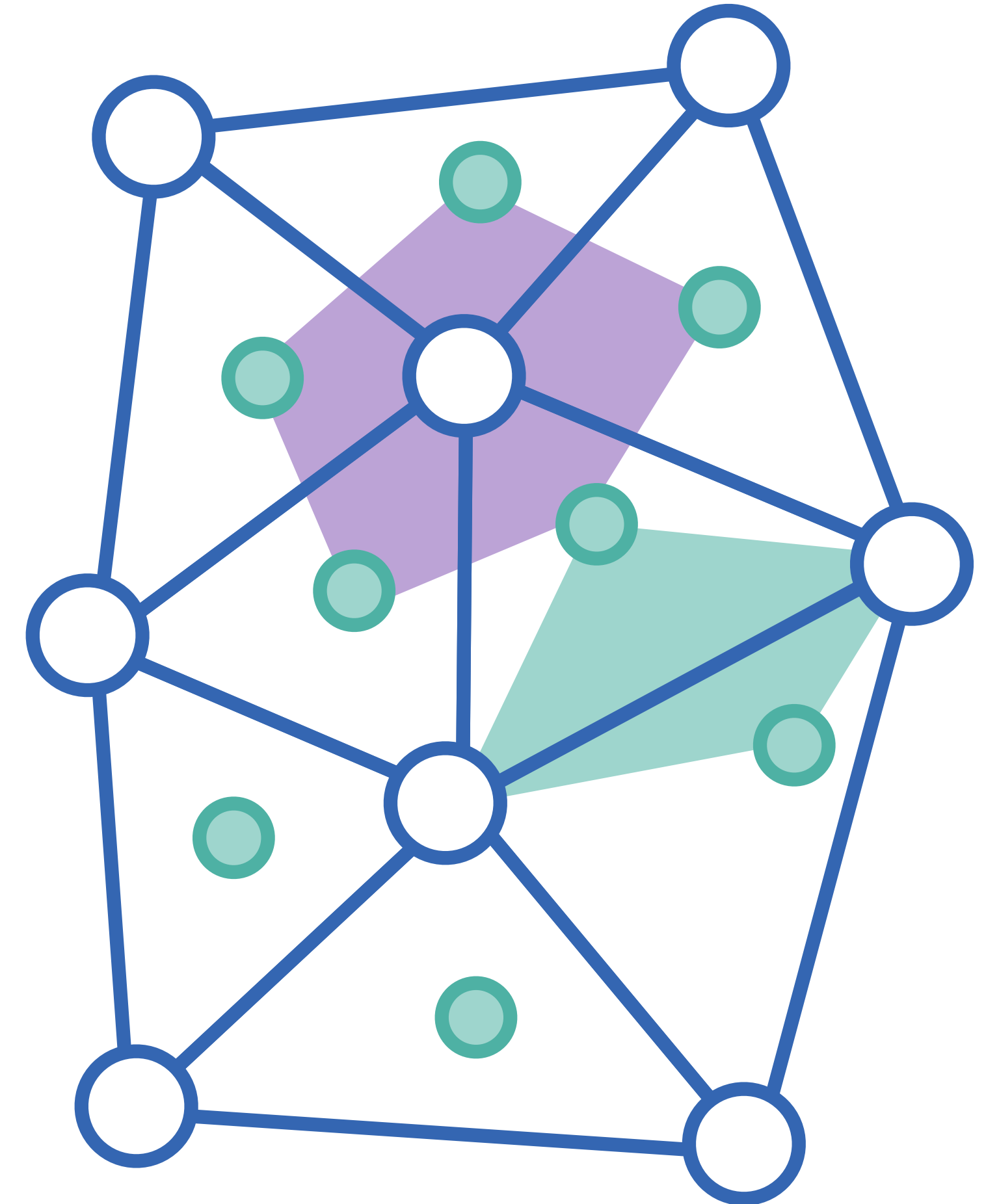
correct



width



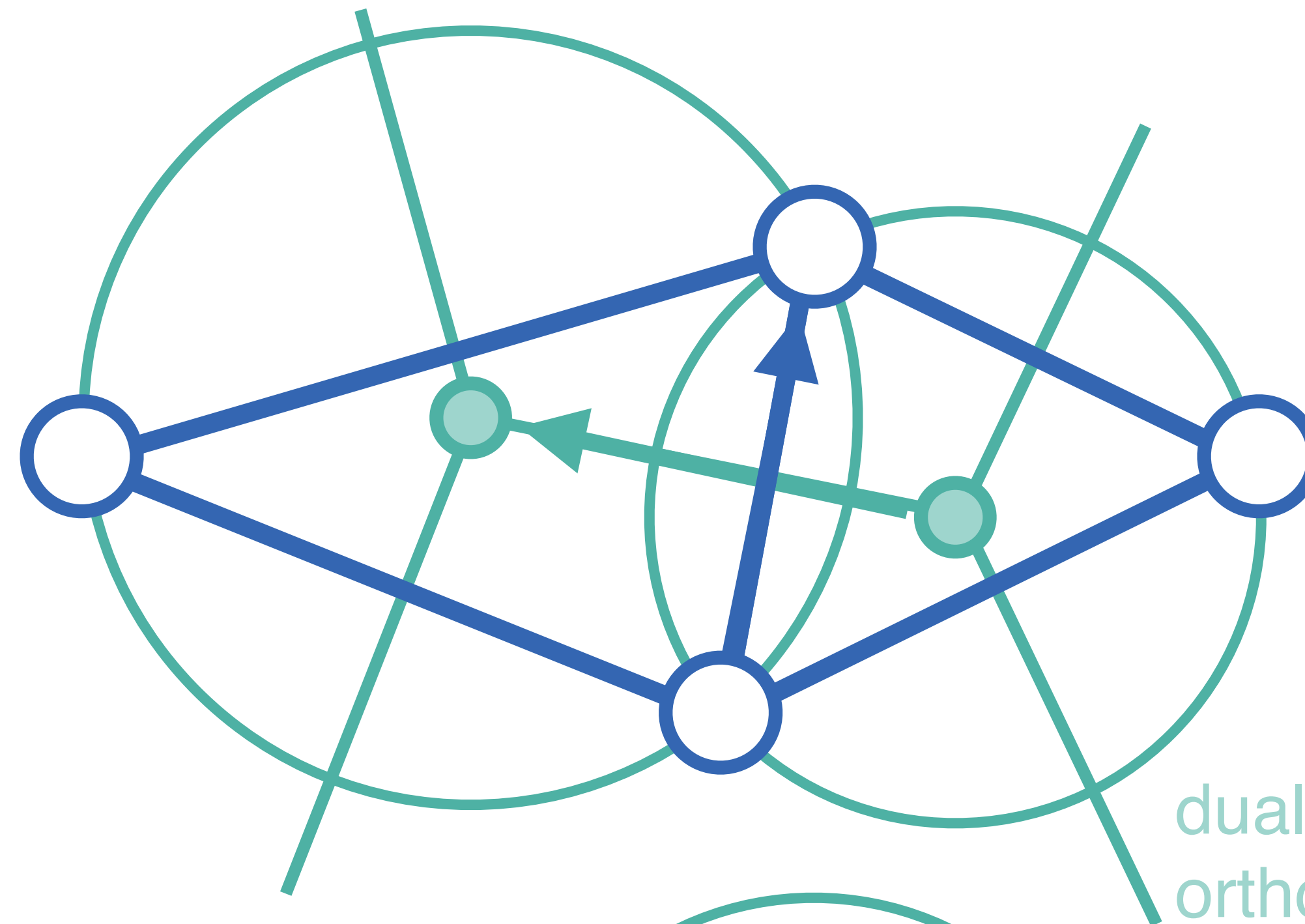
length



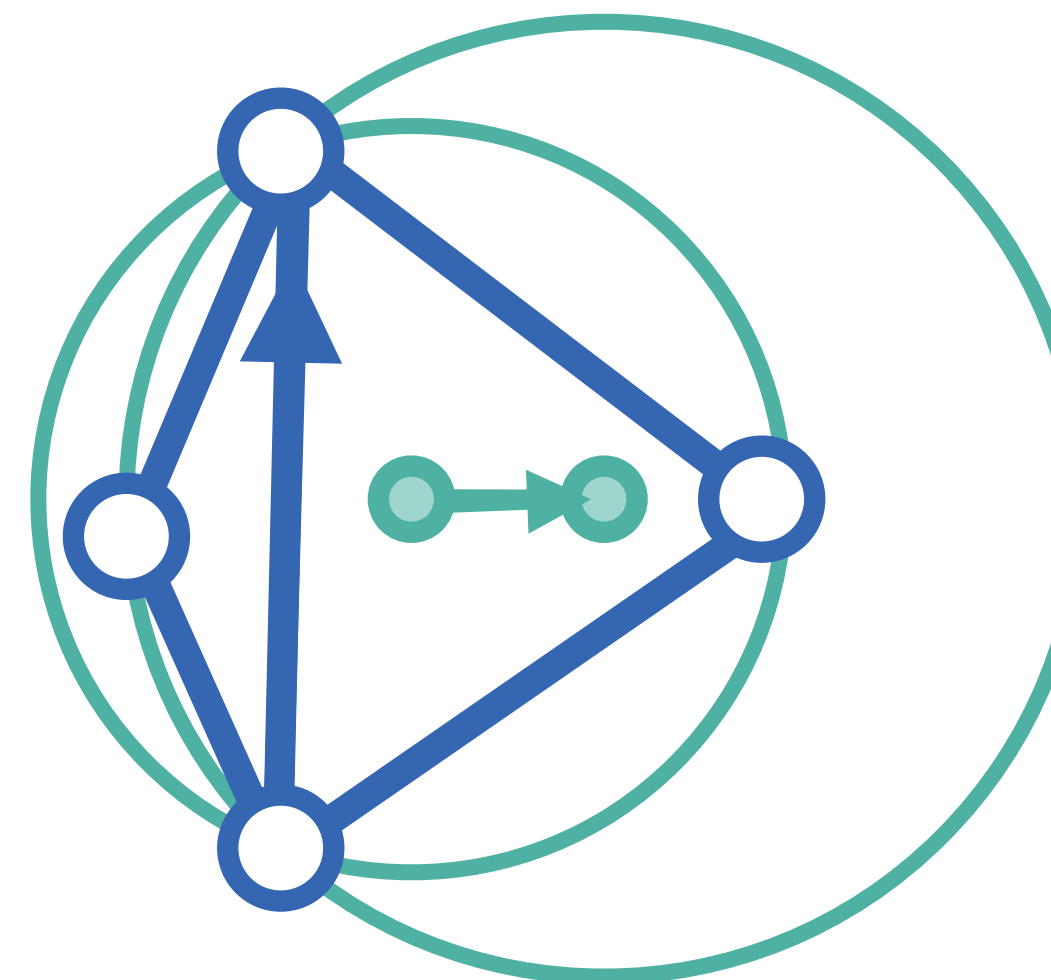
Where to put dual center

- Circumcenter

- ▶ edge width =
dual edge length



- ▶ Relies on good triangle mesh



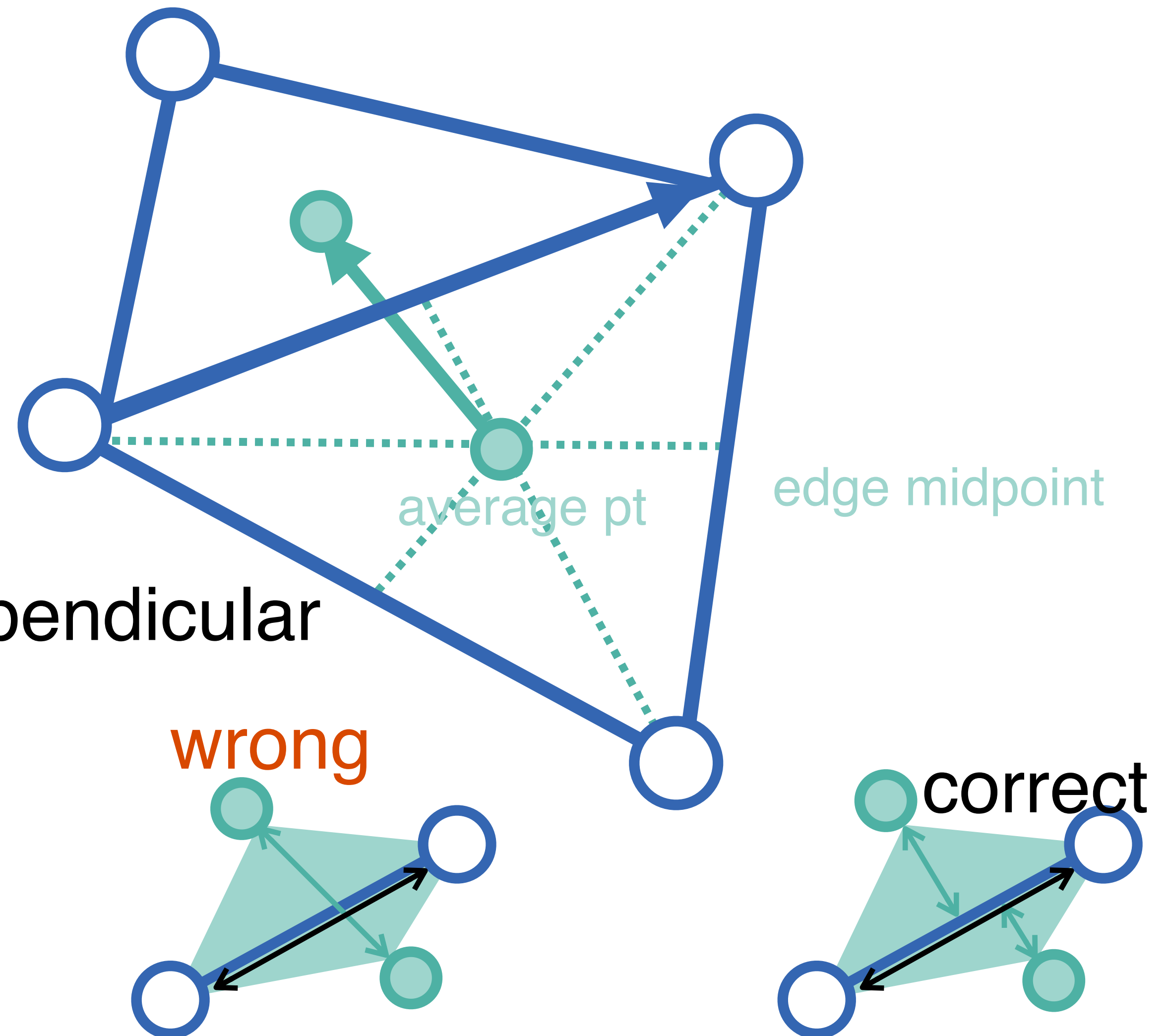
Non-Delaunay
mesh: edge weight
becomes negative

Where to put dual center

- Barycenter

- ▶ Always in the interior of the triangle

- ▶ Dual edge is no longer perpendicular to the primal edge (the “edge width” requires projection)



Discrete Hodge star

Back to Hodge star

Discrete Hodge star

Definition

Define $\star_k : C^k(M) \xrightarrow{\text{linear}} C^{n-k}(M^*)$ by a diagonal matrix with

$$(\star_k \alpha)_{\sigma^*} = \frac{|\mathbf{P}_{\sigma^\perp} \sigma^*|}{|\sigma|} \alpha_\sigma$$

$$\star_k = \begin{bmatrix} w_{\sigma_1} & & & \\ & w_{\sigma_2} & & \\ & & \ddots & \\ & & & w_{\sigma_{|M_k|}} \end{bmatrix} \quad w_\sigma = \frac{|\mathbf{P}_{\sigma^\perp} \sigma^*|}{|\sigma|}$$

On triangle mesh

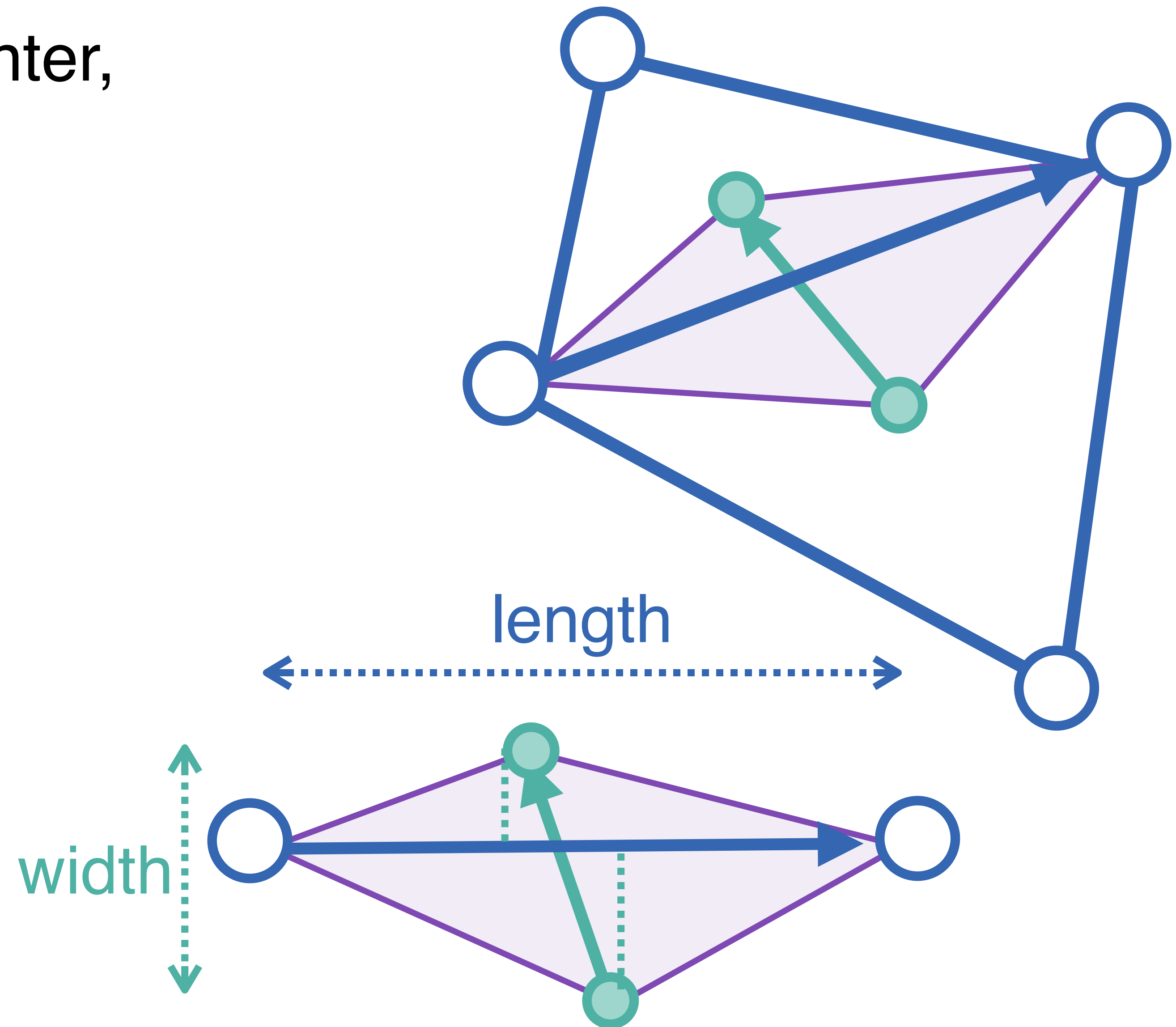
$$\begin{aligned} \star_0 &= \begin{bmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_{|\mathcal{P}|} \end{bmatrix} \quad \begin{array}{l} \text{point masses,} \\ \text{point weights} \end{array} \\ \star_1 &= \begin{bmatrix} w_{e_1} & & & \\ & w_{e_2} & & \\ & & \ddots & \\ & & & w_{e_{|\mathcal{E}|}} \end{bmatrix} \quad \begin{array}{l} \text{edge weights,} \\ \text{edge conductivity,} \\ \text{edge stiffness} \end{array} \\ \star_2 &= \begin{bmatrix} 1/a_{f_1} & & & \\ & 1/a_{f_2} & & \\ & & \ddots & \\ & & & 1/a_{f_{|\mathcal{F}|}} \end{bmatrix} \quad \begin{array}{l} 1/(\text{face area}) \end{array} \end{aligned}$$

Star on 1-form

- Once we have picked dual center, we build the edge weights

$$w_e = \frac{\text{width}}{\text{length}}$$

$$\star_1 = \begin{bmatrix} w_{e_1} & & & \\ & w_{e_2} & & \\ & & \ddots & \\ & & & w_{e_{|\mathcal{E}|}} \end{bmatrix}$$



Star on 1-form

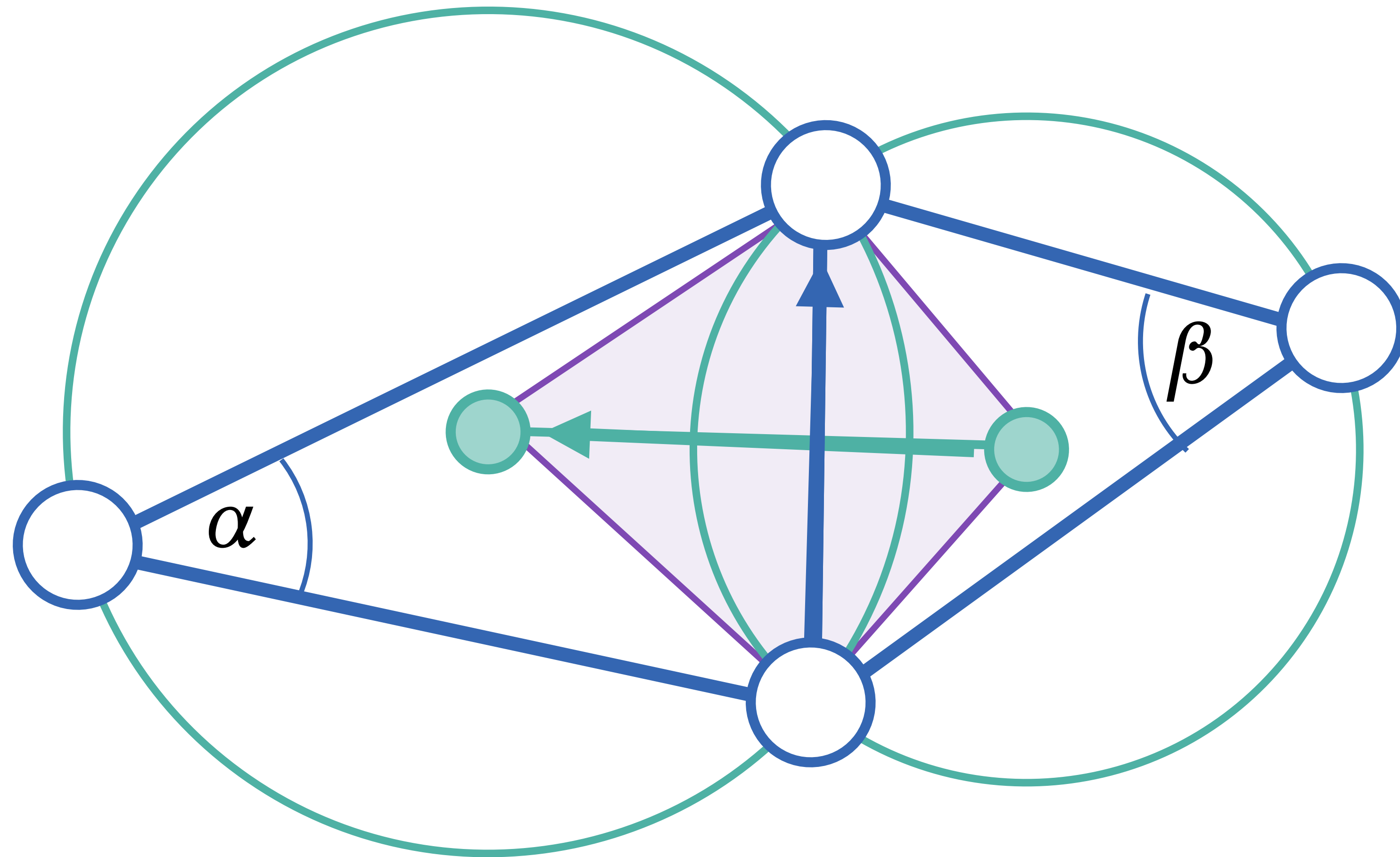
- For circumcentric dual:

$$w_e = \frac{\text{width}}{\text{length}}$$

$$= \frac{1}{2}(\cot \alpha + \cot \beta)$$

(exercise)

known as the
cotangent weight



Star on 0-form

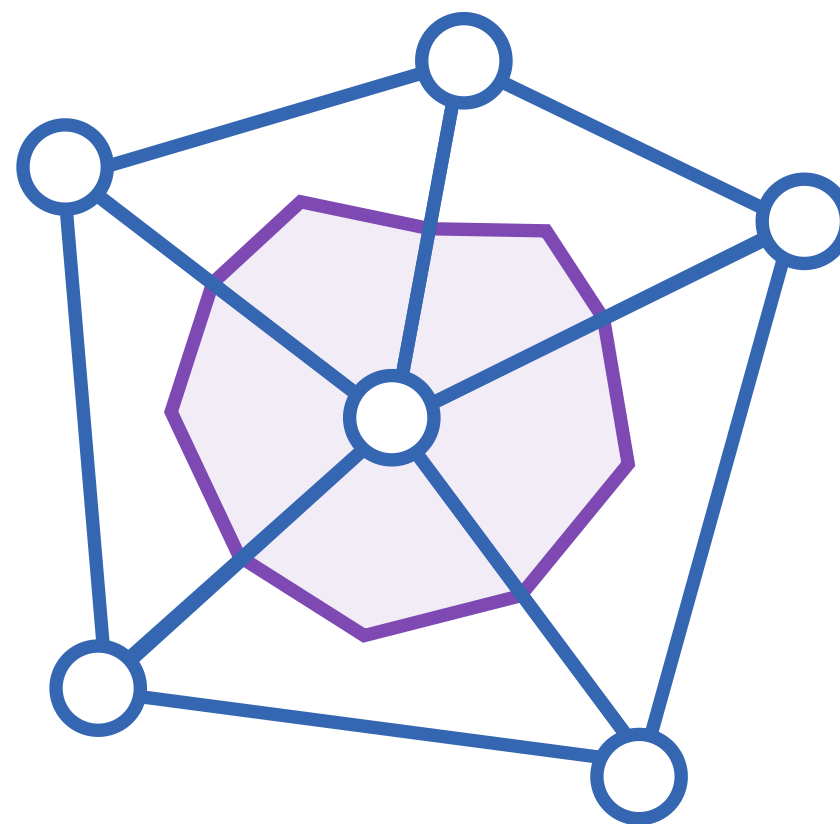
- As for point weight

$$m_i = \text{Area}_{\text{pt}_i}$$

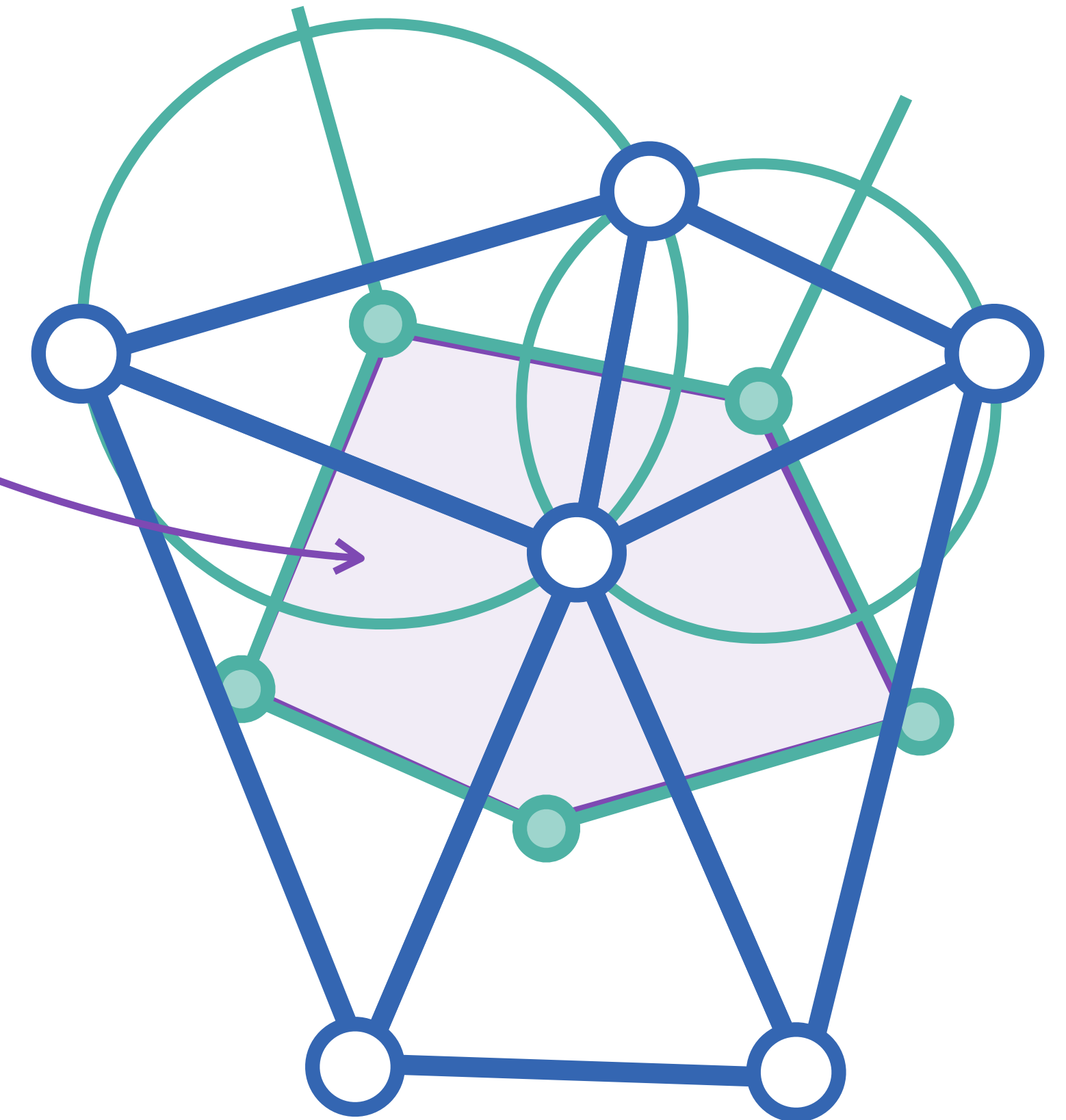
- For circumcentric:
area of the Voronoi cell

- For barycentric (much simpler formula):

$$\frac{1}{3} \sum_{f \succ i} \text{Area}_f$$



Voronoi cell



$$\star_0 = \begin{bmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_{|\mathcal{P}|} \end{bmatrix} \in \mathbb{R}^{|\mathcal{P}| \times |\mathcal{P}|}$$

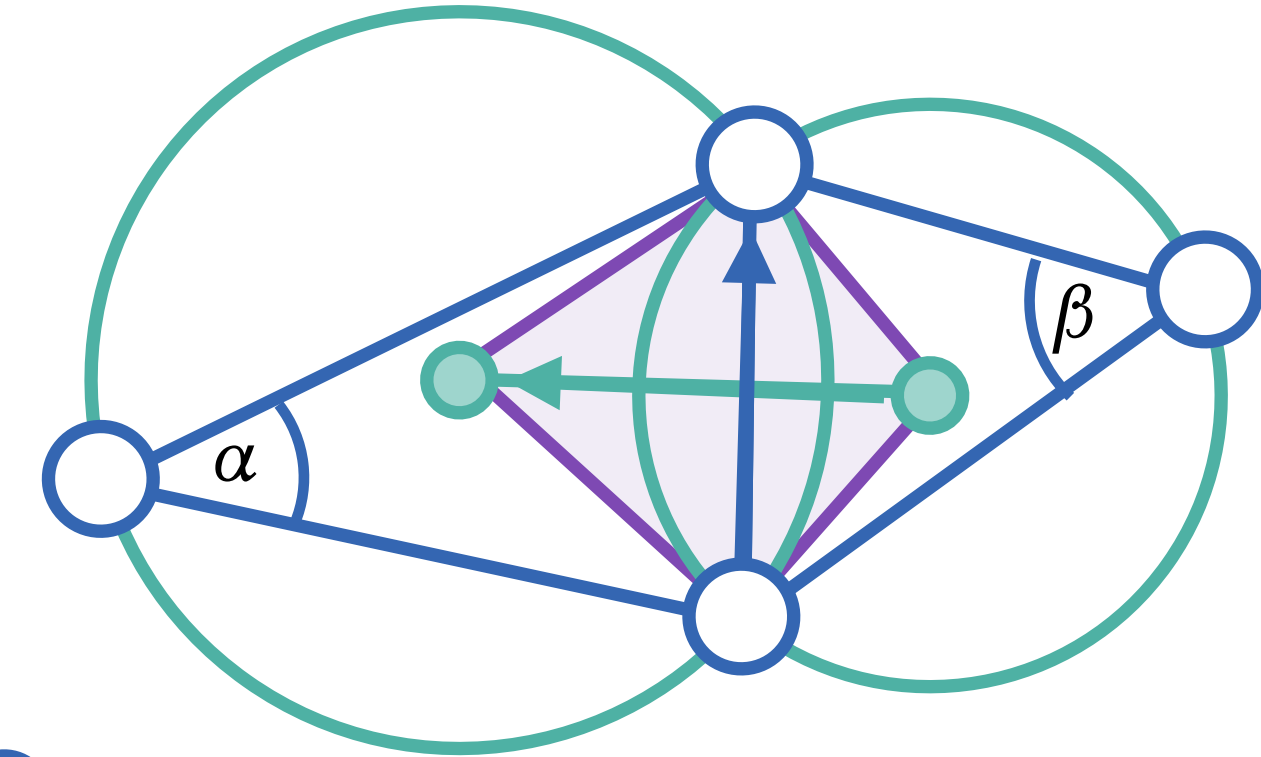
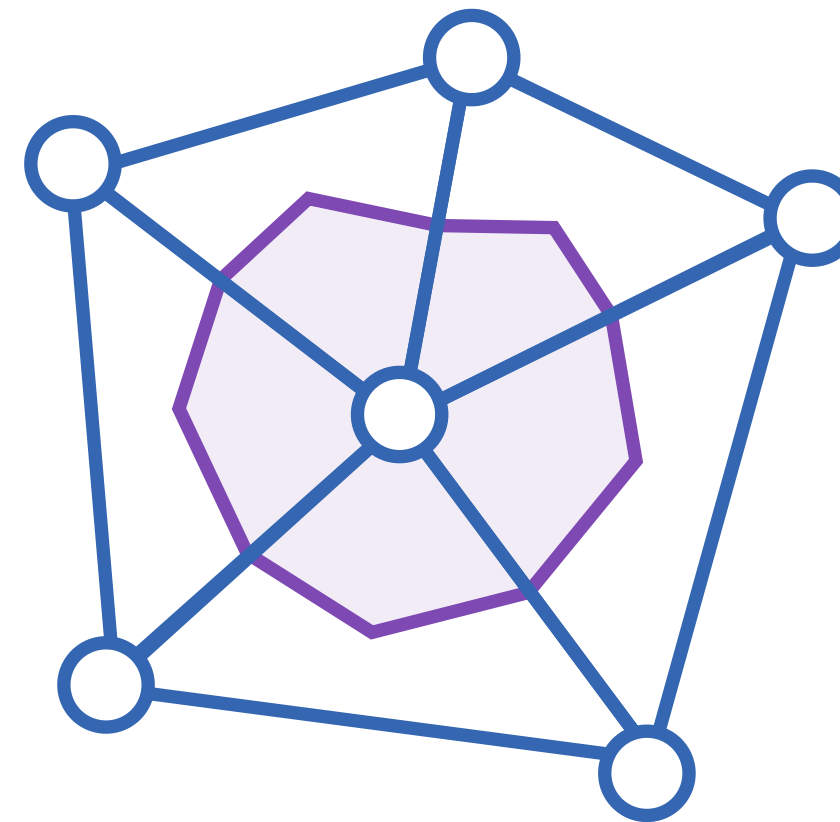
These two yield the same order of accuracy

Most common setup

- In practice, we often use circumcentric “cotan” edge weights and the barycentric point area.

$$w_e = \frac{1}{2}(\cot \alpha + \cot \beta)$$

$$m_i = \frac{1}{3} \sum_{f \succ i} \text{Area}_f$$



- (It is equivalent to the “piecewise linear Finite Element Method with lumped mass” which has good numerical guarantees.)

L2 inner product

Relation to norm-squared
of a differential form

L2 inner product

- The L2 inner product between two k -forms:

$$\langle\langle \alpha, \beta \rangle\rangle = \int_M \langle \alpha, \beta \rangle dA \quad \alpha, \beta \in \Omega^k(M)$$

- Discretely

$$\langle\langle \alpha, \beta \rangle\rangle = \alpha^T [\star_k] \beta = \sum_{i=1}^{M_k} w_{\sigma_k} \alpha_{\sigma_k} \beta_{\sigma_k}$$

- We shall see roughly $\langle\langle \alpha, \beta \rangle\rangle \approx \langle\langle \alpha, \beta \rangle\rangle$

L2 inner product

- For 0-forms and 2-forms on triangle mesh

$$\phi \in C^0 \quad \|\phi\|^2 = \sum_{i=1}^{|\mathcal{P}|} m_i |\phi_i|^2 \approx \int_M |\phi|^2 dA$$

$$\omega \in C^2 \quad \|\omega\|^2 = \sum_{f \in \mathcal{F}} \frac{1}{a_f} |\omega_f|^2 = \sum_{f \in \mathcal{F}} a_f \left| \frac{\omega_f}{a_f} \right|^2 \approx \int_M |\omega|^2 dA$$

L2 inner product

- For 1-forms

$$\|\alpha\|^2 = \sum_{e \in \mathcal{E}} \frac{|\text{edge width}|}{|\text{edge length}|} |\alpha_e|^2 \stackrel{?}{\approx} \int_M |\alpha|^2 dA$$

- ▶ Let $\hat{e}$ be normalized edge vector, $\hat{e}^\perp$ its 90° rotation

- ▶ Pointwise $|\alpha|^2 = \alpha[\hat{e}]^2 + \alpha[\hat{e}^\perp]^2$

- ▶ Now, $\int_M |\alpha|^2 dA \approx \sum_{e \in \mathcal{E}} \frac{\text{width}_e \text{length}_e}{2} |\alpha|^2$

$$= \sum_{e \in \mathcal{E}} \frac{\text{width}_e \text{length}_e}{2} \left(\alpha[\hat{e}]^2 + \alpha[\hat{e}^\perp]^2 \right)$$

$$\approx \sum_{e \in \mathcal{E}} \frac{\text{width}_e \text{length}_e}{2} \left(2 \cdot \alpha[\hat{e}]^2 \right) = \sum_{e \in \mathcal{E}} \frac{\text{width}_e}{\text{length}_e} \alpha[e]^2$$

(assuming
edge orientation
are sufficiently
random)

More general Hodge stars

- Consider the cochain complex $C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \xrightarrow{d_2} \dots$
- Its dual complex $C_0 \xleftarrow{d_0^\top} C_1 \xleftarrow{d_1^\top} C_2 \xleftarrow{d_2^\top} \dots$

$$C^{k*} \cong C_k \cong C^{n-k}(M^*)$$

only when dual mesh can be built
(need manifold-ness)

- Define a “norm” for each k -cochain: $\|\alpha\|^2 = \alpha^\top [\star_k] \alpha$
- This defines a symmetric positive-definite matrix called Hodge star

$$\star_k : C^k \rightarrow C_k$$

doesn't need to be
diagonal

Chain complex with with norms

- We obtain a chain complex with stars (norms)

$$\begin{array}{ccccccc} C^0 & \xrightarrow{d_0} & C^1 & \xrightarrow{d_1} & C^2 & \xrightarrow{d_2} & \dots \\ \star_0 \downarrow & & \star_1 \downarrow & & \star_2 \downarrow & & \\ C_0 & \xleftarrow{d_0^\top} & C_1 & \xleftarrow{d_1^\top} & C_2 & \xleftarrow{d_2^\top} & \dots \end{array}$$

- You can build this diagram as long as you have a chain complex with some notion of $\|\cdot\|^2$.

Hodge Laplacian for k -forms

$$\begin{array}{ccccccc}
 \cdots & \rightarrow & C^{k-1} & \xrightarrow{d_{k-1}} & C^k & \xrightarrow{d_k} & C^{k+1} \rightarrow \cdots \\
 & & \star_{k-1} \downarrow & \curvearrowright & \star_k \downarrow & \curvearrowright & \star_{k+1} \downarrow \\
 \cdots & \leftarrow & C_{k-1} & \xleftarrow{d_{k-1}^\top} & C_k & \xleftarrow{d_k^\top} & C_{k+1} \leftarrow \cdots
 \end{array}$$

- Laplacian is the 2nd derivative operator given by

$$\pm \Delta = \star_k^{-1} d_k^\top \star_{k+1} d_k + d_{k-1} \star_{k-1}^{-1} d_{k-1}^\top \star_k$$

- To keep it symmetric and positive (semi)definite, take another $\star_k$

$$\mathbf{L} = d_k^\top \star_{k+1} d_k + \star_k d_{k-1} \star_{k-1}^{-1} d_{k-1}^\top \star_k$$

Example: Heat Equation

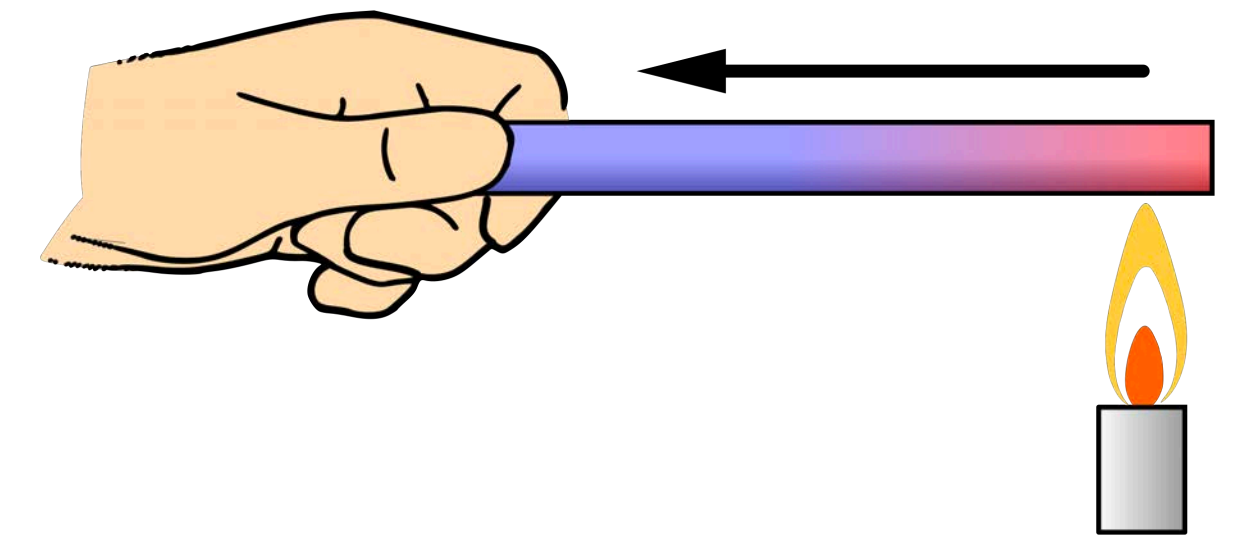
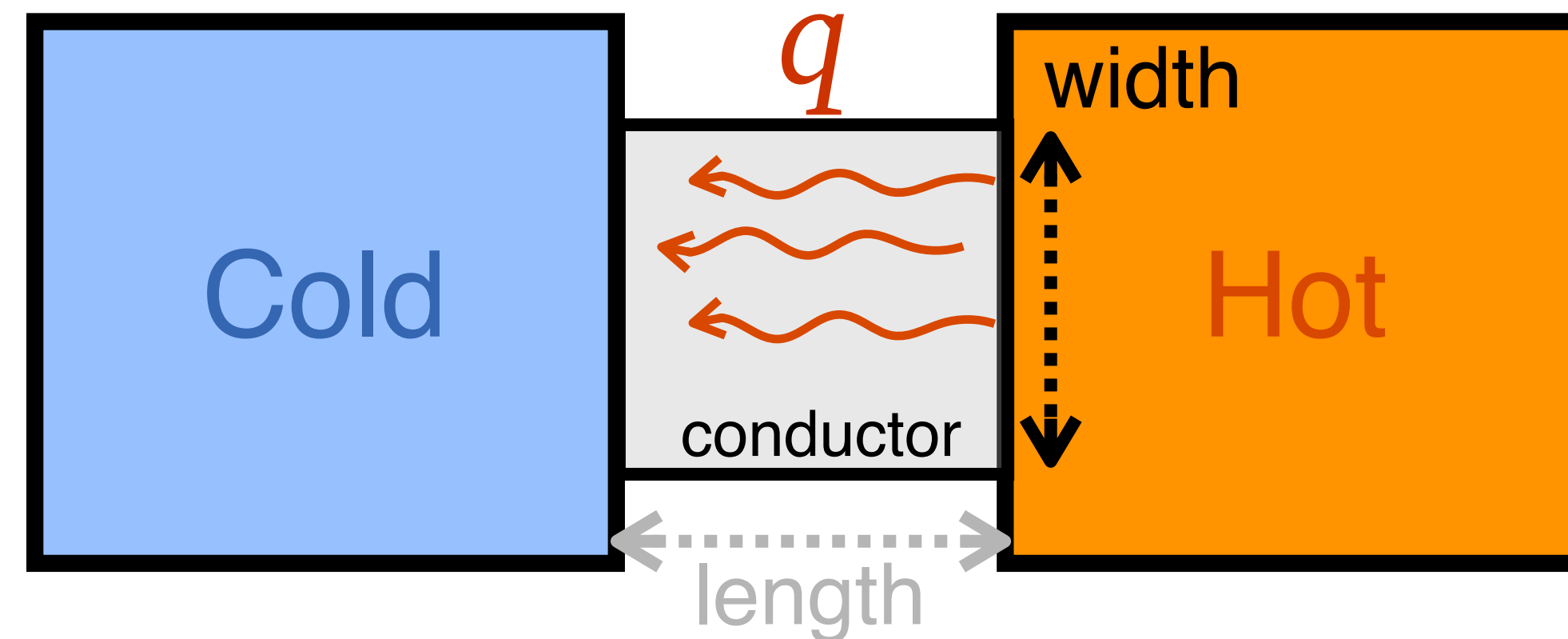
- Laplacian in Physics
- Finite Difference
- Graph Laplacian
- Discrete differential forms
- Discrete Hodge stars
- Example: heat equation

Setup

- Interplay between two physical quantities
 - ▶ Temperature (indication of hot/cold) (K or $^{\circ}C$ or $^{\circ}F$)
 - ▶ Heat (amount of energy) (J or *Calorie*)
- Two physical laws:
 - ▶ Spatial temperature difference induces heat flow
 - ▶ Temperature rises when heat is added to a system

Heat conduction

- Spatial temperature difference induces heat flow



$$\begin{array}{l} \text{heat} \\ \text{flux } q \\ \text{(energy per time)} \end{array} = \underbrace{-\text{Conductivity}}_{\substack{\text{proportional to } \frac{\text{width}}{\text{length}} \\ \text{and a coefficient with unit } \frac{\text{J}}{\text{K s}}}} \cdot \left(\begin{array}{l} \text{temperature} \\ \text{difference} \end{array} \right)$$

Temperature and heat

- Temperature rises when heat is added to a system



$$\underbrace{\left(\begin{array}{c} \text{rise in} \\ \text{temperature} \end{array} \right)}_{\text{(temperature)}} \cdot \text{Heat capacity} = \underbrace{\left(\begin{array}{c} \text{total inflow} \\ \text{of heat} \end{array} \right)}_{\text{(energy)}}$$

$$\underbrace{\left(\begin{array}{c} \text{rate of rise in} \\ \text{temperature} \end{array} \right)}_{\text{(temperature per time)}} \cdot \text{Heat capacity} = \underbrace{\left(\begin{array}{c} \text{total inflow} \\ \text{rate of heat} \end{array} \right)}_{\text{(energy per time)}}$$

proportional to
volume and
coefficient with
unit $\frac{\text{K}}{\text{J}}$

Heat Equation

- Let $M = (M_0, M_1, M_2)$ be a mesh

- Let $u \in C^0(M; \mathbb{R}K)$
temperature-valued 0-form

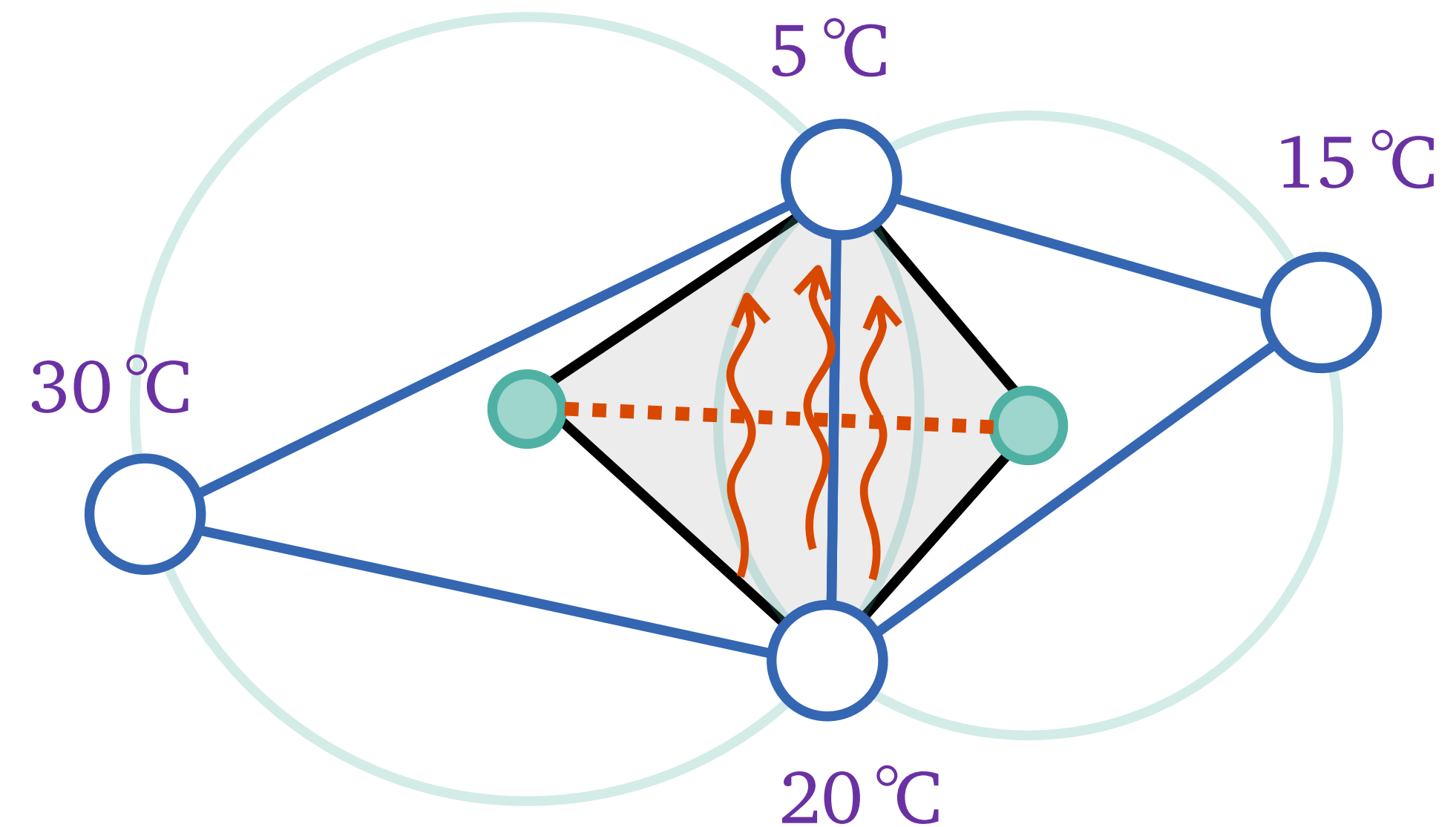
- Temperature difference

$$\mathbf{d}_0 u \in C^1(M; \mathbb{R}K)$$

- Let $w_e \in \mathbb{R}_{>0} \frac{\text{J}}{\text{sK}}$ be the heat conductivity of edge e

- Each edge gets a heat flux $q \in C^1(M^*; \mathbb{R} \text{J/s})$

$$q_e = -w_e (\mathbf{d}_0 u)_e \quad \text{or simply} \quad q = -\star_1 \mathbf{d}_0 u$$

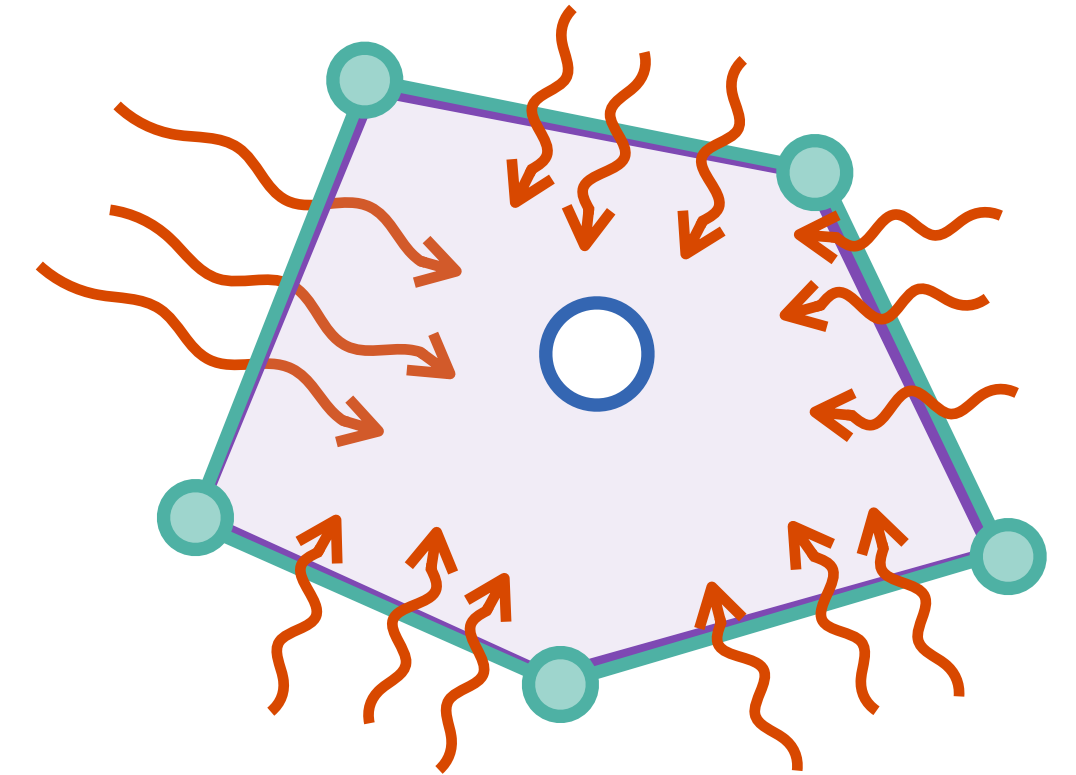


Heat Equation

- Give heat flux $\mathbf{q} \in C^1(M^*; \mathbb{R}^{J/s})$

$$(\mathbf{d}_0^\top \mathbf{q})_i = \sum_{e \succ i} \begin{cases} -q_e & \text{if } i = \text{src}(e) \\ +q_e & \text{if } i = \text{dst}(e) \end{cases}$$

= Total heat flowing
into cell i per unit time



- Let $m_i \in \mathbb{R}_{>0} \frac{\text{K}}{\text{J}}$ be the heat capacity

- Then $m_i \frac{\partial u_i}{\partial t} = (\mathbf{d}_0^\top \mathbf{q})_i$ or simply $\star_0 \frac{\partial u}{\partial t} = \mathbf{d}_0^\top \mathbf{q}$

Heat Equation

- Putting together

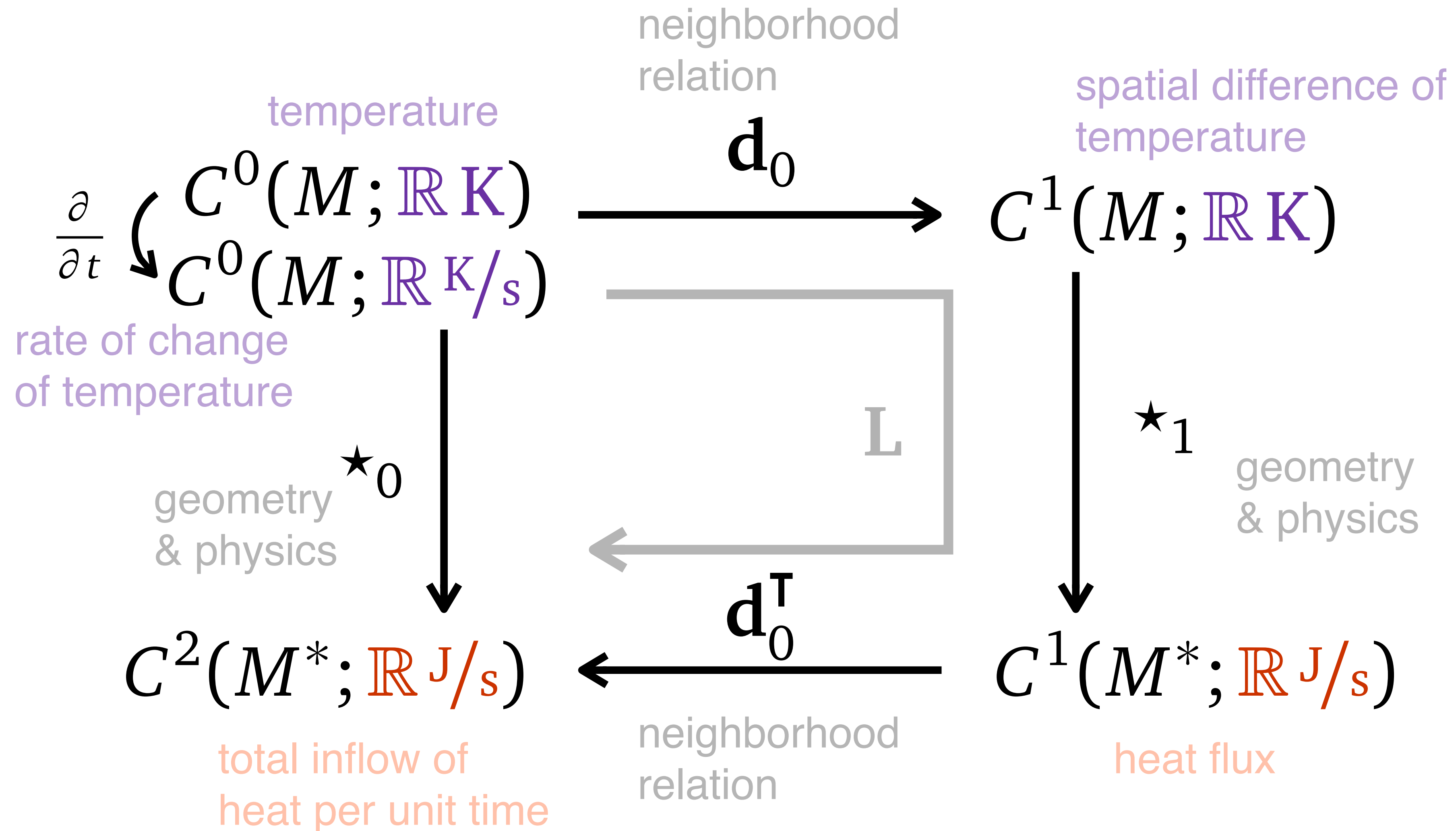
$$\mathbf{q} = -\star_1 \mathbf{d}_0 \mathbf{u} \qquad \star_0 \frac{\partial \mathbf{u}}{\partial t} = \mathbf{d}_0^\top \mathbf{q}$$

- We get

$$\implies \star_0 \frac{\partial \mathbf{u}}{\partial t} = -\mathbf{d}_0^\top \star_1 \mathbf{d}_0 \mathbf{u} \qquad \text{Heat equation}$$

$$\text{i.e.} \quad \star_0 \frac{\partial \mathbf{u}}{\partial t} = -\mathbf{L} \mathbf{u}$$

Heat Equation



$$\star_0 \frac{\partial u}{\partial t} = -\mathbf{d}_0^\top \star_1 \mathbf{d}_0 u$$