CSE 270 (FA25)
Additional Topics

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Geometric Measure Theory

- Geometric measure theory
- More about pullback formula
  - Pullback formulae in 3D
  - Lie derivative
  - Intuition behind Cartan’s formula
  - Advection

Geometric Measure Theory

- Glimpse into geometric measure theory
- In this theory, curves and surfaces are represented by “occupancy” or “density” in the ambient 3D space.
- It offers the official language speaking about “delta functions” using exterior calculus
  - A **Dirac delta function** is formally defined as an impulse

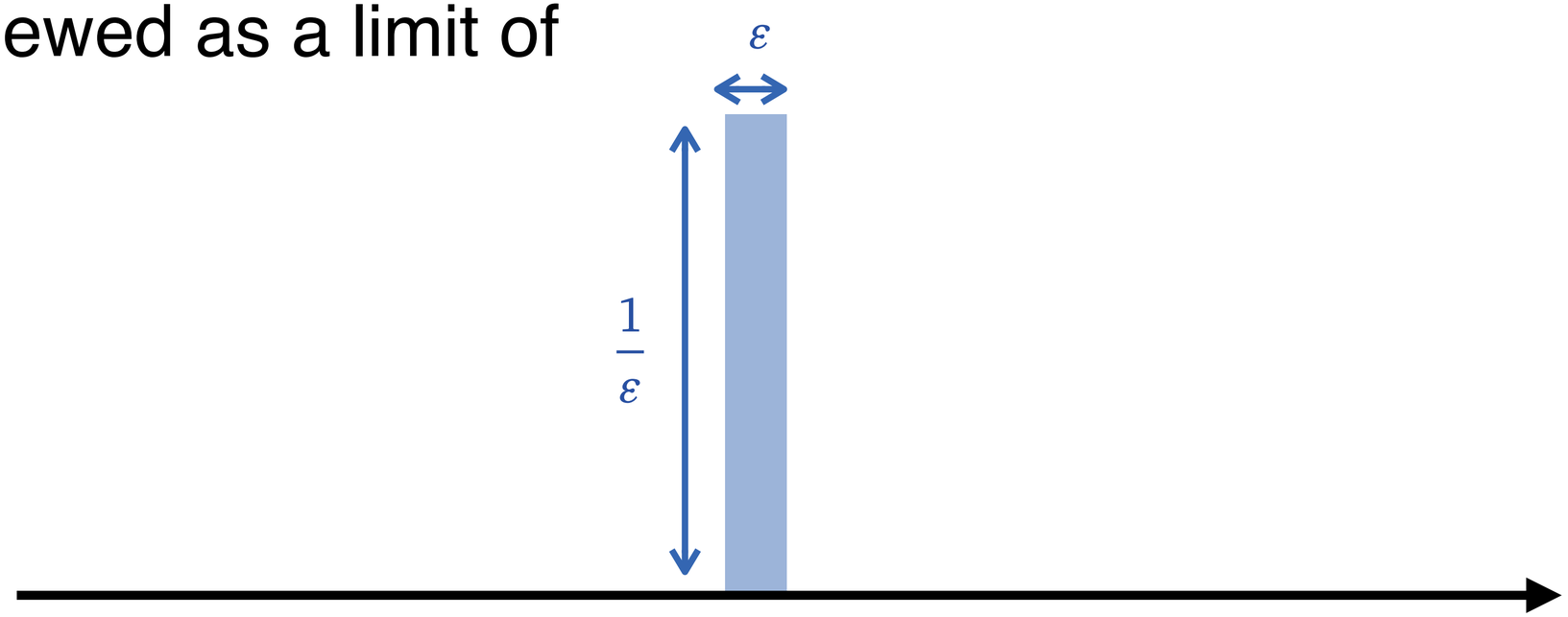
$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases} \qquad \int_a^b \delta(x)\,dx = \begin{cases} 1 & [a,b] \ni 0 \\ 0 & \text{otherwise} \end{cases}$$

Geometric Measure Theory

- A **Dirac delta function** is formally defined as an impulse

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- It is viewed as a limit of



Geometric Measure Theory

- A **Dirac delta function** is formally defined as an impulse

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases} \qquad \int_a^b \delta(x)\,dx = \begin{cases} 1 & [a,b] \ni 0 \\ 0 & \text{otherwise} \end{cases}$$

- Equivalently:

$$\int_{\mathbb{R}} f(x)\delta(x)\,dx = f(0)$$

for any continuous function *f*.

Delta measures

- More generally, for a point *p* on an *n*-dimension manifold

$$\delta_p \in \Omega^n(M)$$

(i.e. it only makes sense to be integrated over regions)

defined by

$$\int_U \delta_p = \begin{cases} 1 & p \in U \\ 0 & \text{otherwise} \end{cases} \qquad \text{or} \qquad \int_M f\,\delta_p = f(p)$$

- (Think of what limiting function this delta function is)

Delta currents (integral currents)

- For each *k*-chain (*k*-dim surface) γ

  define $\delta_\gamma \in \Omega^{n-k}(M)$

  so that for each $\alpha \in \Omega^k(M)$

$$\int_M \alpha \wedge \delta_\gamma = \int_\gamma \alpha$$

- Equivalently, for all (n-k)-surface σ: $\int_\sigma \delta_\gamma = \#(\gamma \cap \sigma)$ (count in total signed intersections)

- (Think of what limiting function this delta function is)

Delta currents (integral currents)

- Check:

$$\delta_A \wedge \delta_B = \delta_{A \cap B}$$

$$d\delta_A = (-1)^{n-\dim(A)+1} \delta_{\partial A}$$

$$\sup_{|\alpha| \leq 1} \int_M \alpha \wedge \delta_A = |A|$$

Total surface area

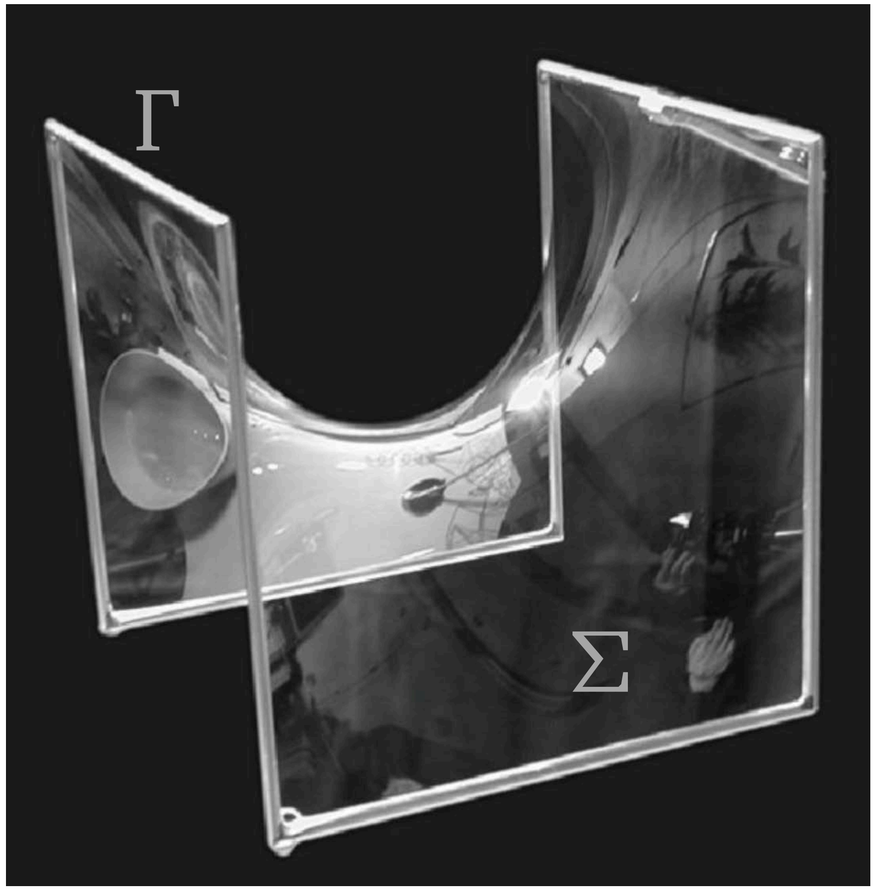
Application

Convexify seemingly non-convex geometric optimization problems

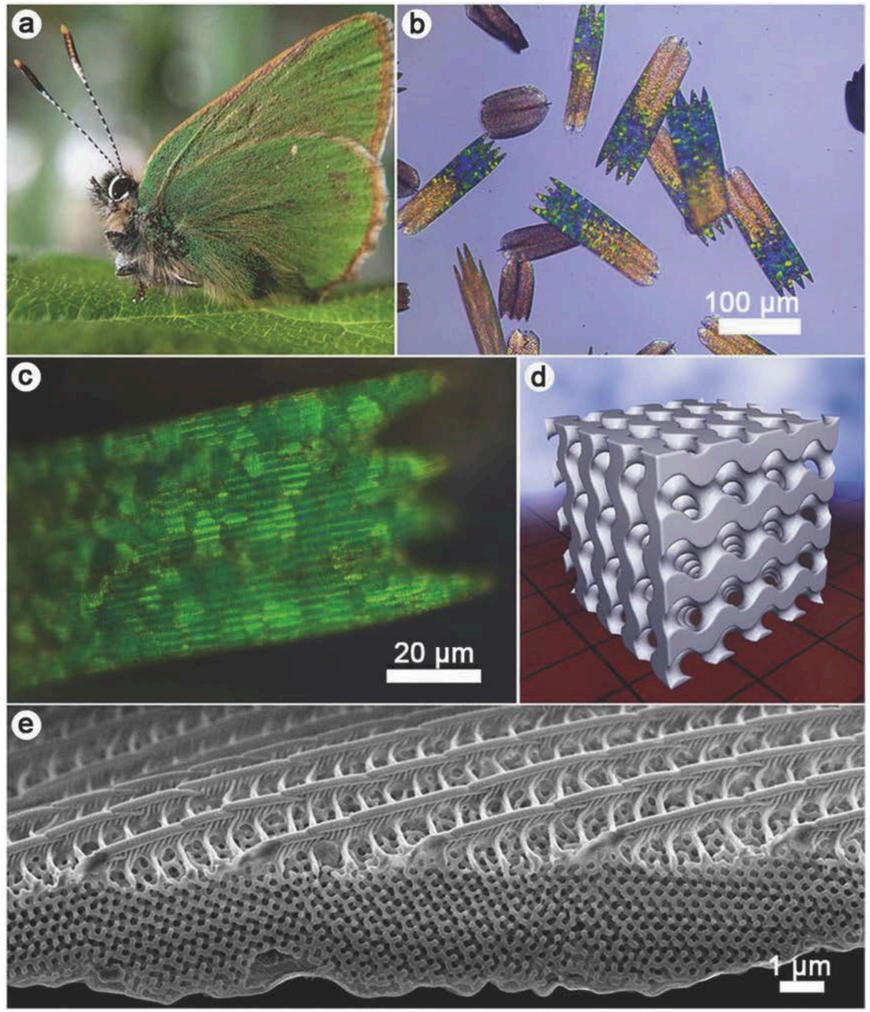
Minimal Surface Problem

Problem. (Lagrange 1760, Plateau 1873)

Extend a given boundary curve Γ *to a surface* Σ *with minimal area.*



Minimal surfaces



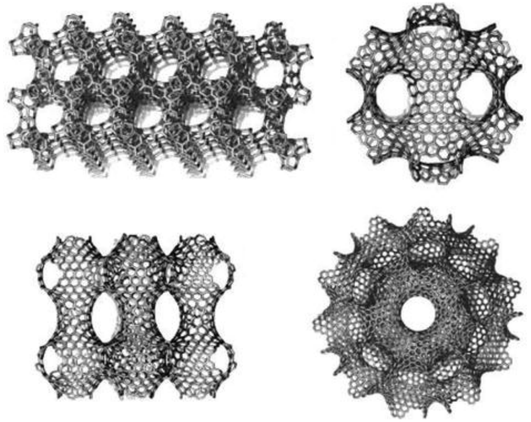
Microbiology



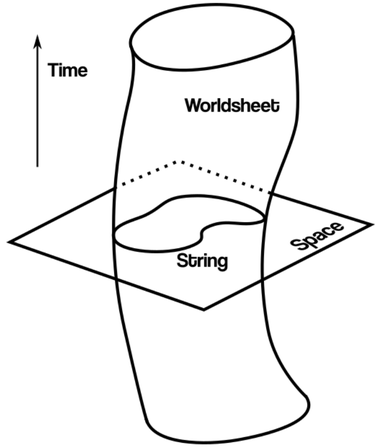
Self-organizing structure



Architecture



Carbon nanostructure



String theory

Standard approach

Traditional method: gradient descent

$$\frac{\partial}{\partial t}(\text{Position}) = -(\text{MeanCurvature})\vec{N}$$

surface tension

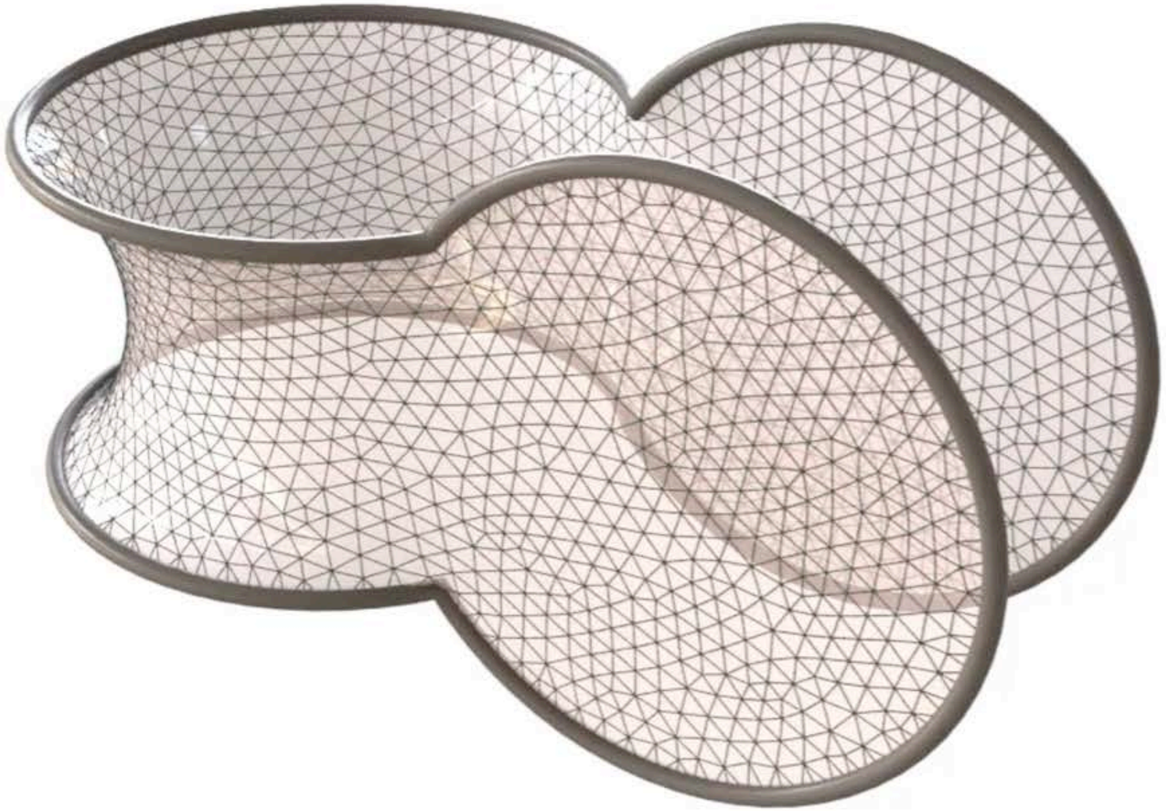


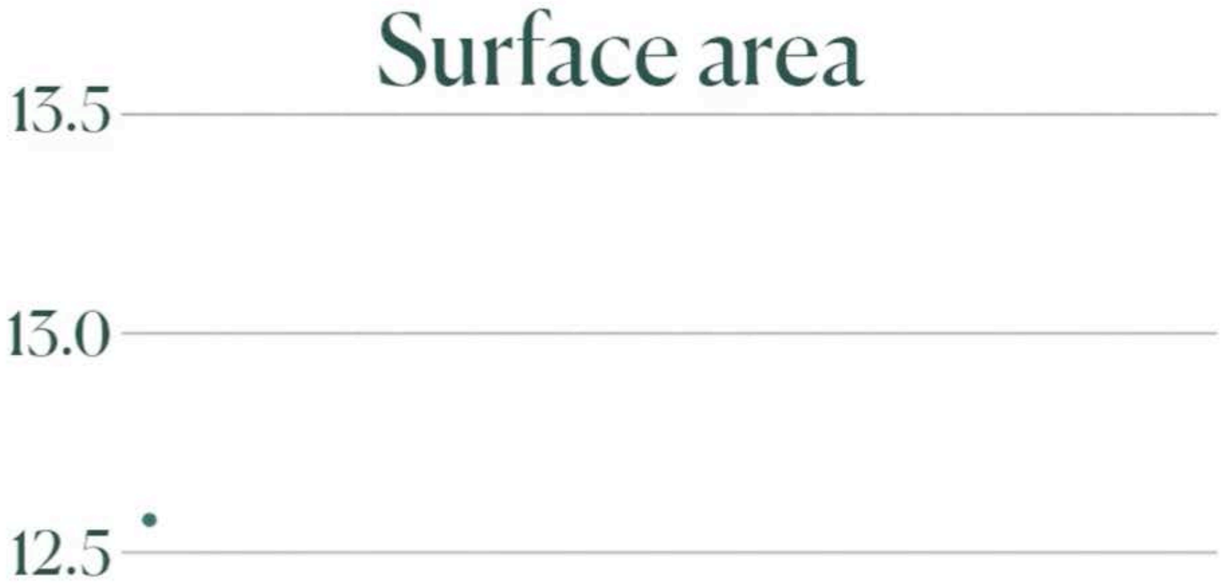
Challenge



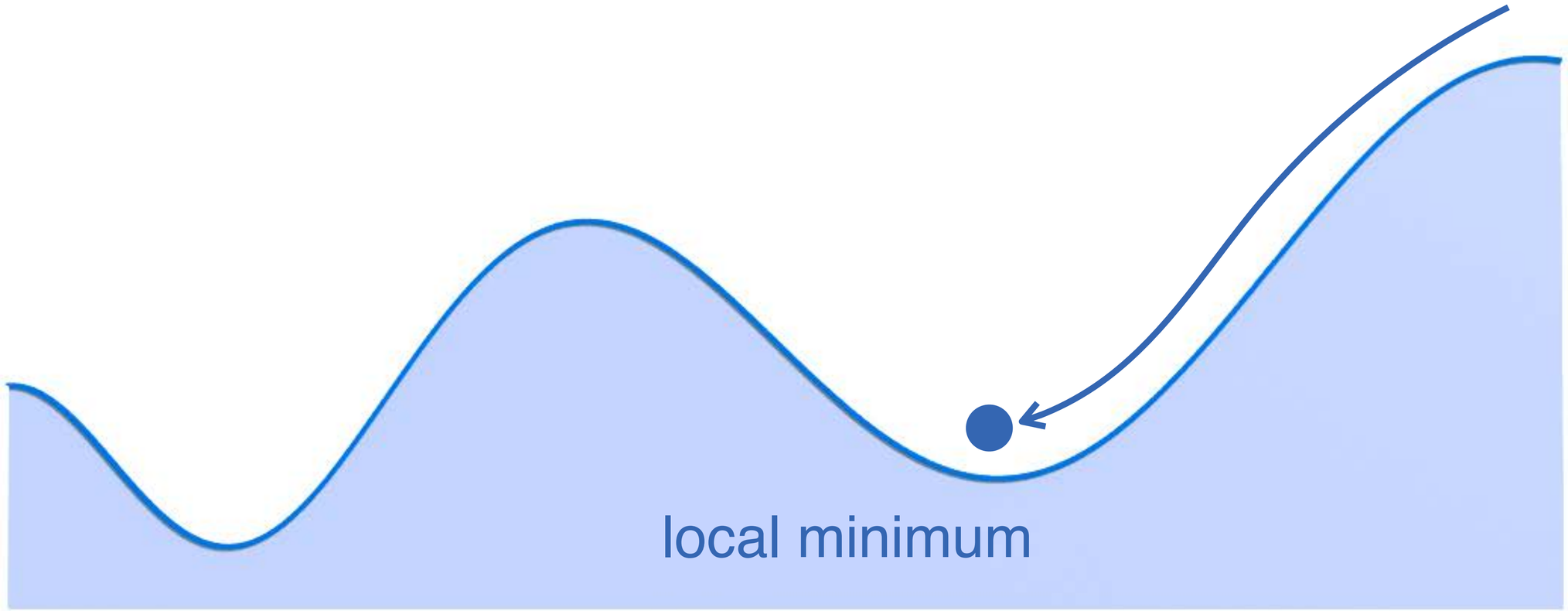
multiple local minima

The area loss function is not convex

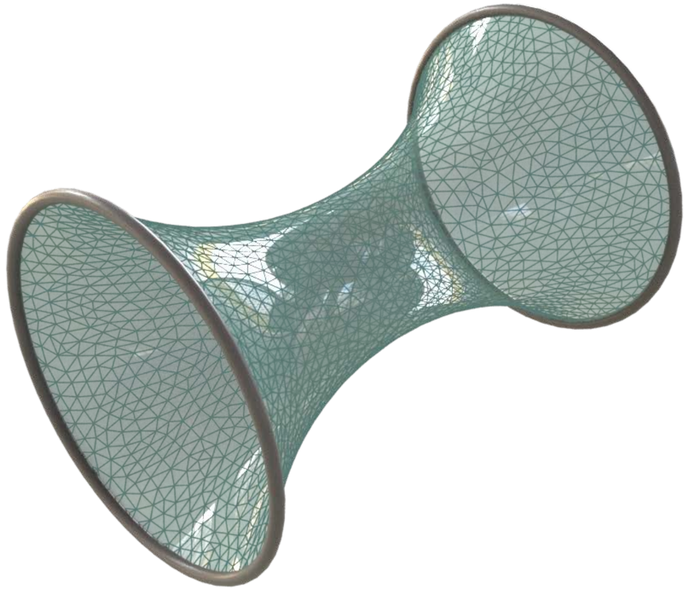


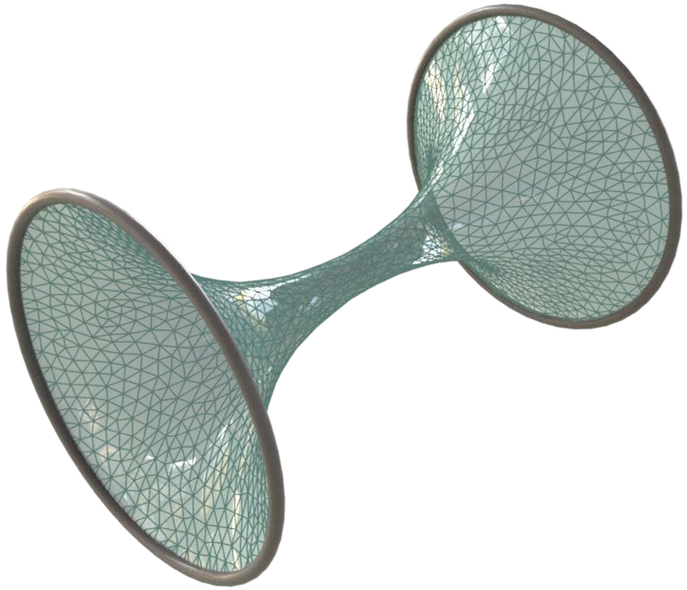


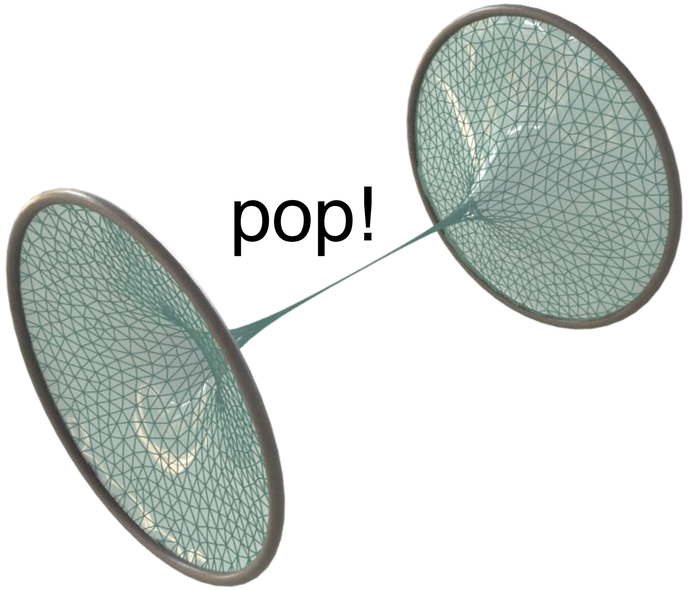
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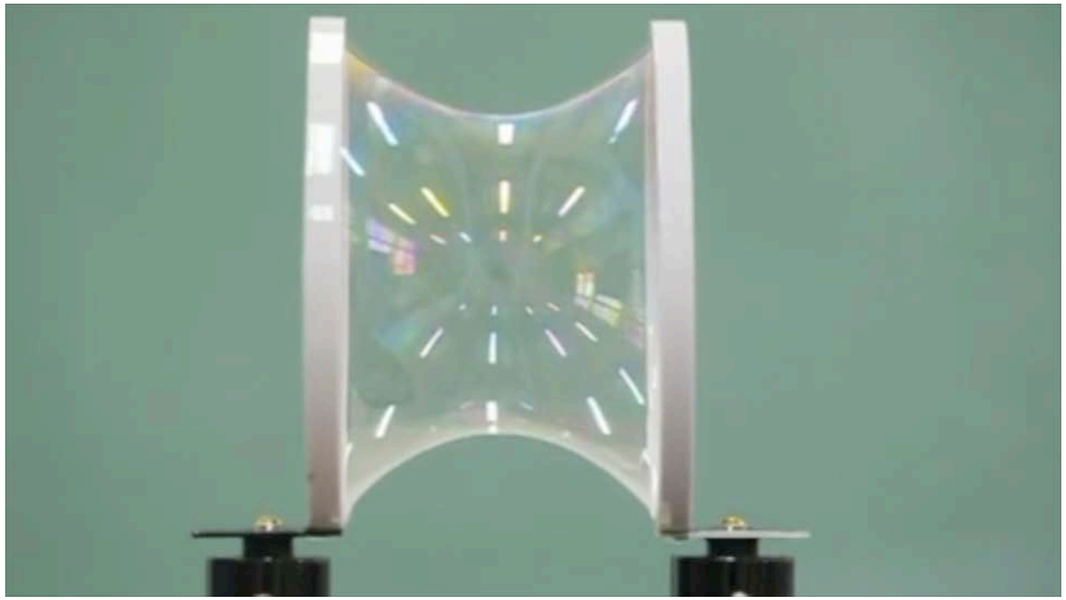


Change of topology is catastrophic

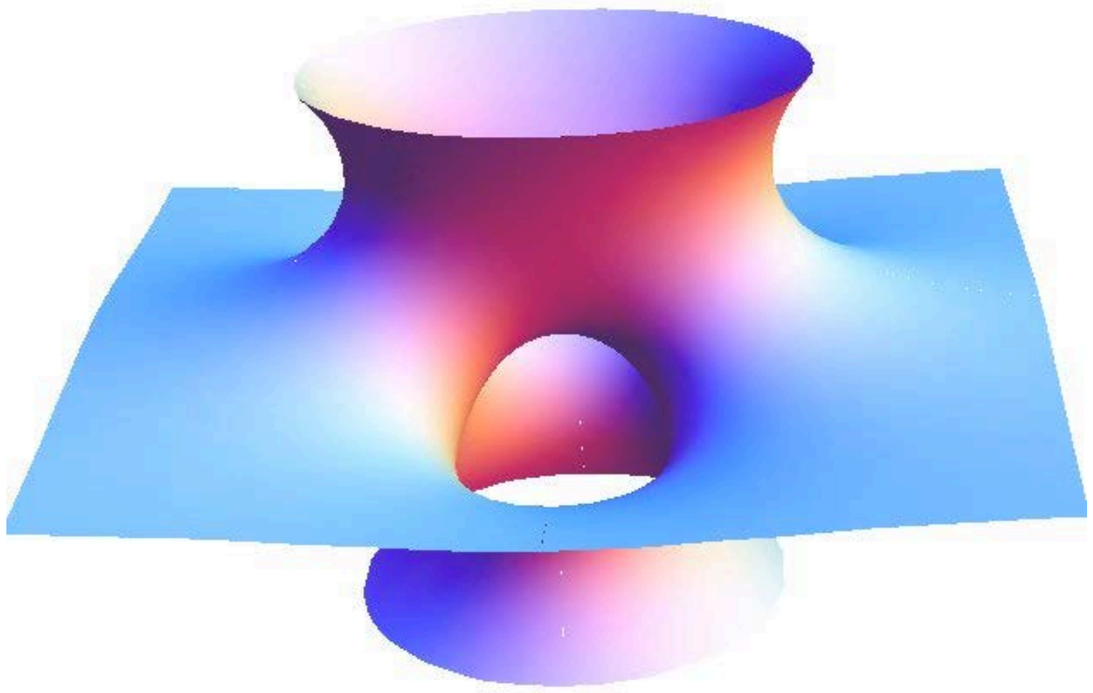


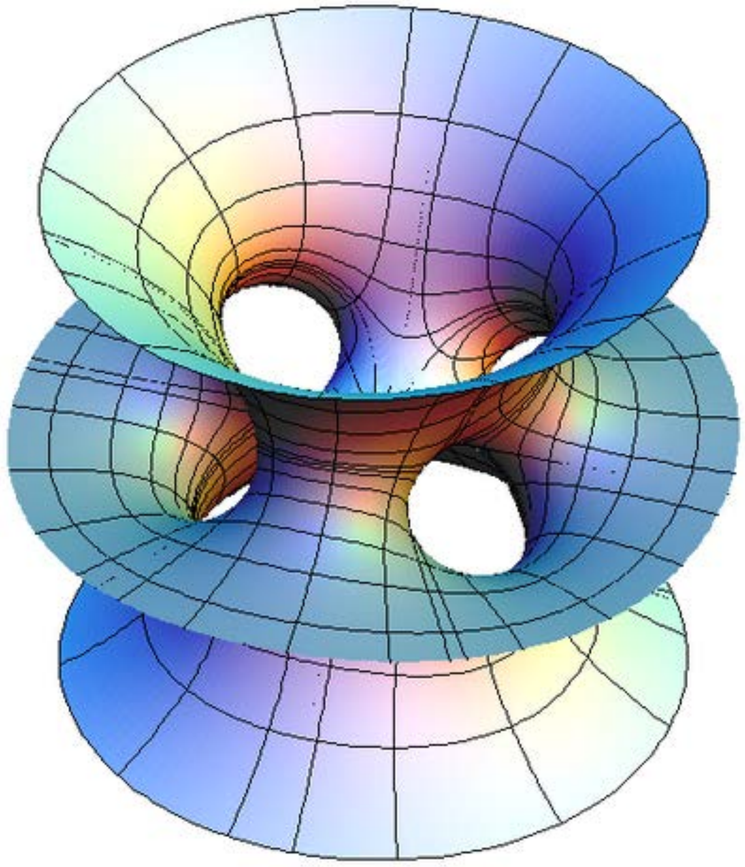






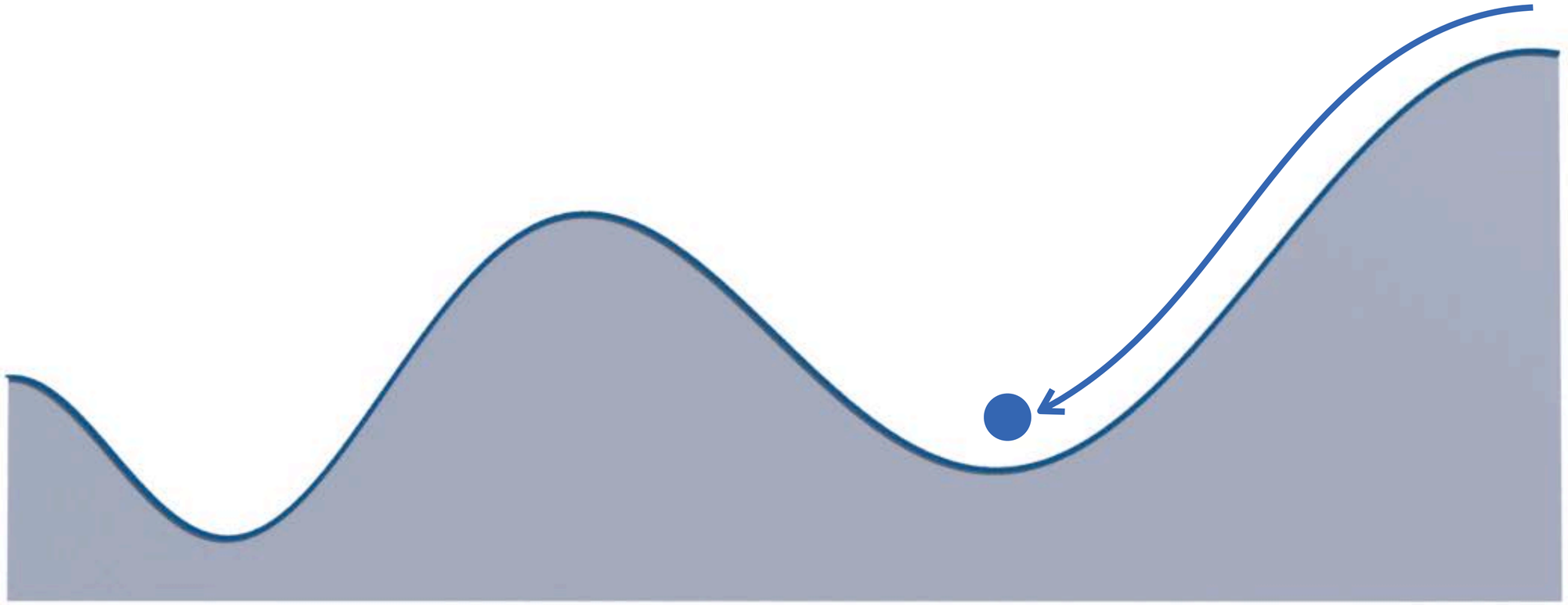
Predicting topology is hard!





Costa’s minimal surfaces 1982

Convex problem in disguise

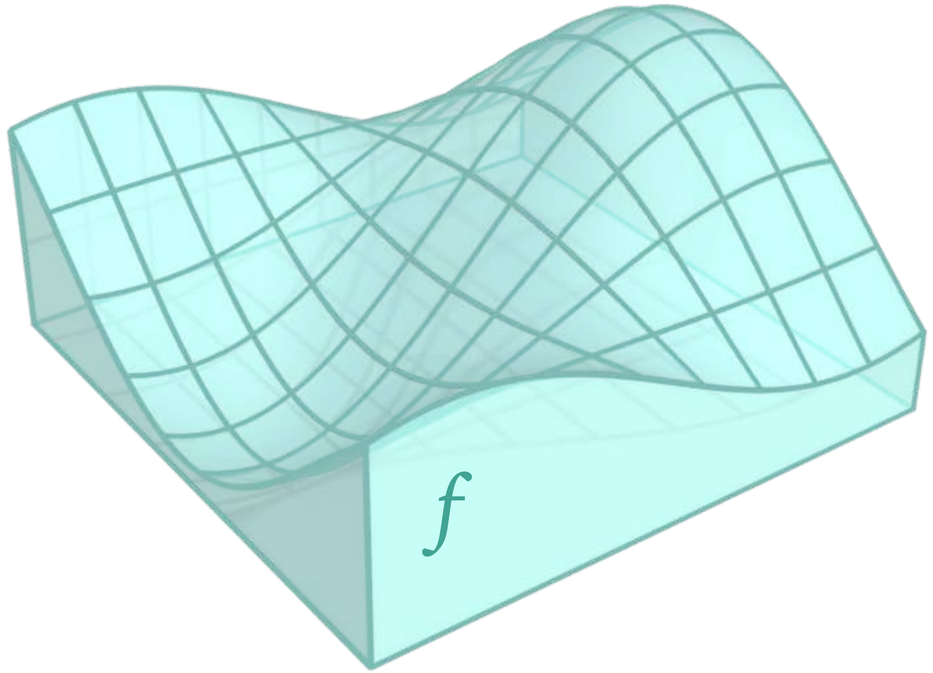


Recall: Measure Theory

- Let M be a manifold.
- Space of measures = dual space of space of continuous functions

$$\mathcal{M}(M) \quad \overset{\text{dual}}{\longleftrightarrow} \quad C(M)$$

$$\int_M f\,\mu$$

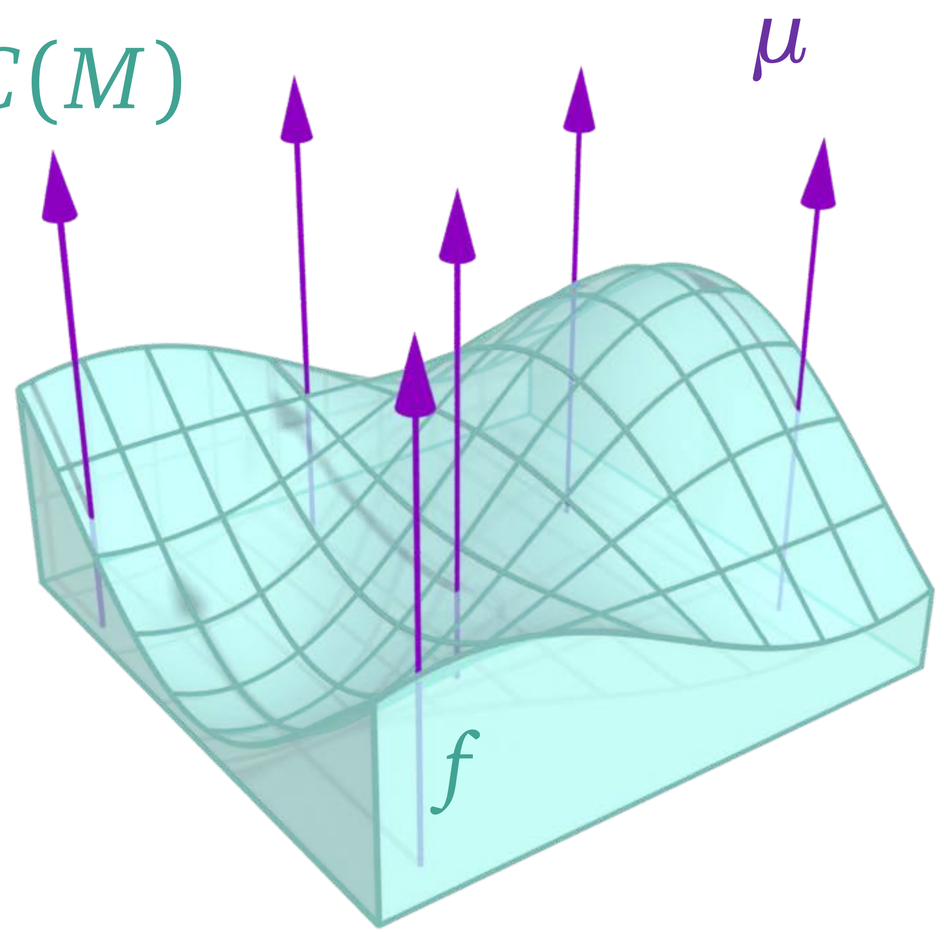


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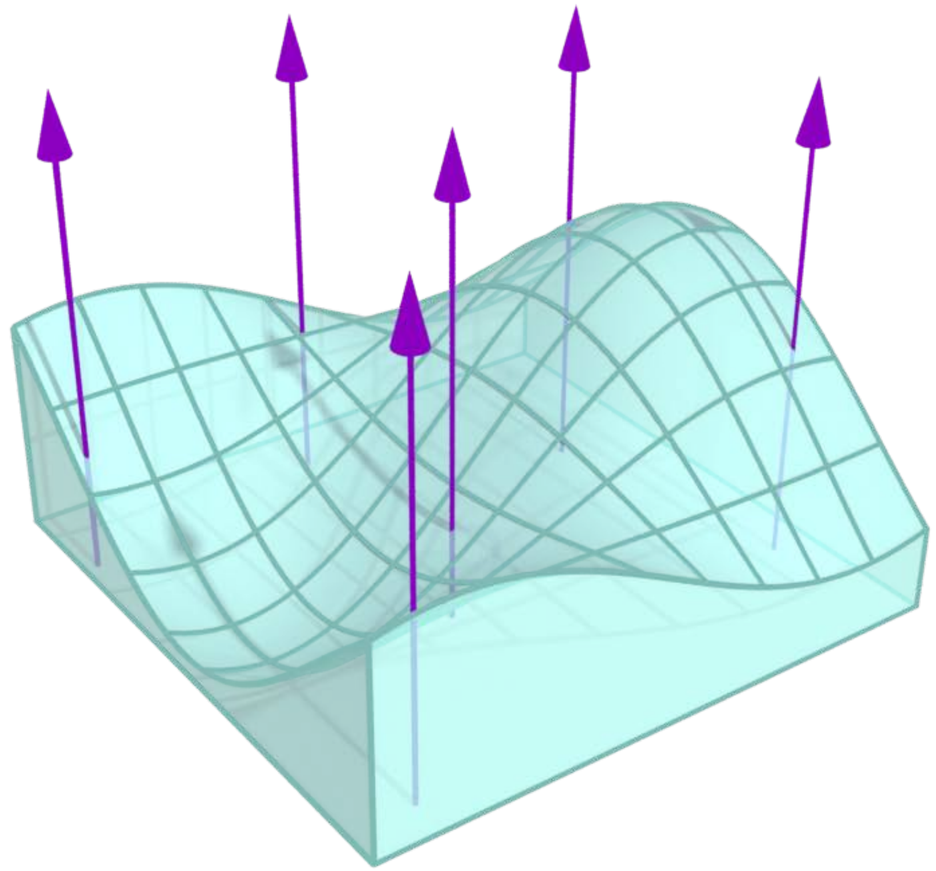
$$\int_M f\,\mu$$

- For each $p \in M$

  the Dirac δ-measure is defined by

$$\delta_p \in \mathcal{M}(M)$$

$$\int_M f\,\delta_p = f(p)$$

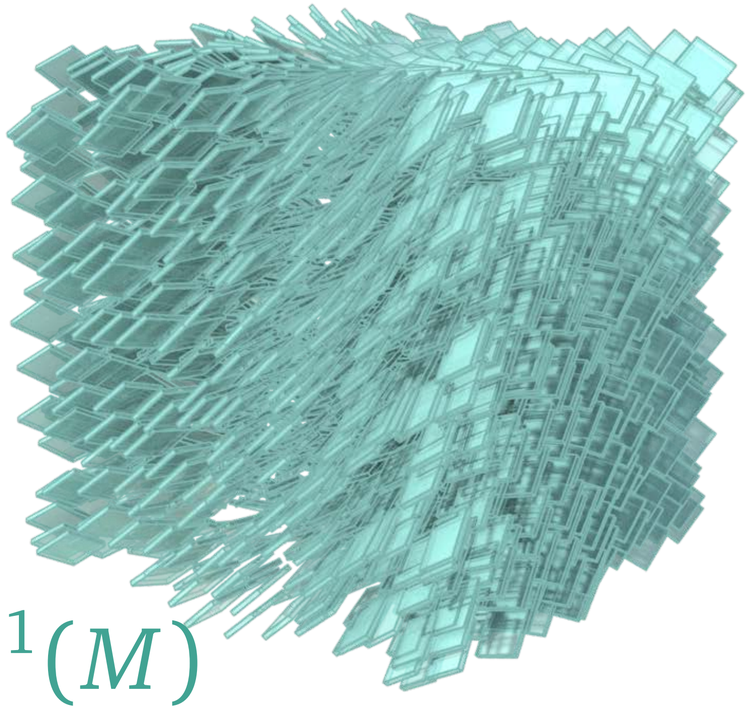


Currents

- Space of k-currents = dual space of space of continuous k-forms

$$\text{“}\Omega^{n-k}(M)\text{”} = \mathscr{C}_k(M) \quad \overset{\text{dual}}{\longleftrightarrow} \quad \Omega^k(M)$$

$$\int_M \alpha \wedge \omega$$



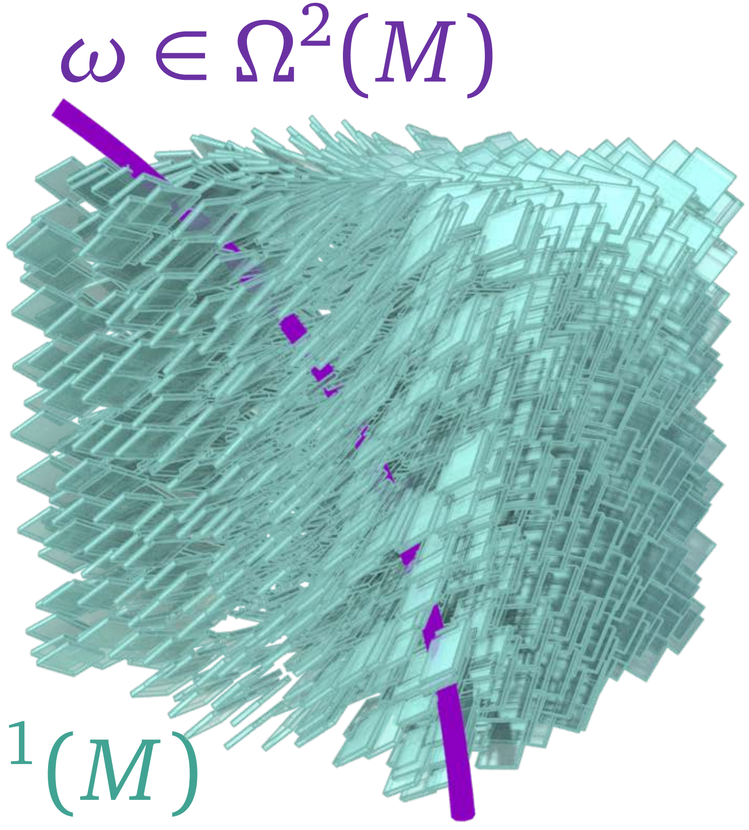
$$\alpha \in \Omega^1(M)$$

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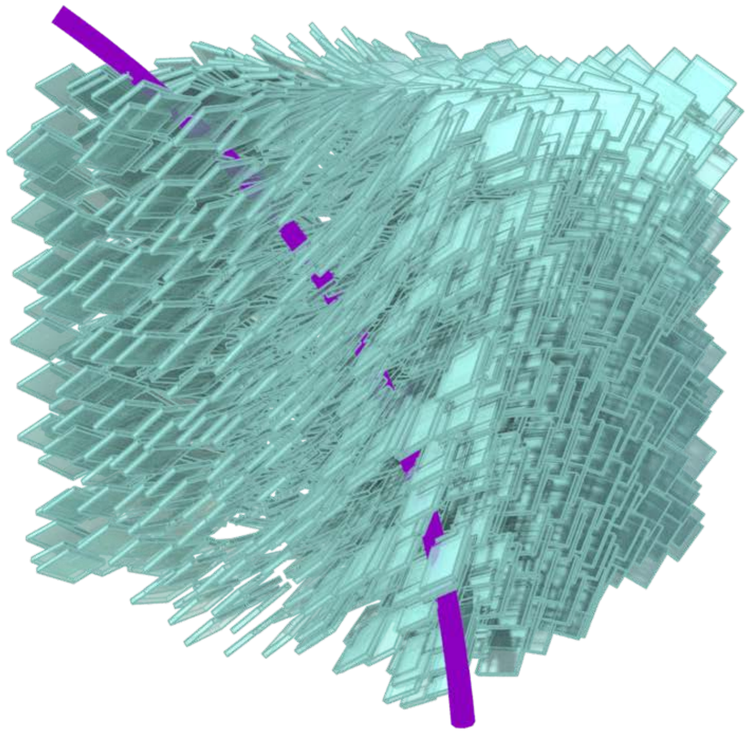
$$\text{“}\Omega^{n-k}(M)\text{”} = \mathscr{C}_k(M) \quad \overset{\text{dual}}{\longleftrightarrow} \quad \Omega^k(M)$$

$$\int_M \alpha \wedge \omega$$

- Each codimension-*k* submanifold Γ defines a Dirac δ-form

$$\delta_\Gamma \in \Omega^{n-k}(M)$$

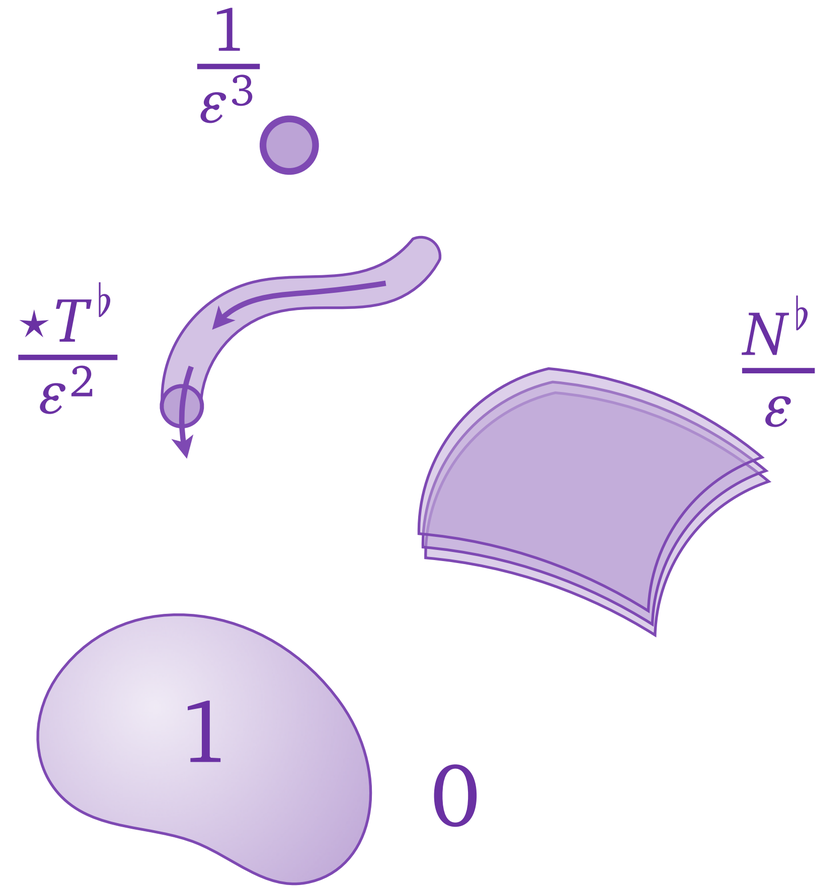
$$\int_M \alpha \wedge \delta_\Gamma = \int_\Gamma \alpha$$



Points, curves, surfaces, volumes

In 3D

	Geometry	Dirac δ-form
• Point	$p \in M$	$\delta_p \in \Omega^3(M)$
• Curve	$\Gamma \subset M$	$\delta_\Gamma \in \Omega^2(M)$
• Surface	$\Sigma \subset M$	$\delta_\Sigma \in \Omega^1(M)$
• Volume	$U \subset M$	$\delta_U \in \Omega^0(M)$



Points, curves, surfaces, volumes

- Surface $\Sigma \subset M$ $\quad\quad$ $\delta_\Sigma \in \Omega^1(M)$



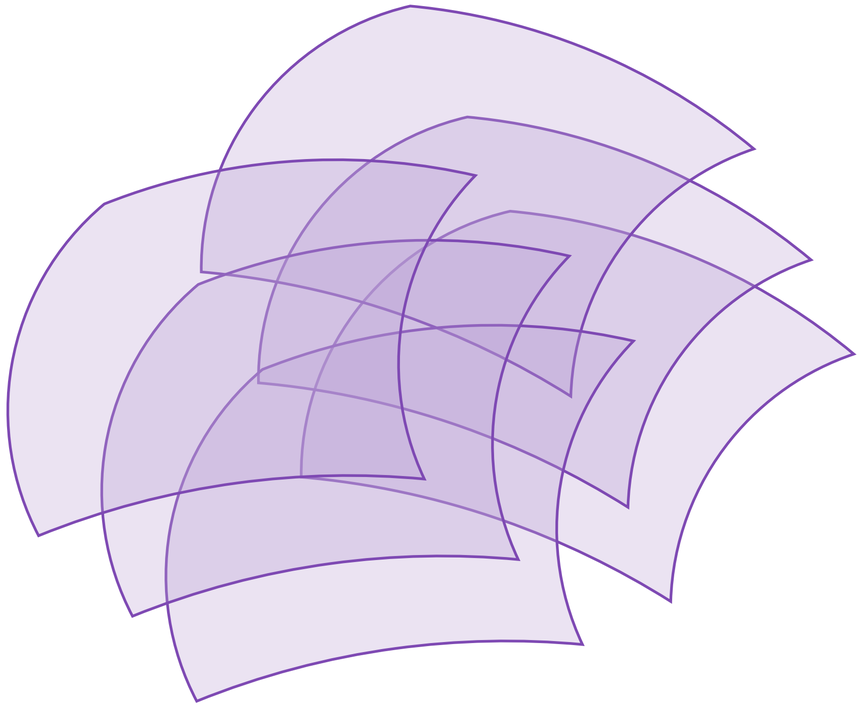
Points, curves, surfaces, volumes

- ~~Surface $\Sigma \in M$~~

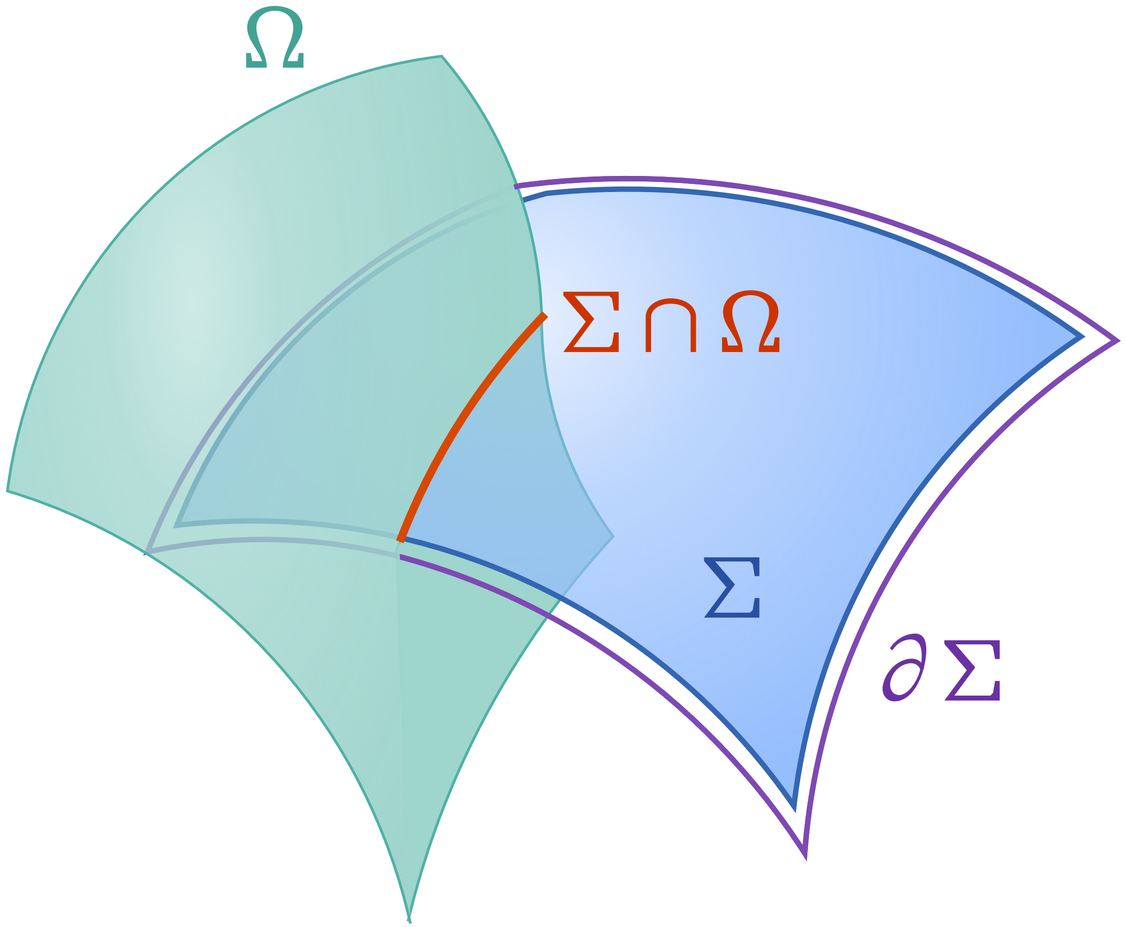
  Superposition of surfaces

Smooth 1-form

$\omega \in \Omega^1(M)$



Representing surfaces by forms



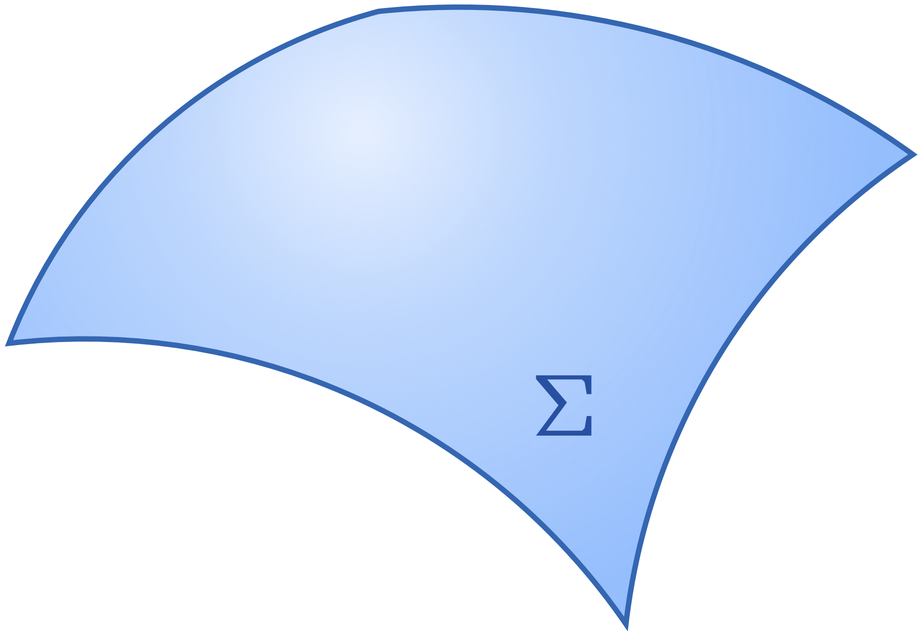
geometric operations | *linear algebra on k-forms*

$$\delta_{\Sigma\cap\Omega} = \delta_{\Sigma} \wedge \delta_{\Omega}$$

$$\delta_{\partial\Sigma} = d\delta_{\Sigma}$$

$$\vdots$$

Representing surfaces by forms

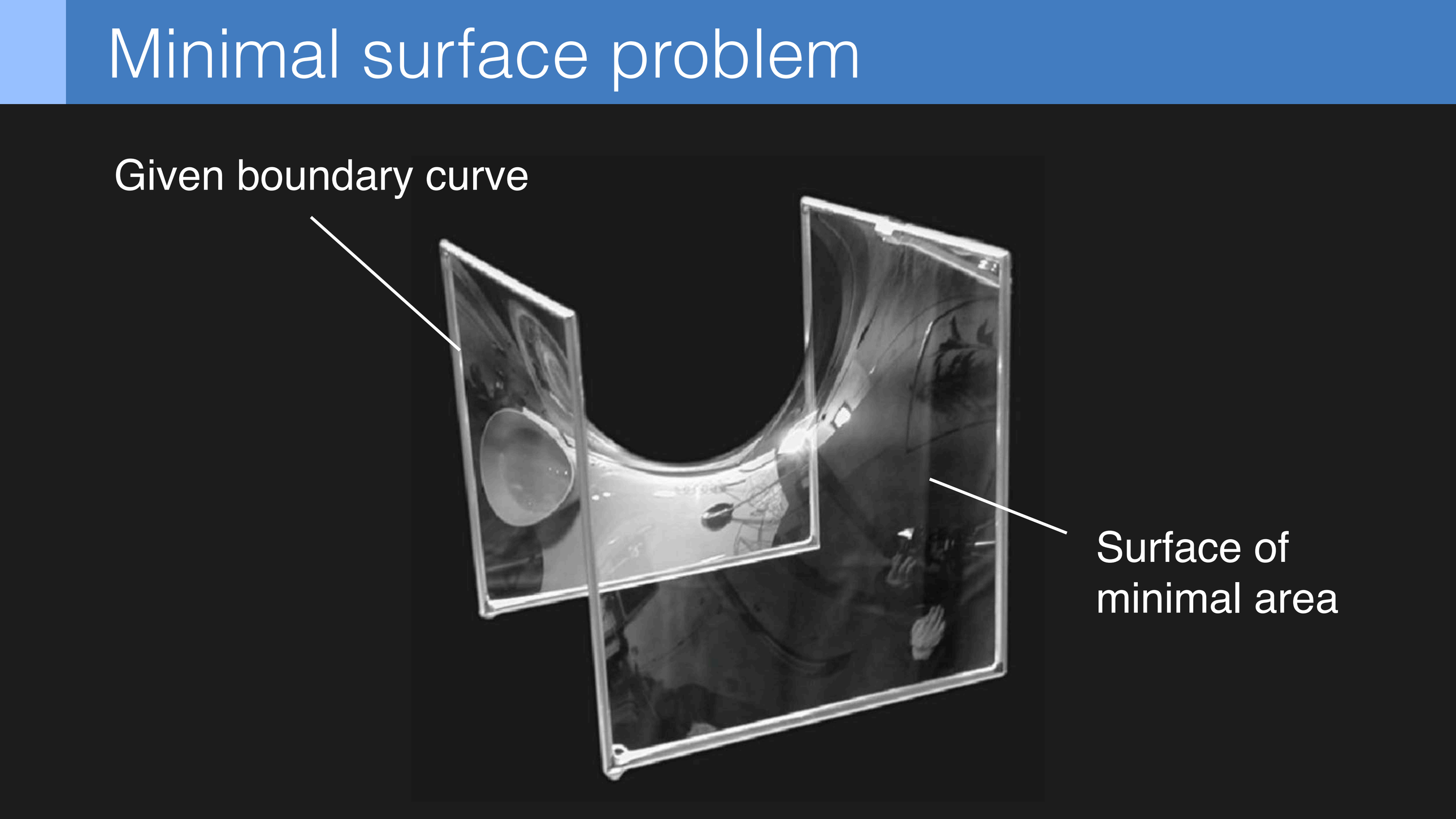


geometric measurement

convex functional

$$\text{Area}(\Sigma) = \int_{\mathbb{R}^3} |\delta_\Sigma|$$

Minimal surface problem



Minimal surface problem

Problem I. (Lagrange 1760, Plateau 1873)

Extend a given boundary curve Γ *to a surface* Σ *with minimal area.*

Non-convex

Problem II. (Federer–Fleming 1960) In terms of currents,

$$\underset{\Sigma:\text{surface}}{\text{minimize}} \int_M |\delta_\Sigma| \quad \text{s.t.} \quad d\delta_\Sigma = \delta_\Gamma$$

Problem III. An equivalent convex problem:

$$\underset{\eta \in \Omega^1(M)}{\text{minimize}} \int_M |\eta| \quad \text{s.t.} \quad d\eta = \delta_\Gamma$$

Convex

Minimal surface problem

$$\underset{\eta \in \Omega^1(M)}{\text{minimize}} \int_M |\eta| \quad \text{s.t.} \quad d\eta = \delta_\Gamma$$

Lemma. Given any $\eta \in \Omega^1(M)$ s.t. $d\eta = \delta_\Gamma$,
there exists a δ-form $\delta_\Sigma \in \Omega^1(M), \partial\Sigma = \Gamma$, so that

$$\int_M |\delta_\Sigma| \leq \int_M |\eta|$$

Minimal surface problem

Problem II. (Minimal Dirac δ-form)

$$\underset{\Sigma:\text{surface}}{\text{minimize}} \int_M |\delta_\Sigma| \quad \text{s.t.} \quad d\delta_\Sigma = \delta_\Gamma$$



Problem III. (Convex relaxation)

$$\underset{\eta \in \Omega^1(M)}{\text{minimize}} \int_M |\eta| \quad \text{s.t.} \quad d\eta = \delta_\Gamma$$

Convex

Minimal surface problem

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Minimal surface problem

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Proof. $d\eta = \delta_\Gamma \implies \eta = d\theta$ for some $\mathbb{S}^1$-valued function θ branched at Γ.

Minimal surface problem

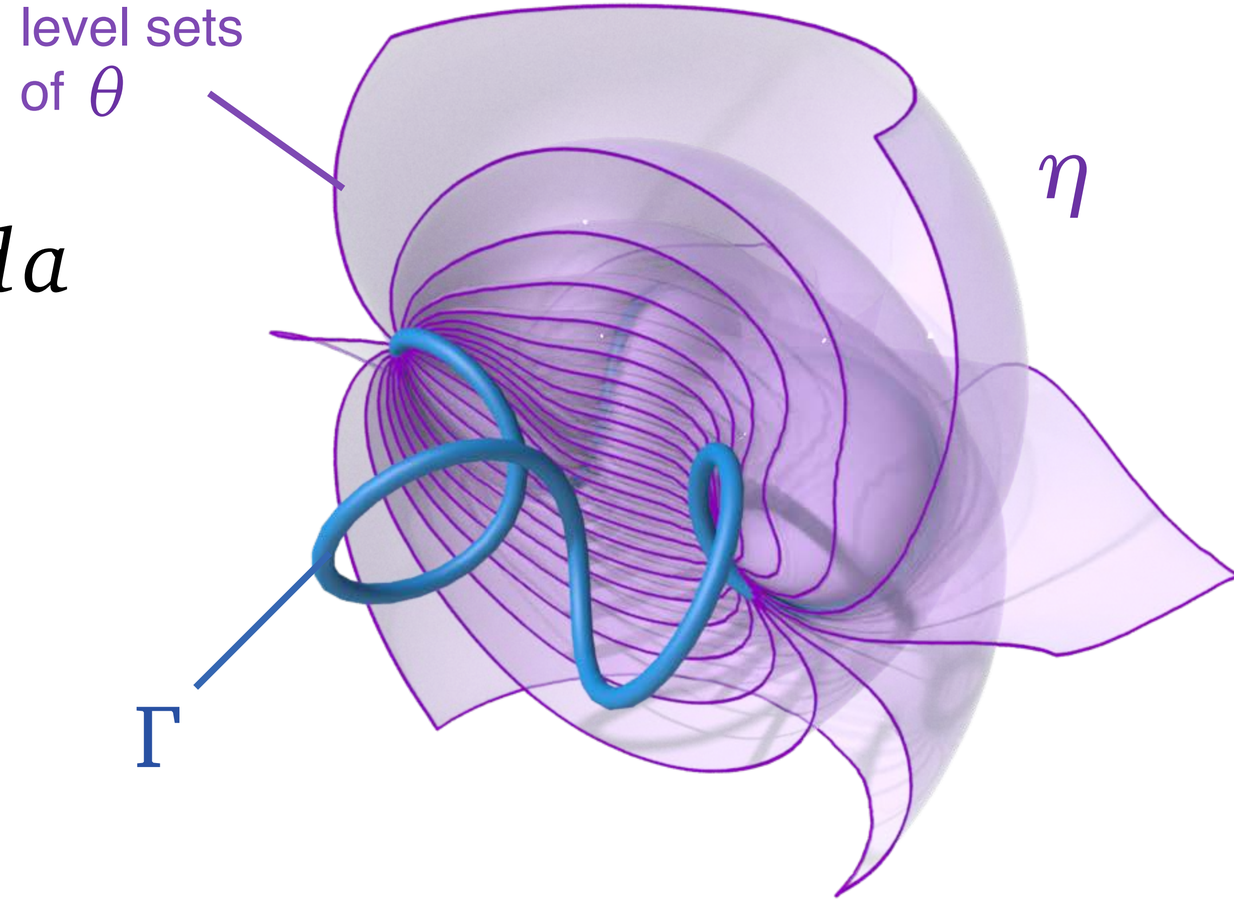
Proof. $d\eta = \delta_\Gamma \implies \eta = d\theta$ for some $\mathbb{S}^1$-valued function θ branched at Γ.

Fubini theorem

$$\int_M |\eta| = \frac{1}{2\pi} \int_{\mathbb{S}^1} \left(\int_M |\delta_{\{\theta = a\}}| \right) da$$

$$\geq \frac{1}{2\pi} \int_{\mathbb{S}^1} da \left(\int_M |\delta_{\{\theta = a^*\}}| \right)$$

$$= \int_M |\delta_{\{\theta = a^*\}}| \qquad \square$$



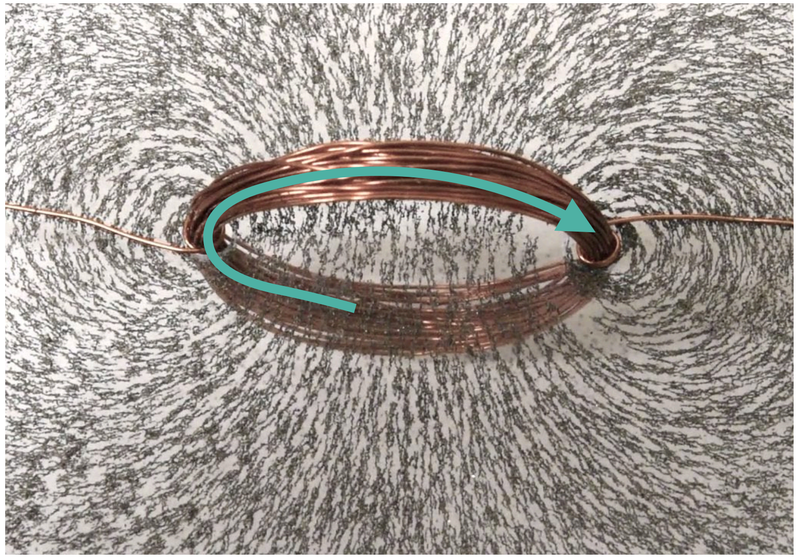
Physical interpretation

$$\underset{\eta \in \Omega^1(M)}{\text{minimize}} \int_M |\eta| \quad \text{s.t.} \quad d\eta = \delta_\Gamma$$

Physical interpretation

Magnetostatics

- Given electric current density $\mathbf{J}$
- It generates a magnetic field $\mathbf{B}$ s.t.
  - Ampère's Law: $\nabla \times \mathbf{B} = \mathbf{J}$
  - Minimal magnetic energy

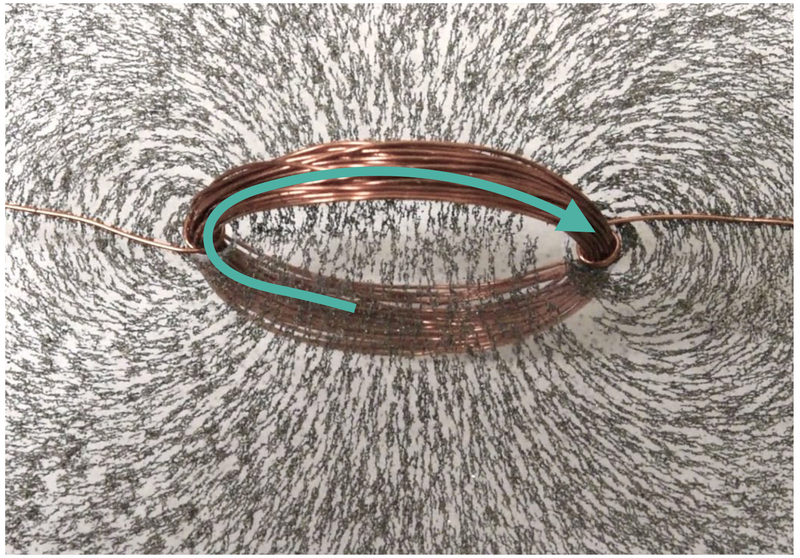


Physical interpretation

Magnetostatics

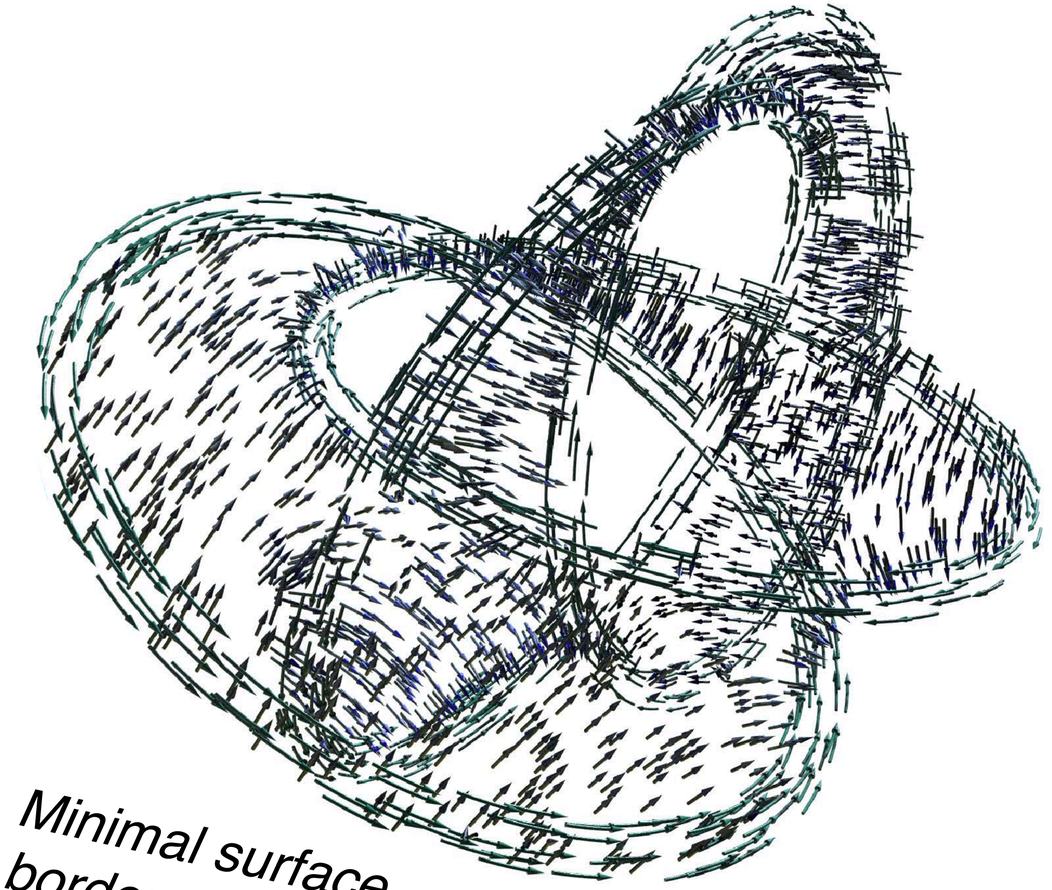
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  - ~~Minimal magnetic energy~~

    Optimal sparsity



Physical interpretation

$$\min_{\mathbf{B}} \int_{\mathbb{R}^3} |\mathbf{B}|$$
$$\text{s.t.} \nabla \times \mathbf{B} = \mathbf{J}$$

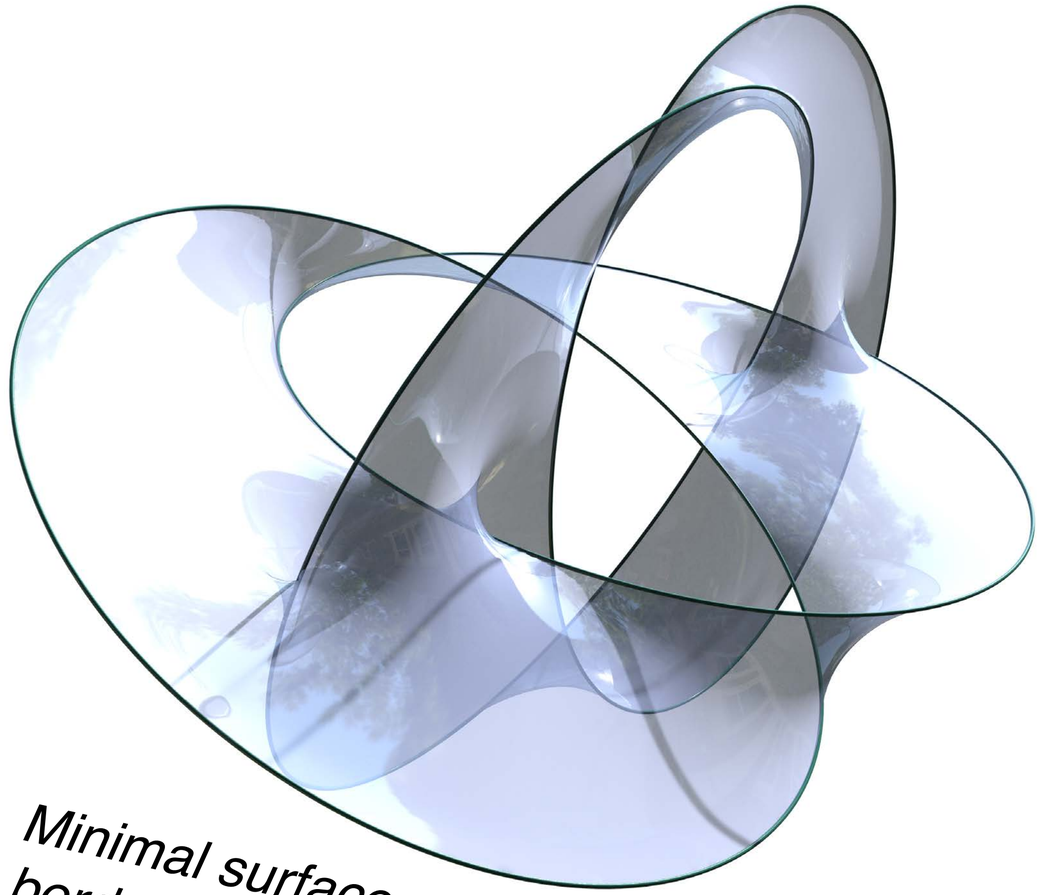


Minimal surface bordered by given boundary curve

Physical interpretation

$$\min_{\mathbf{B}} \int_{\mathbb{R}^3} |\mathbf{B}|$$
$$\text{s.t.} \nabla \times \mathbf{B} = \mathbf{J}$$

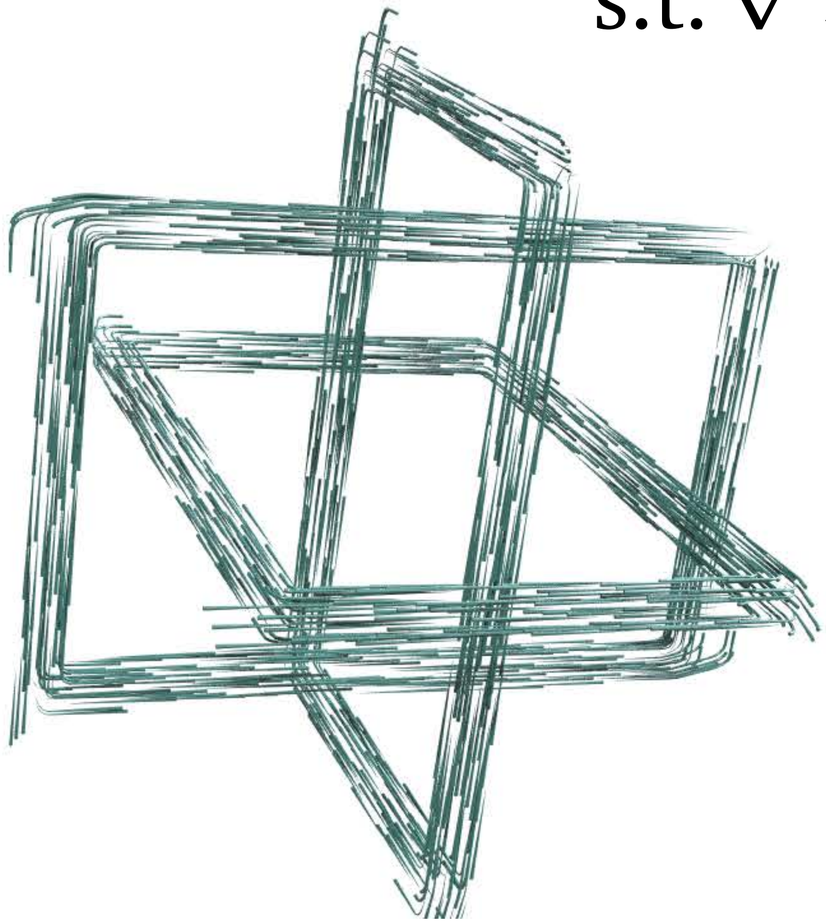
Convex problem



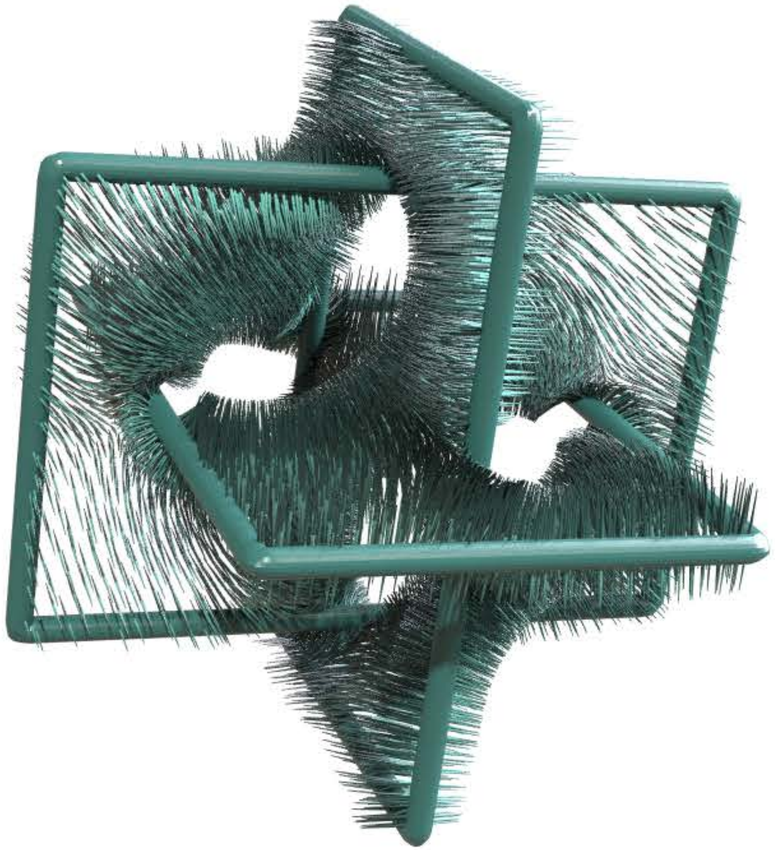
Minimal surface bordered by given boundary curve

Convex minimal surface problem

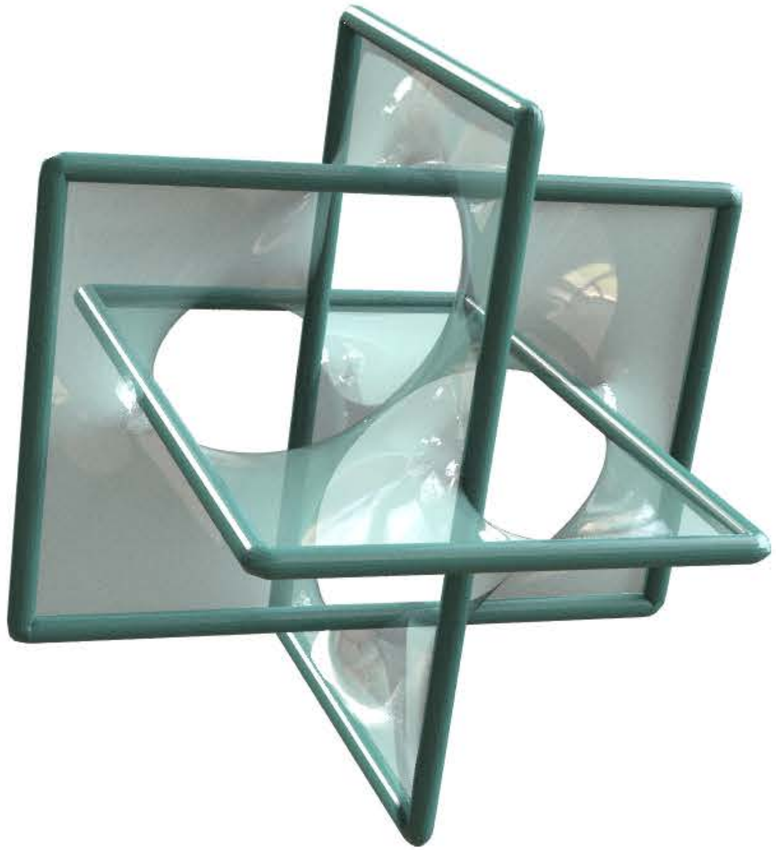
$$\min_{\mathbf{B}} \int_{\mathbb{R}^3} |\mathbf{B}|$$
$$\text{s.t.} \nabla \times \mathbf{B} = \mathbf{J}$$



$\mathbf{J}$



$\mathbf{B}$



Numerical solution

- The **minimal surface problem** is equivalent to the L^1 optimization with linear constraint

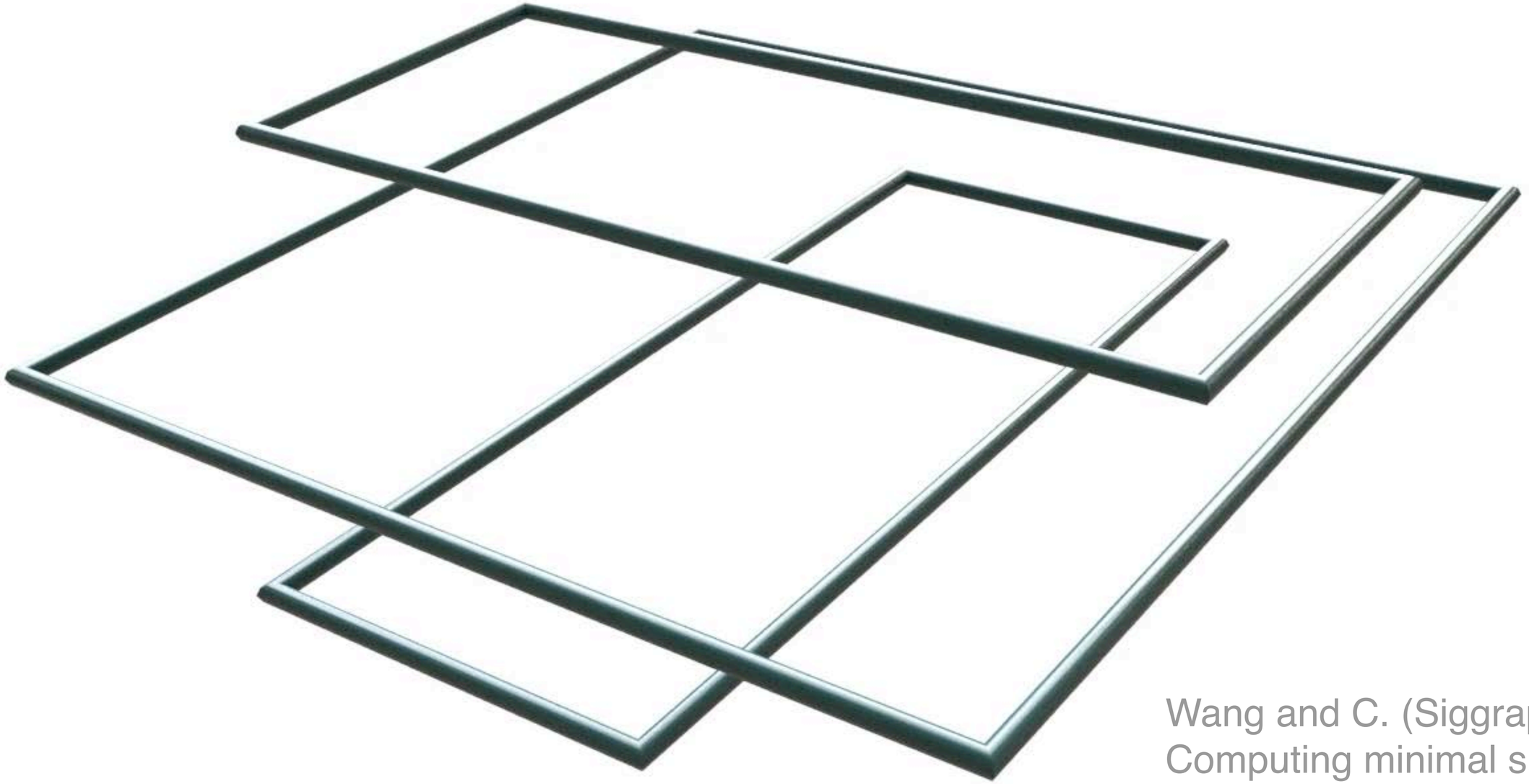
$$\underset{\eta \in \Omega^1(M)}{\text{minimize}} \int_M |\eta| \quad \text{s.t.} \quad d\eta = \delta_\Gamma$$

- Compressed sensing, sparse reconstruction, LASSO, …
- Numerical methods: ADMM, Bregmann iter., proximal grad, etc.
- Simulate

$$\begin{cases} \dot{\eta} = -\frac{\eta}{|\eta|} - \delta p \\ \dot{p} = \Delta^{-1}(d\eta - \delta_\Gamma) \end{cases}$$

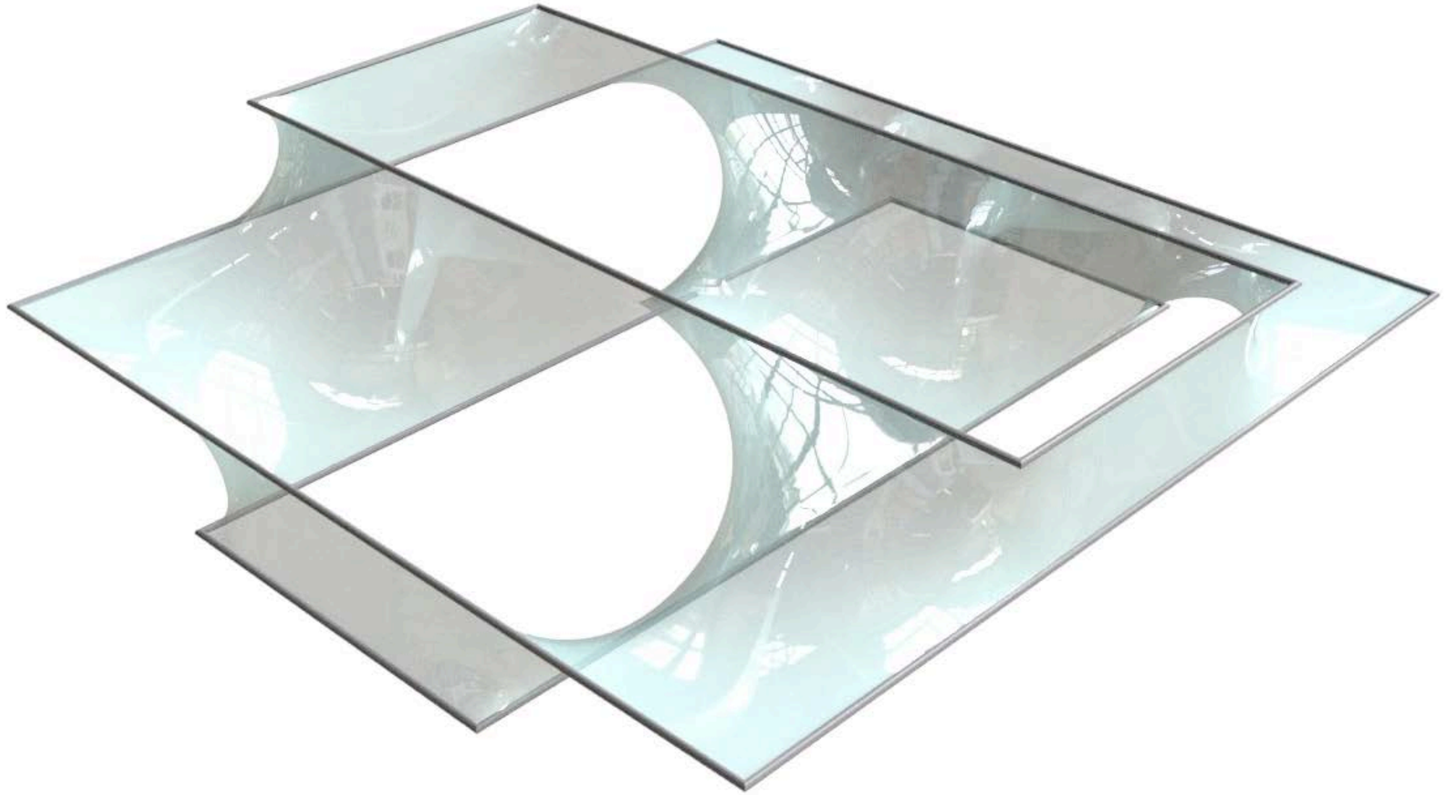
  - Local shrinkage
  - Global Poisson solve (by FFT)

Convex minimal surface problem



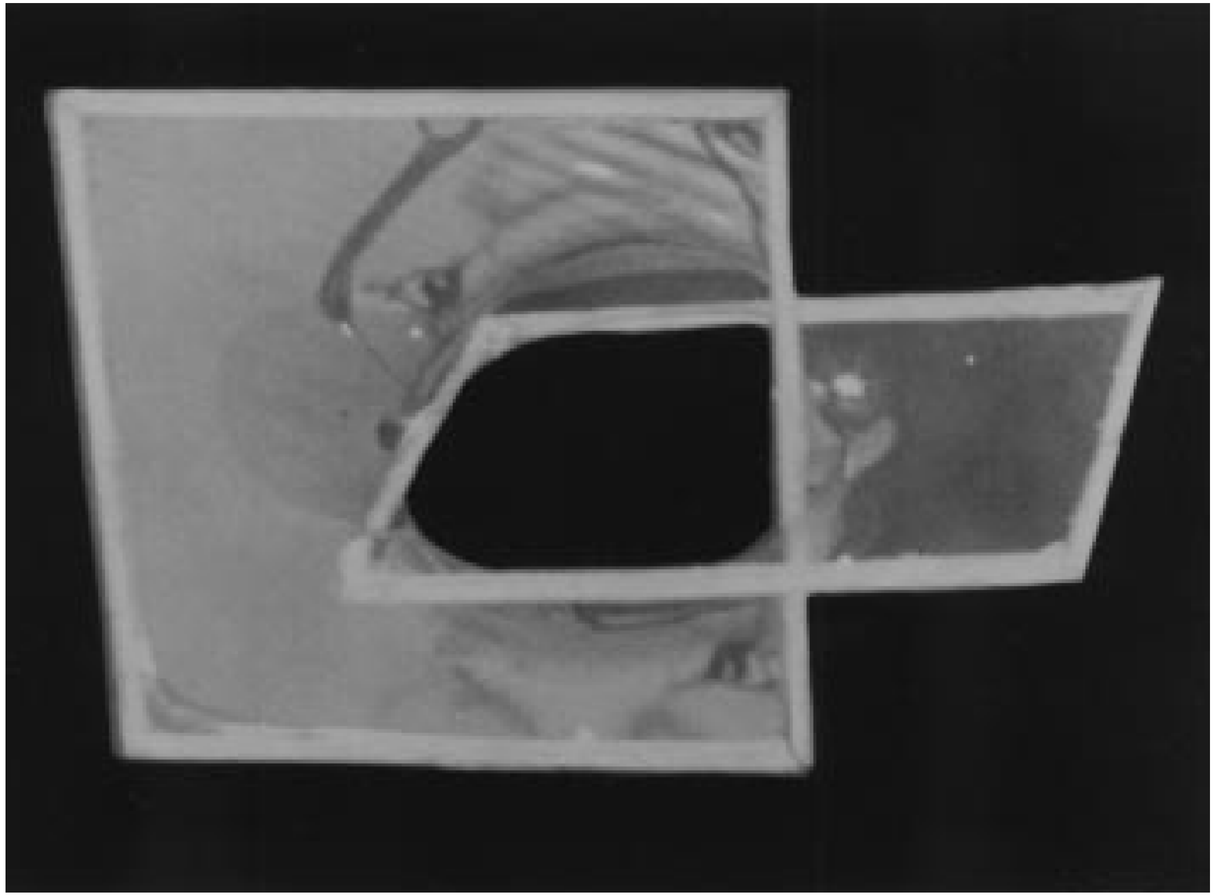
Wang and C. (Siggraph 2021)
Computing minimal surfaces
with differential forms

Costa’s minimal surface

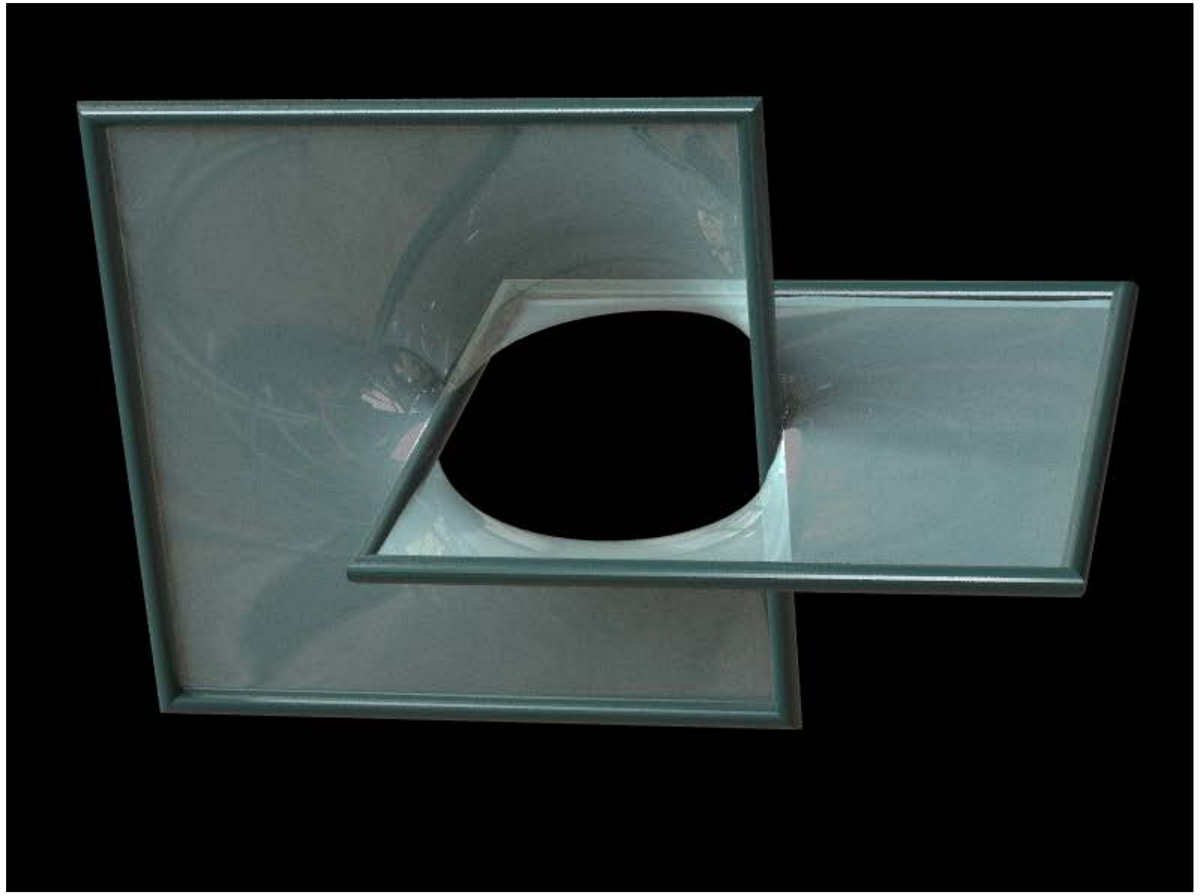


Wang and C. (Siggraph 2021)
Computing minimal surfaces
with differential forms

Soap film



Photograph of a real soap film

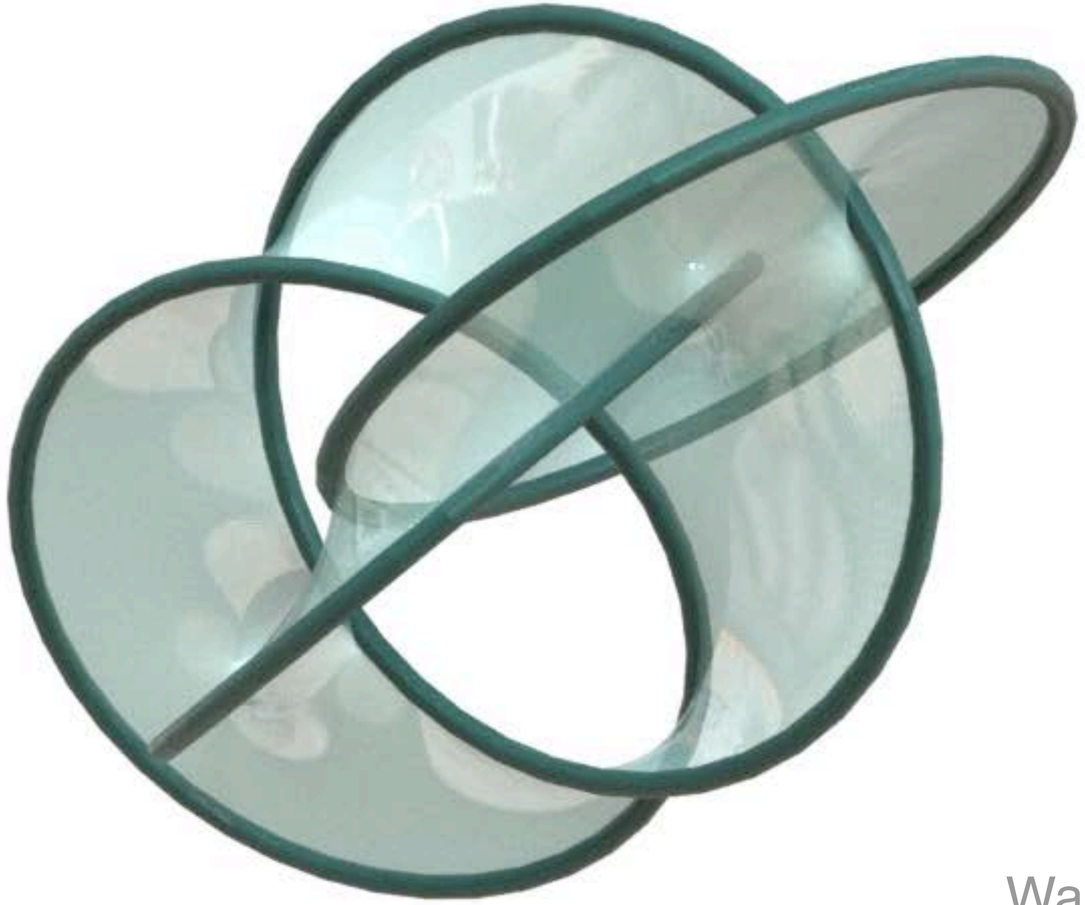


Our result

Scherk's periodic minimal surface



Arbitrary boundary



Wang and C. (Siggraph 2021)
Computing minimal surfaces
with differential forms

String interpolation



Wang and C. (Siggraph 2021)
Computing minimal surfaces
with differential forms

Hodge decomposition

- Classical Hodge decomposition

(k+1)-form

$$\eta = d\phi + \delta\psi + h$$

k-form *(k-1)-form* *harmonic k-form*

Hodge decomposition

- Classical Hodge decomposition

coexact

$$\eta = d\phi + \boxed{\delta\psi} + h$$

$$= \operatorname*{argmin}_{\substack{\xi \in \Omega^k(M) \\ d\xi = d\eta}} \int_M |\xi|^2$$

Hodge decomposition

- Classical Hodge decomposition

$$\eta = d\phi + \boxed{\delta\psi + h} \quad \textit{coclosed}$$

$$= \operatorname{argmin}_{\substack{\xi \in \Omega^k(M) \\ d\xi = d\eta \\ [\xi] = [\eta]}} \int_M |\xi|^2$$

Hodge decomposition = Optimality condition
*for **least squares** problems*

Generalize to L^1 *problems!*

Hodge decomposition

- Classical Hodge decomposition $\eta = d\phi + \delta\psi + h$
- L^1-based decomposition $\eta = d\phi + \xi + \zeta$

globally minimal submanifold

$$\xi = \operatorname*{argmin}_{\substack{\xi \in \Omega^k(M) \\ d\xi = d\eta}} \int_M |\xi|$$

homologically constrained minimal submanifold

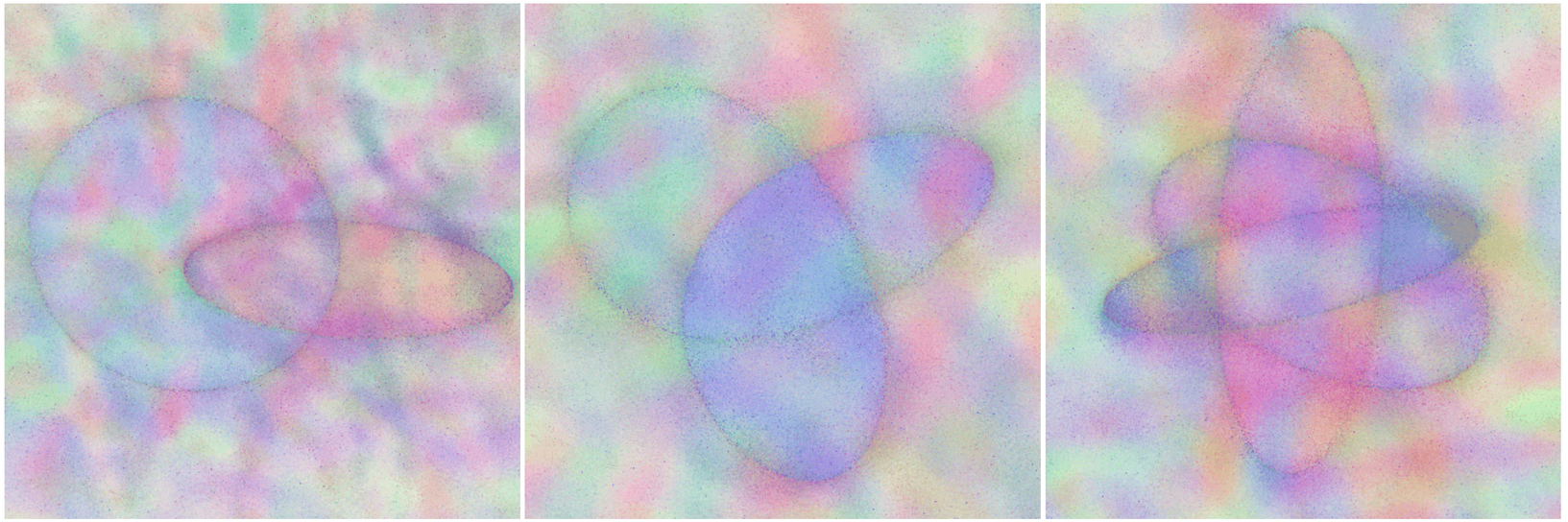
$$\xi + \zeta = \operatorname*{argmin}_{\substack{\xi \in \Omega^k(M) \\ d\xi = d\eta \\ [\xi] = [\eta]}} \int_M |\xi|$$

- Coclosed/coexact-ness are replaced by:

calibration $\boxed{\star \frac{\xi}{|\xi|}}$ is closed (resp. exact) on $\operatorname{supp}(\xi)$

Deep currents

- The surfaces are represented by $\eta \in \Omega^1(M)$
- Represent functions and forms by deep neural network (neural field)
  - Input: (x,y,z) position
  - Output: function or form value
- Infinite grid resolution



Palmer, Smirnov, Wang, C., Solomon
CVPR 2022
DeepCurrents

More about Pullback, Lie derivatives

- Geometric measure theory
- More about pullback formula
  - Pullback formulae in 3D
  - Lie derivative
  - Intuition behind Cartan's formula
  - Advection

Calculus identities not taught in calculus

- In single-variable calculus

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(s)\,ds = b'\,f(b) - a'\,f(a)$$

- What is the formula for

$$\frac{d}{dt}\int_{S_t} \omega = ?$$

Calculus identities not taught in calculus

- Differentiating integrals

$$\frac{d}{dt}\int_{C_t} \mathbf{f}\cdot\, d\mathbf{l} = \int_{C_t} [(\nabla\times\mathbf{f})\times\dot{C}_t]\cdot d\mathbf{l} + (\mathbf{f}\cdot\dot{C}_t)|_{\partial C_t}$$

rate of change of line integral

$$\frac{d}{dt}\iint_{S_t} \mathbf{f}\cdot\mathbf{n}\,dA = \iint_{S_t} (\nabla\cdot\mathbf{f})\dot{S}_t\cdot\mathbf{n}\,dA + \oint_{\partial S_t} (\mathbf{f}\times\dot{S}_t)\cdot d\mathbf{l}$$

rate of change of surface integral

$$\frac{d}{dt}\iiint_{\Omega_t} f\, dV = \oiint_{\partial\Omega_t} f\,\dot{\Omega}_t\cdot\mathbf{n}\,dA$$

rate of change of volume integral (Reynolds' transport theorem)

Calculus identities not taught in calculus

- Cofactor matrix and cross product

$$\mathrm{cof}(\mathbf{A})^{\mathsf{T}}(\mathbf{u}\times\mathbf{v}) = (\mathbf{A}^{\mathsf{T}}\mathbf{u})\times(\mathbf{A}^{\mathsf{T}}\mathbf{v})$$

$$\mathbf{A}^{\mathsf{T}}(\mathbf{u}\times(\mathbf{A}\mathbf{v})) = (\mathrm{cof}(\mathbf{A})^{\mathsf{T}}\mathbf{u})\times\mathbf{v}$$

- Curl of cross & cross of curl

$$\nabla\times(\mathbf{a}\times\mathbf{u}) = \mathbf{u}\cdot\nabla\mathbf{a} - \mathbf{a}\cdot\nabla\mathbf{u} + (\nabla\cdot\mathbf{u})\mathbf{a} - (\nabla\cdot\mathbf{a})\mathbf{u}$$

$$(\nabla\times\mathbf{a})\times\mathbf{u} = \mathbf{u}\cdot\nabla\mathbf{a} + \nabla(\mathbf{a}\cdot\mathbf{u}) - (\nabla\mathbf{u})^{\mathsf{T}}\mathbf{a}$$

Calculus identities not taught in calculus

- Backpropagation through $\boldsymbol{\phi} : \mathbb{R}^3 \to \mathbb{R}^3 \quad \mathbf{F} = d\boldsymbol{\phi} : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$

$$\mathbf{F}^\intercal(\nabla f \circ \boldsymbol{\phi}) = \nabla(f \circ \boldsymbol{\phi})$$ *Back-propagation of gradient*

$$\operatorname{cof}(\mathbf{F})^\intercal((\nabla \times \mathbf{u}) \circ \boldsymbol{\phi}) = \nabla \times (\mathbf{F}^\intercal \mathbf{u} \circ \boldsymbol{\phi})$$ *Back-propagation of curl*

$$\det(\mathbf{F})((\nabla \cdot \mathbf{u}) \circ \boldsymbol{\phi}) = \nabla \cdot (\operatorname{cof}(\mathbf{F})^\intercal \mathbf{u} \circ \boldsymbol{\phi})$$ *Back-prop. of div*

Pullback formulae in 3D

- Geometric measure theory
- More about pullback formula
  - Pullback formulae in 3D
  - Lie derivative
  - Intuition behind Cartan's formula
  - Advection

Pullback

Definition For linear operator $A\colon U \xrightarrow{\text{linear}} V$

$$\underset{k\text{-form}}{A^*} : \mathrm{Alt}^k(V) \xrightarrow{\text{linear}} \mathrm{Alt}^k(U)$$

$$\left(\underset{k\text{-form}}{A^*}\ \alpha\right)[\![\vec{v}_1, \ldots, \vec{v}_k]\!] := \alpha[\![A\vec{v}_1, \ldots, A\vec{v}_k]\!]$$

Definition For map between manifolds $\phi : M \to N$

$$\underset{k\text{-form}}{\phi^*} : \Omega^k(N) \to \Omega^k(M)$$

$$\left(\underset{k\text{-form}}{\phi^*}\ \alpha\right)[\![\vec{v}_1, \ldots, \vec{v}_k]\!] := \alpha[\![\underset{\text{vec}}{\phi_*}\vec{v}_1, \ldots, \underset{\text{vec}}{\phi_*}\vec{v}_k]\!]$$

In Euclidean/Cartesian 3D

- Using the conversion

$$(r)_{0\text{-form}} := r \qquad (\mathbf{v})_{1\text{-form}}[\![\mathbf{a}]\!] := \mathbf{v}^\intercal \mathbf{a}$$

$$(\mathbf{v})_{2\text{-form}}[\![\mathbf{a},\mathbf{b}]\!] := \det\begin{bmatrix} | & | & | \\ \mathbf{v} & \mathbf{a} & \mathbf{b} \\ | & | & | \end{bmatrix} \qquad (r)_{3\text{-form}}[\![\mathbf{a},\mathbf{b},\mathbf{c}]\!] := r\det\begin{bmatrix} | & | & | \\ \mathbf{a} & \mathbf{b} & \mathbf{c} \\ | & | & | \end{bmatrix}$$

- The pullbacks become

$$A^*(\mathbf{v})_{1\text{-form}} = (\mathbf{A}^\intercal\mathbf{v})_{1\text{-form}}$$

$$A^*(\mathbf{v})_{2\text{-form}} = (\mathrm{cof}(\mathbf{A}^\intercal)\mathbf{v})_{2\text{-form}}$$

$$A^*(r)_{3\text{-form}} = \det(\mathbf{A})(r)_{3\text{-form}}$$

$$\mathrm{cof}(\mathbf{A}^\intercal) = \det(\mathbf{A})\mathbf{A}^{-1}$$

cofactor matrix works for non-invertible square matrices too

$$[\mathrm{cof}(\mathbf{M})]_{ij}$$

$$= (-1)^{i+j}\det\begin{pmatrix}\mathbf{M}\text{ with } i\text{th} \\ \text{row and } j\text{th} \\ \text{col removed}\end{pmatrix}$$

In Euclidean/Cartesian 3D

- Elementary facts in exterior calculus:

$$A^*(\alpha \wedge \beta) = (A^*\alpha) \wedge (A^*\beta)$$

$$\phi^*(i_{\phi_* \mathbf{u}}\omega) = i_{\mathbf{u}}\phi^*\omega$$

$$\phi^* \circ d = d \circ \phi^*$$

- Translate them to 3D vector identities

In Euclidean/Cartesian 3D

- If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, and $\mathbf{A} \in \mathbb{R}^{3\times 3}$
  - Show that $\text{cof}(\mathbf{A})^\mathsf{T}(\mathbf{u} \times \mathbf{v}) = (\mathbf{A}^\mathsf{T}\mathbf{u}) \times (\mathbf{A}^\mathsf{T}\mathbf{v})$
  - Show that $\mathbf{A}^\mathsf{T}(\mathbf{u} \times (\mathbf{A}\mathbf{v})) = (\text{cof}(\mathbf{A})^\mathsf{T}\mathbf{u}) \times \mathbf{v}$

In Euclidean/Cartesian 3D

- Given map $\boldsymbol{\phi} : \mathbb{R}^3 \to \mathbb{R}^3$ and let $\mathbf{F} = d\boldsymbol{\phi} : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$
  - For function $f : \mathbb{R}^3 \to \mathbb{R}$

$$\mathbf{F}^\intercal(\nabla f \circ \boldsymbol{\phi}) = \nabla(f \circ \boldsymbol{\phi})$$

  - For vector field $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3$

$$\operatorname{cof}(\mathbf{F})^\intercal((\nabla \times \mathbf{u}) \circ \boldsymbol{\phi}) = \nabla \times (\mathbf{F}^\intercal \mathbf{u} \circ \boldsymbol{\phi})$$

  - (Piola identity)

$$\det(\mathbf{F})((\nabla \cdot \mathbf{u}) \circ \boldsymbol{\phi}) = \nabla \cdot (\operatorname{cof}(\mathbf{F})^\intercal \mathbf{u} \circ \boldsymbol{\phi})$$

In Euclidean/Cartesian 3D

- Given map $\boldsymbol{\phi} : \mathbb{R}^3 \to \mathbb{R}^3$ and let $\mathbf{F} = d\boldsymbol{\phi} : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$
  - Show that rows of $\mathrm{cof}(\mathbf{F})$ must be divergence-free

- **Corollary**
  If a square matrix field has curl-free columns, its cofactor matrix must have div-free rows.

Lie derivative

- Geometric measure theory
- More about pullback formula
  - Pullback formulae in 3D
  - Lie derivative
  - Intuition behind Cartan's formula
  - Advection

Flow

Definition

Let $\mathbf{u}$ be a vector field on a manifold M.

The **flow** generated by $\mathbf{u}$ is a time-dependent map $\varphi_t : M \to M$ satisfying

$$\frac{\partial}{\partial t}\varphi_t = \mathbf{u} \circ \varphi_t \qquad \varphi_0 = \mathrm{id}_M$$

The flow map gives rise to pullback for *k*-forms

$$\varphi_t{}^* : \Omega^k(M) \xrightarrow{\text{linear}} \Omega^k(M)$$

k-form

Lie derivative

Definition

Let $\mathbf{u}$ be a vector field on a manifold M.

The **Lie derivative** for *k*-forms with respect to $\mathbf{u}$ is defined by

$$\underset{k\text{-form}}{\mathscr{L}_{\mathbf{u}}} : \Omega^k(M) \xrightarrow{\text{linear}} \Omega^k(M)$$

$$\underset{k\text{-form}}{\mathscr{L}_{\mathbf{u}}\ \omega} := \frac{\partial}{\partial t}\Big|_{t=0} \Big(\underset{k\text{-form}}{\varphi_t{}^*\ \omega} \Big)$$

where φ_t is the flow map generated by $\mathbf{u}$.

Lie derivative

- The property

$$\varphi_t{}^*(\alpha \wedge \beta) = (\varphi_t{}^*\alpha) \wedge (\varphi_t{}^*\beta)$$

  implies $\mathscr{L}_{\mathbf{u}}(\alpha \wedge \beta) = (\mathscr{L}_{\mathbf{u}}\,\alpha) \wedge \beta + \alpha \wedge (\mathscr{L}_{\mathbf{u}}\,\beta)$

  i.e. the Lie derivative is a derivation.

- The property $\varphi_t{}^* \circ d = d \circ \varphi_t{}^*$

  implies $\mathscr{L}_{\mathbf{u}} \circ d = d \circ \mathscr{L}_{\mathbf{u}}$

Lie derivative (alt. definition)

Definition

Let $\mathbf{u}$ be a vector field on a manifold M.

The **Lie derivative** for *k*-forms with respect to $\mathbf{u}$ is defined by the unique operator $\mathscr{L}_{\mathbf{u}} : \Omega^k(M) \xrightarrow{\text{linear}} \Omega^k(M)$ satisfying

(k-form)

- derivation: $\mathscr{L}_{\mathbf{u}}(\alpha \wedge \beta) = (\mathscr{L}_{\mathbf{u}}\,\alpha) \wedge \beta + \alpha \wedge (\mathscr{L}_{\mathbf{u}}\,\beta)$
- commute with *d*: $\mathscr{L}_{\mathbf{u}} \circ d = d \circ \mathscr{L}_{\mathbf{u}}$
- when applied on 0-form $\mathscr{L}_{\mathbf{u}} f = df\,[\![\mathbf{u}]\!]$

Cartan's magic formula

Theorem

Lie derivatives for *k*-forms have the following explicit formula

$$\mathscr{L}_{\mathbf{u}}\,\omega = i_{\mathbf{u}} d\,\omega + d\,i_{\mathbf{u}}\omega$$

That is,

$$\mathscr{L}_{\mathbf{u}} = i_{\mathbf{u}} \circ d + d \circ i_{\mathbf{u}}$$

(Cartan's magic formula)

(*Proof*)

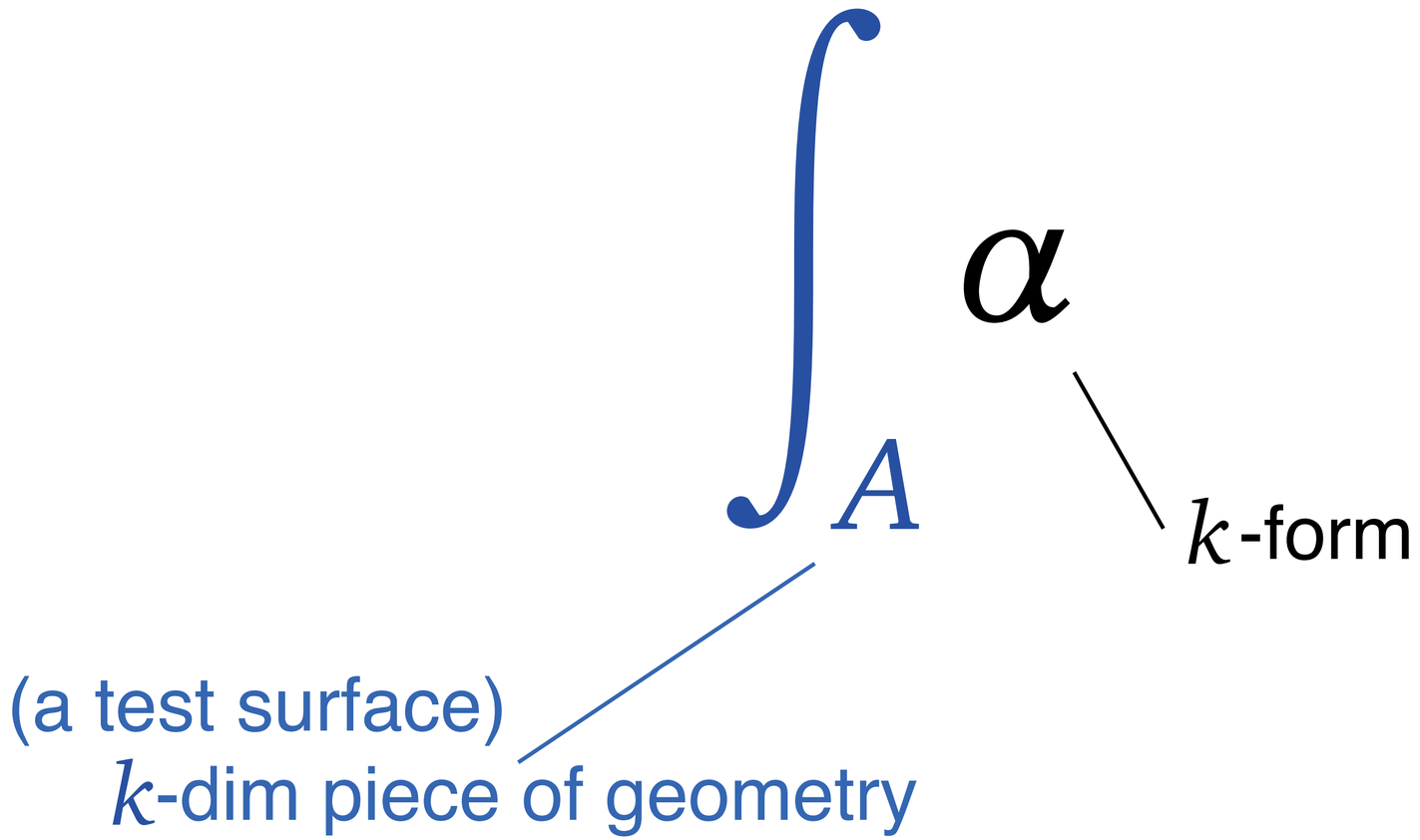
Check that the RHS satisfy the 3 properties that uniquely define the Lie derivative.

□

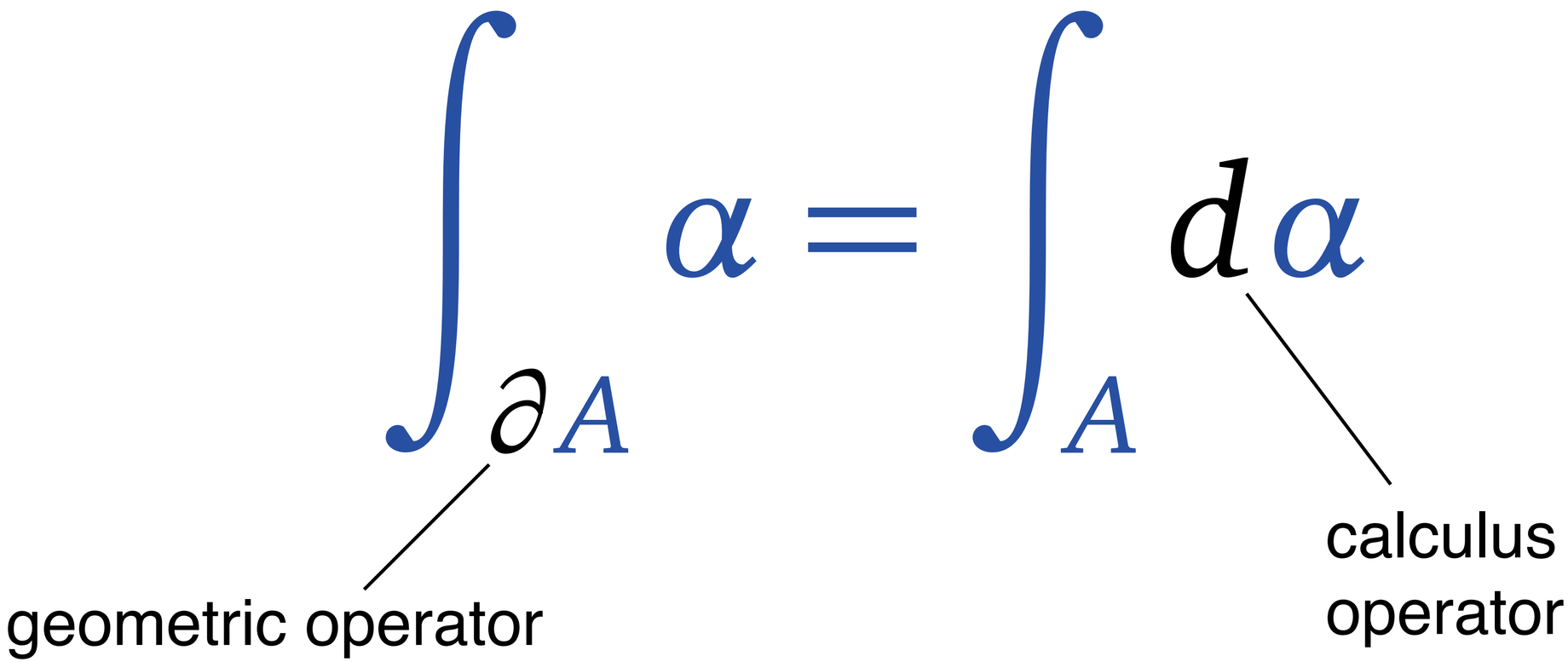
Intuition using Stokes-like theorems

- Geometric measure theory
- More about pullback formula
  - Pullback formulae in 3D
  - Lie derivative
  - Intuition behind Cartan's formula
  - Advection

Integral formulation

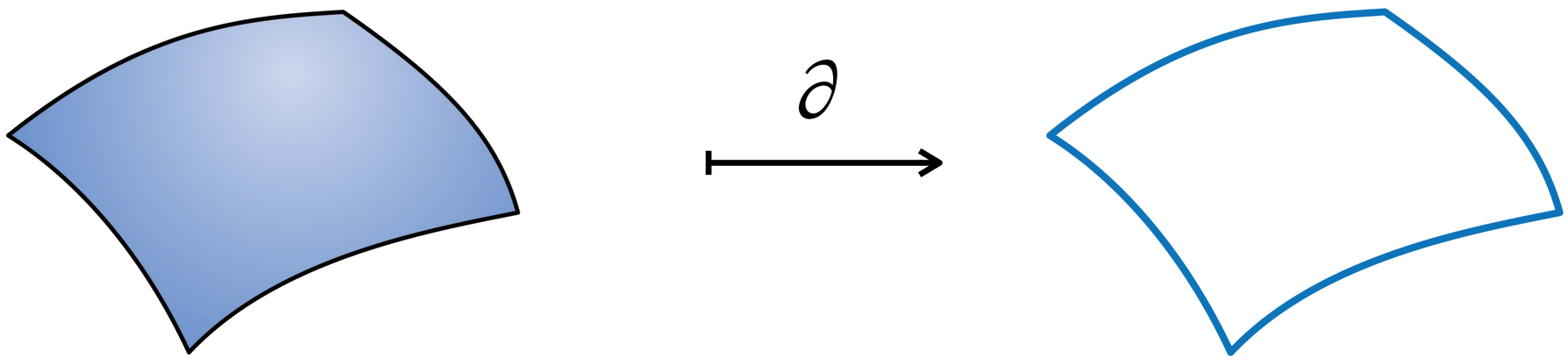


Integral formulation



Operations

- Boundary

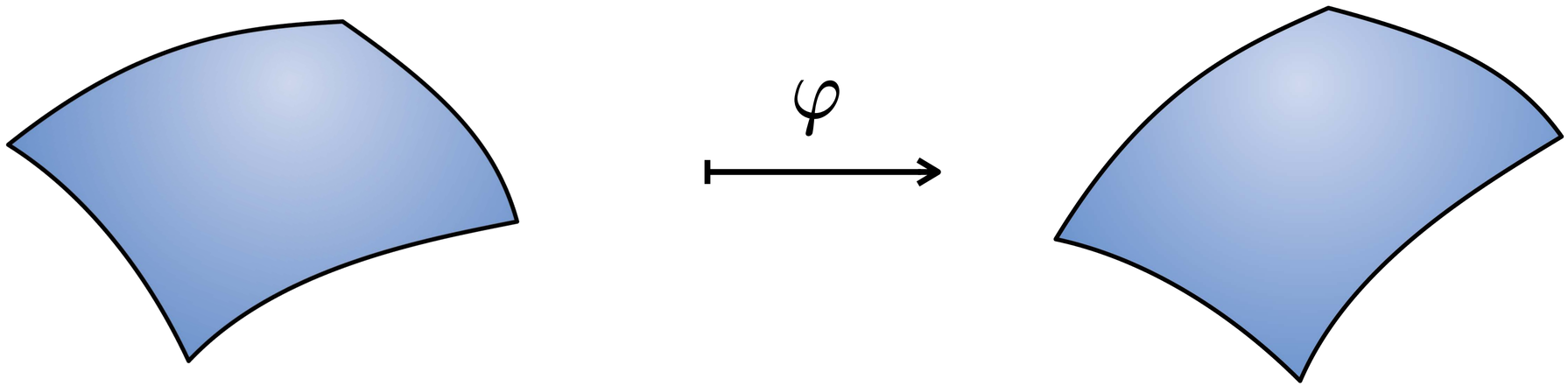


k-surface

$(k-1)$-surface

Operations

- Map

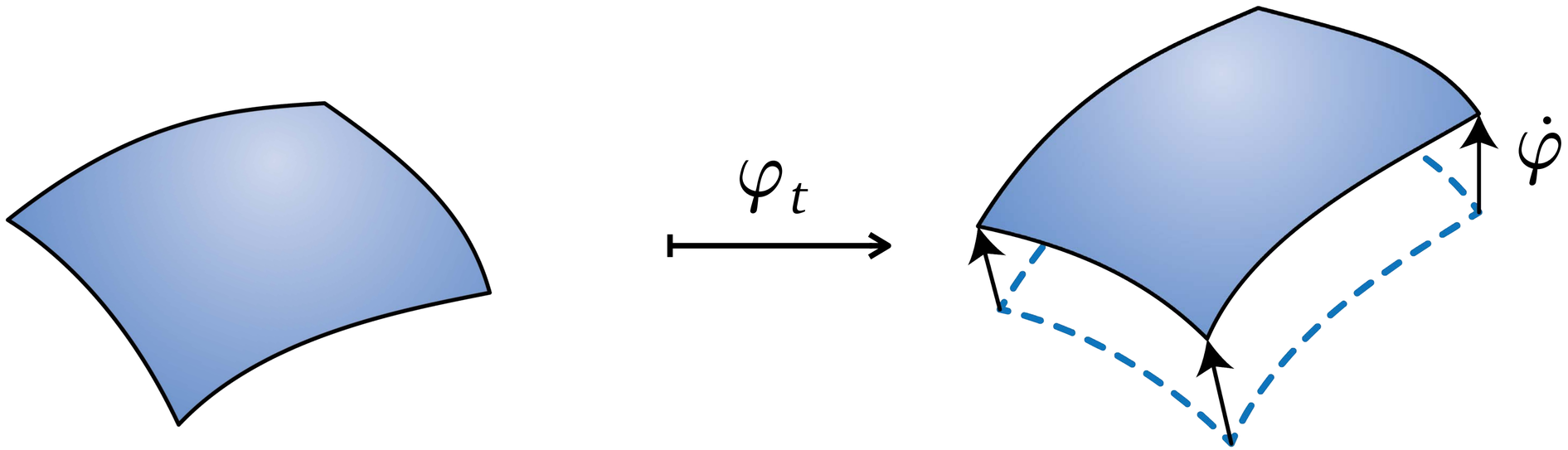


k-surface

k-surface possibly in another space

Operations

- Time-dependent map

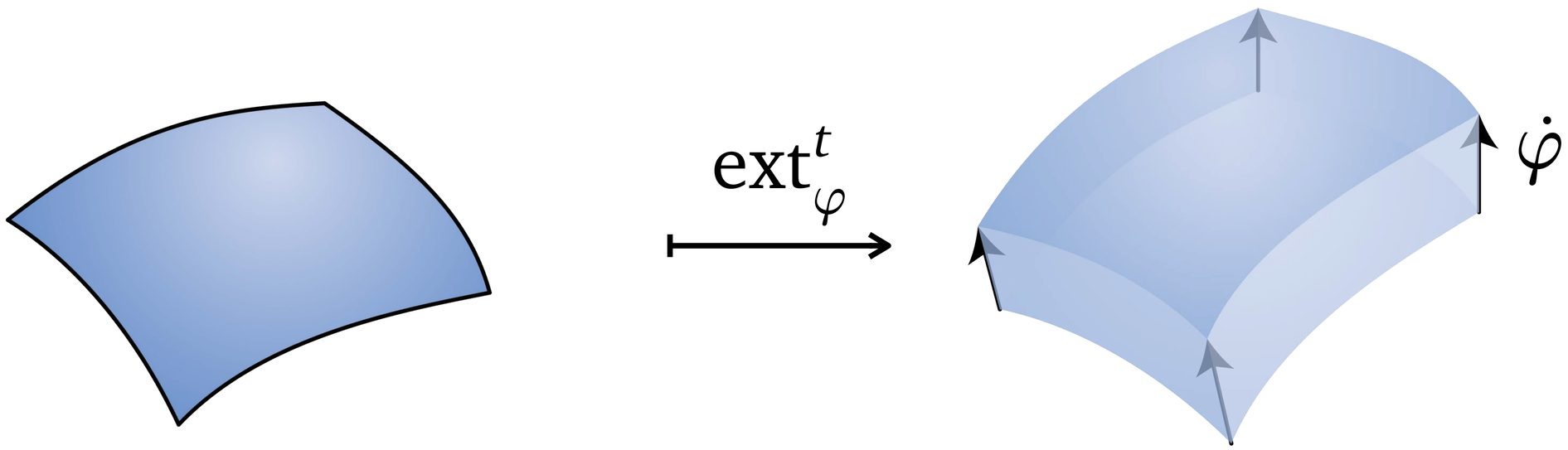


k-surface

moving k-surface

Operations

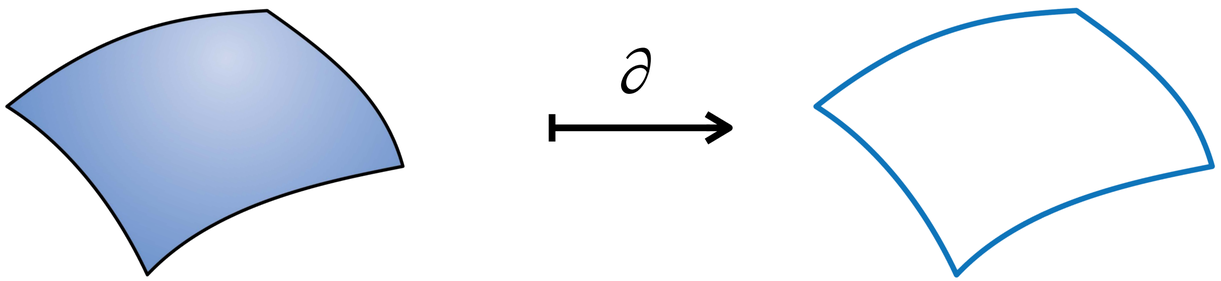
- Extrusion



k-surface

$(k+1)$-surface

Stokes-like Theorems



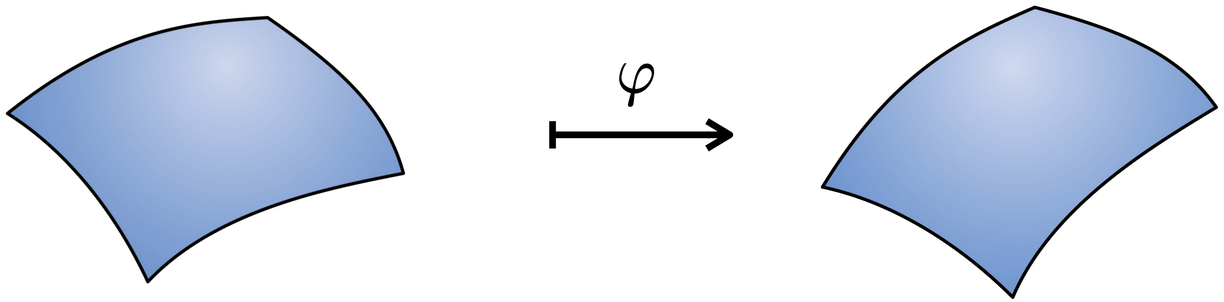
Boundary / Exterior derivative

∂ : k-surfaces $\longrightarrow$ $(k-1)$-surfaces

d : $(k-1)$-forms $\longrightarrow$ k-forms

$$\int_{\partial A} \alpha = \int_{A} d\alpha$$

Stokes-like Theorems



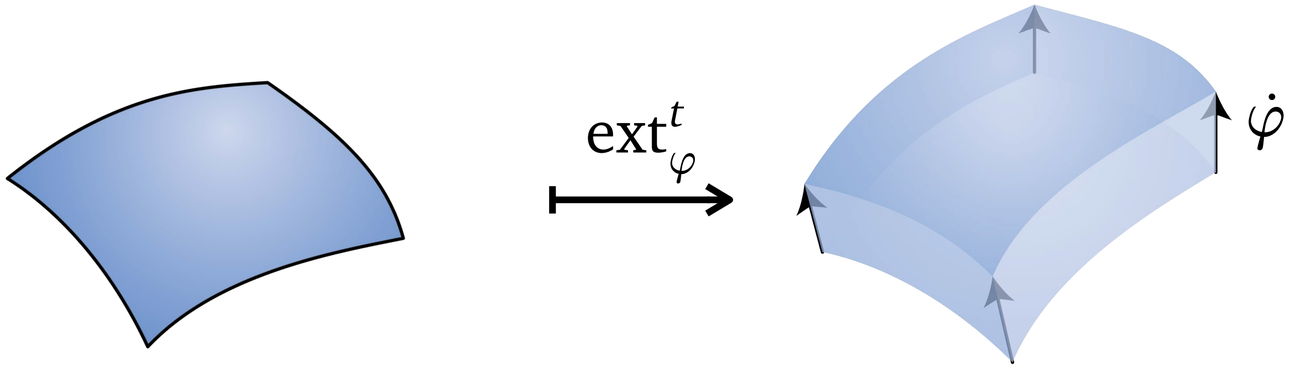
Map / Pullback operator

φ : k-surfaces $\longrightarrow$ k-surfaces

φ^* : k-forms $\longrightarrow$ k-forms

$$\int_{\varphi(A)} \alpha = \int_A \varphi^* \alpha$$

Stokes-like Theorems

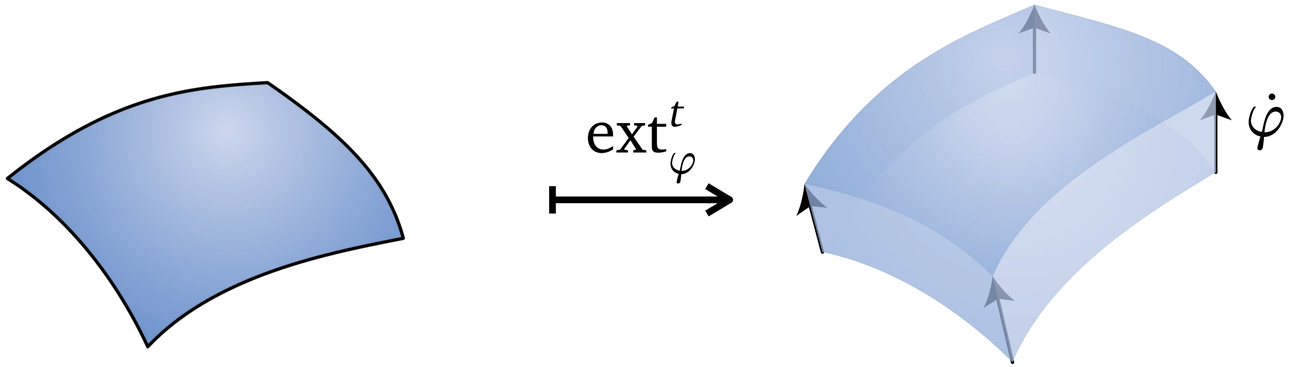


Extrusion /

$$\mathrm{ext}_\varphi^t : k\text{-surfaces} \longrightarrow (k+1)\text{-surfaces}$$
$$: (k+1)\text{-forms} \longrightarrow k\text{-forms}$$

$$\lim_{t \to 0} \frac{1}{t} \int_{\mathrm{ext}_\varphi^t(A)} \alpha =$$

Stokes-like Theorems



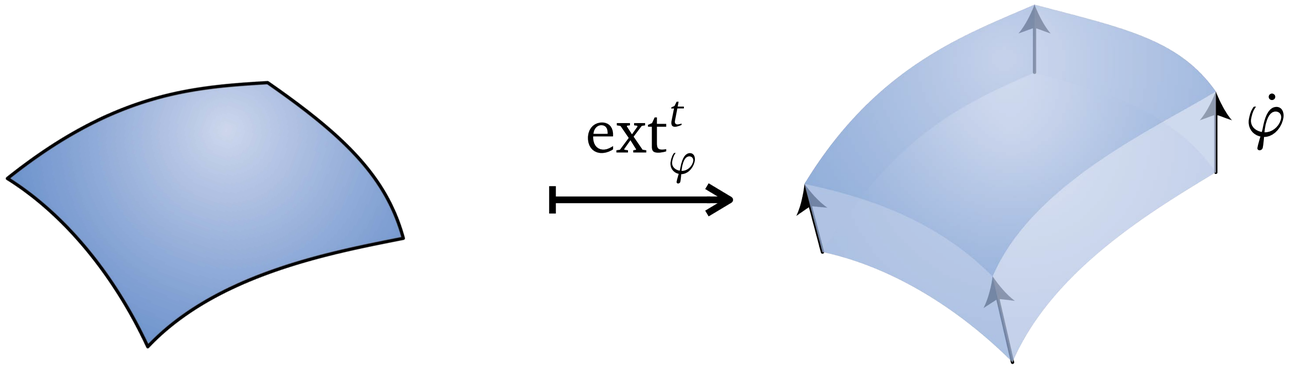
Extrusion /

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{ext}_\varphi^t : k\text{-surfaces} \longrightarrow (k+1)\text{-surfaces}$$

$$: (k+1)\text{-forms} \longrightarrow k\text{-forms}$$

$$\left.\frac{d}{dt}\right|_{t=0} \int_{\mathrm{ext}_\varphi^t(A)} \alpha =$$

Stokes-like Theorems



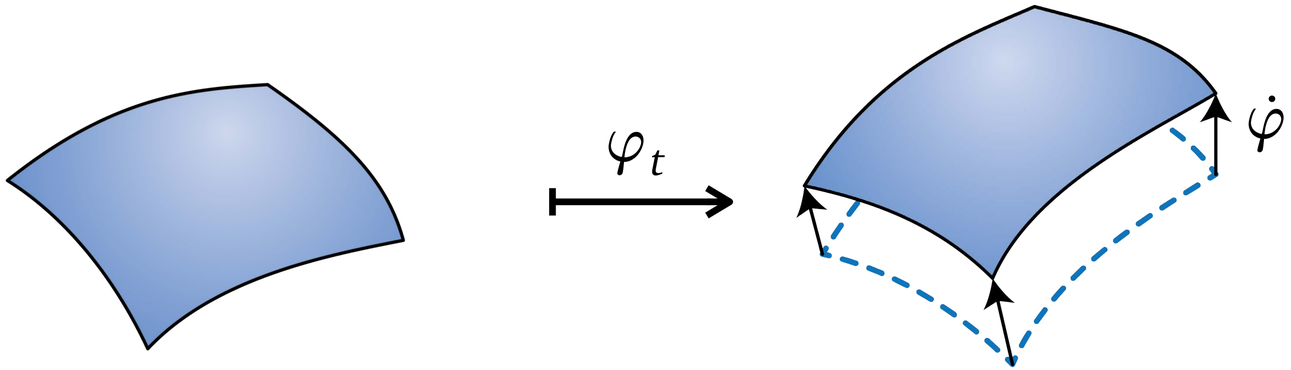
Extrusion / Interior product

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{ext}_\varphi^t : k\text{-surfaces} \longrightarrow (k+1)\text{-surfaces}$$

$$i_{\dot{\varphi}} : (k+1)\text{-forms} \longrightarrow k\text{-forms}$$

$$\left.\frac{d}{dt}\right|_{t=0} \int_{\mathrm{ext}_\varphi^t(A)} \alpha = \int_A i_{\dot{\varphi}}\alpha$$

Stokes-like Theorems



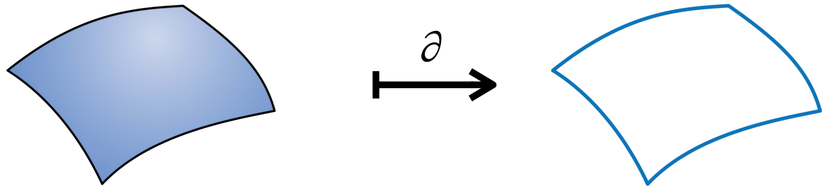
Time-dependent map / Lie derivative

$\left.\frac{d}{dt}\right|_{t=0} \varphi_t$: k-surfaces $\longrightarrow$ k-surfaces

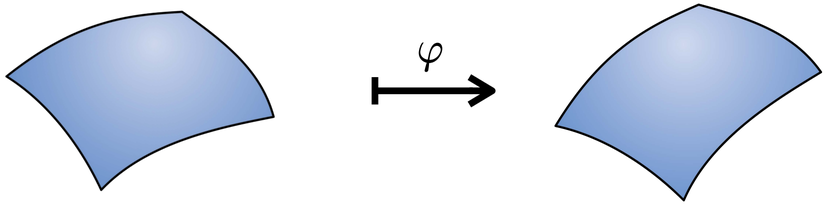
$\mathscr{L}_{\dot{\varphi}}$: k-forms $\longrightarrow$ k-forms

$$\left.\frac{d}{dt}\right|_{t=0} \int_{\varphi_t(A)} \alpha = \int_A \mathscr{L}_{\dot{\varphi}} \alpha$$

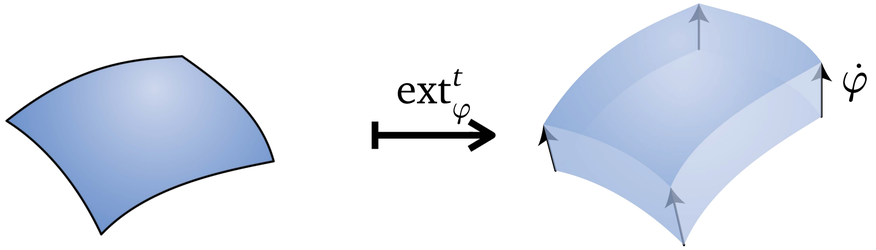
Stokes-like Theorems



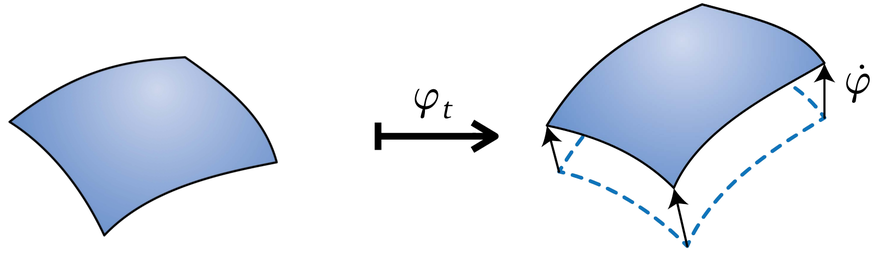
$$\int_{\partial A} \alpha = \int_A d\alpha$$



$$\int_{\varphi(A)} \alpha = \int_A \varphi^* \alpha$$

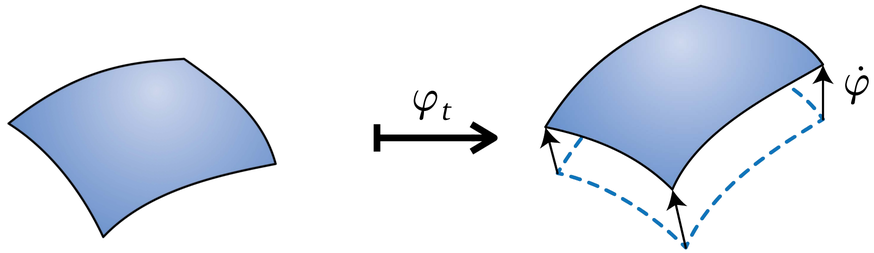


$$\left.\frac{d}{dt}\right|_{t=0} \int_{\mathrm{ext}_\varphi^t(A)} \alpha = \int_A i_{\dot{\varphi}} \alpha$$



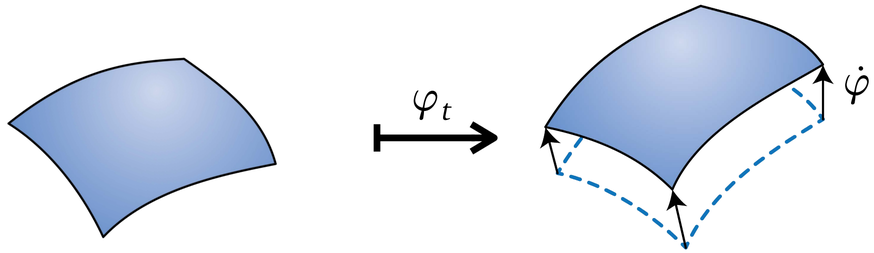
$$\left.\frac{d}{dt}\right|_{t=0} \int_{\varphi_t(A)} \alpha = \int_A \mathcal{L}_{\dot{\varphi}} \alpha$$

Lie derivative



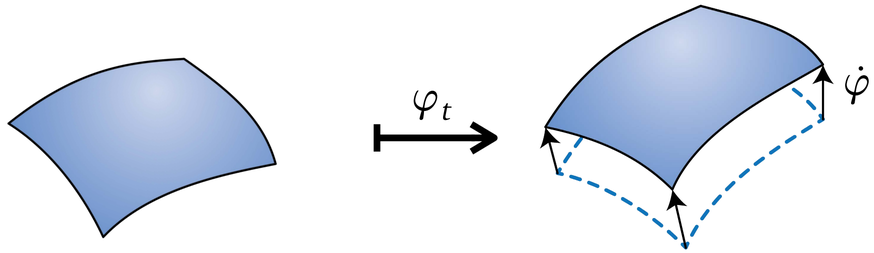
$$\left.\frac{d}{dt}\right|_{t=0} \int_{\varphi_t(A)} \alpha = \int_A \mathscr{L}_{\dot{\varphi}} \alpha$$

Lie derivative



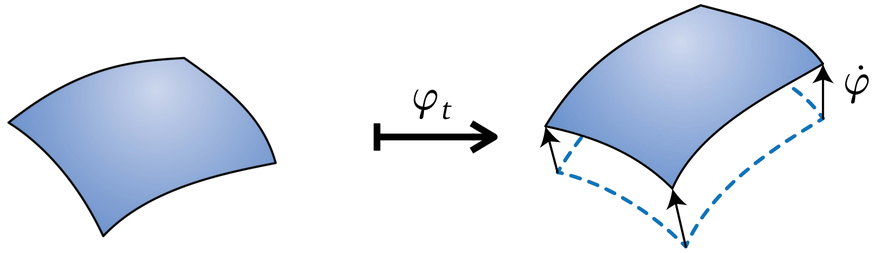
$$\left.\frac{d}{dt}\right|_{t=0} \int_{\varphi_t(A)} \alpha = \left.\frac{d}{dt}\right|_{t=0} \int_A \varphi_t^* \alpha = \int_A \mathcal{L}_{\dot{\varphi}} \alpha$$

Lie derivative



$$\left.\frac{d}{dt}\right|_{t=0} \int_{\varphi_t(A)} \alpha = \int_A \left.\frac{\partial}{\partial t}\right|_{t=0} \varphi_t^* \alpha = \int_A \mathcal{L}_{\dot{\varphi}} \alpha$$

Lie derivative



$$\left.\frac{d}{dt}\right|_{t=0} \int_{\varphi_t(A)} \alpha = \int_A \left.\frac{\partial}{\partial t}\right|_{t=0} \varphi_t^* \alpha = \int_A \mathcal{L}_{\dot{\varphi}} \alpha$$

Recall: The Lie derivative of a differential form α along a vector field $\mathbf{v}$ is defined by

$$\mathcal{L}_{\mathbf{v}} \alpha := \left.\frac{\partial}{\partial t}\right|_{t=0} \left(\varphi_t^* \alpha\right)$$

where $\varphi_t : M \to M$ satisfies $\varphi_0(\mathbf{x}) = \mathbf{x}$, $\frac{\partial}{\partial t}\varphi_0 = \mathbf{v}$

Cartan Magic Formula

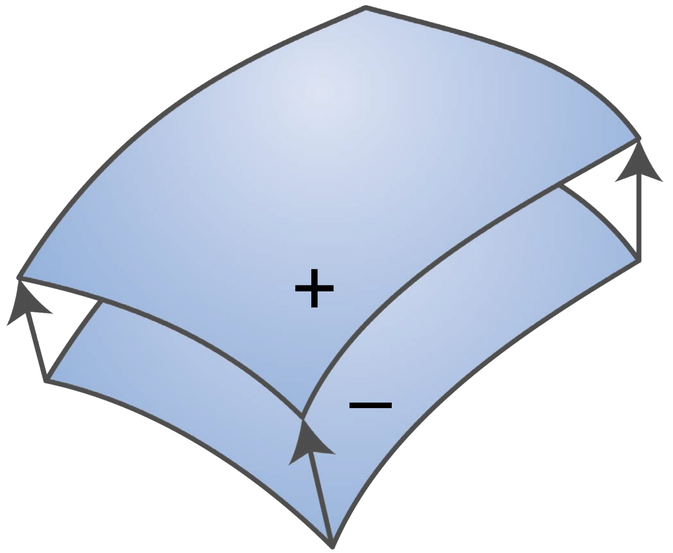
$$\frac{d}{dt}\int_{\varphi_t(A)} \alpha$$

Cartan Magic Formula

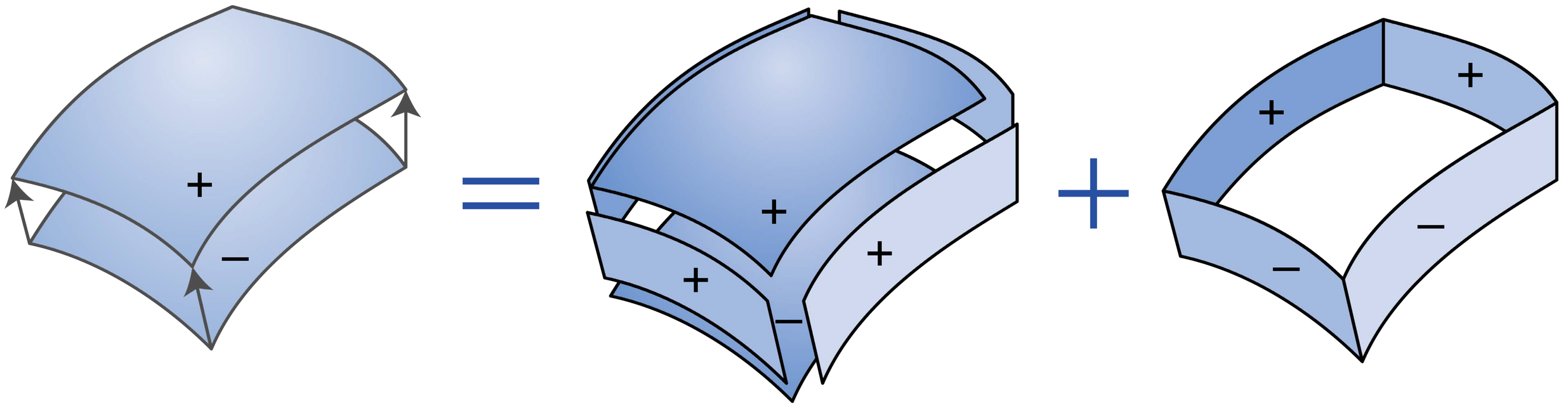
$$\frac{d}{dt}\varphi_t(A)$$

Cartan Magic Formula

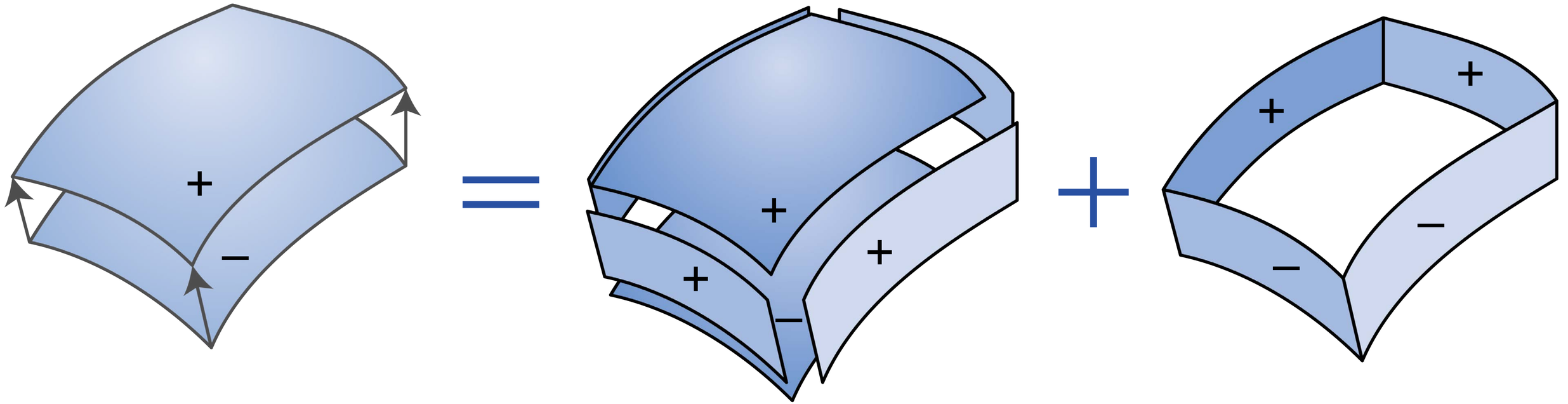
$$\frac{d}{dt}\varphi_t(A) =$$



Cartan Magic Formula



Cartan Magic Formula



$$\frac{d}{dt}\varphi_t(A) \qquad \frac{d}{dt}\,\partial\left(\mathrm{ext}^t_\varphi(A)\right) \qquad \frac{d}{dt}\,\mathrm{ext}^t_\varphi(\partial(A))$$

Cartan Magic Formula

Cartan Magic Formula

$$\mathscr{L}_{\dot{\varphi}}\alpha = i_{\dot{\varphi}}d\alpha + di_{\dot{\varphi}}\alpha$$

Lie derivatives written in vector calculus

0-form

$$\mathscr{L}_{\mathbf{u}}\,(a)_{0\text{-form}} = (\mathbf{u}\cdot\nabla a)_{0\text{-form}}$$

1-form

$$\begin{aligned}\mathscr{L}_{\mathbf{u}}\,(\mathbf{a})_{1\text{-form}} &= (\mathbf{u}\cdot\nabla\mathbf{a} + (\nabla\mathbf{u})^{\mathsf{T}}\mathbf{a})_{1\text{-form}} \\ &= \big((\nabla\times\mathbf{a})\times\mathbf{u} + \nabla(\mathbf{u}\cdot\mathbf{a})\big)_{1\text{-form}}\end{aligned}$$

2-form

$$\begin{aligned}\mathscr{L}_{\mathbf{u}}\,(\mathbf{a})_{2\text{-form}} &= (\mathbf{u}\cdot\nabla\mathbf{a} - \mathbf{a}\cdot\nabla\mathbf{u} + (\nabla\cdot\mathbf{u})\mathbf{a})_{2\text{-form}} \\ &= \big((\nabla\cdot\mathbf{a})\mathbf{u} + \nabla\times(\mathbf{a}\times\mathbf{u})\big)_{2\text{-form}}\end{aligned}$$

3-form

$$\mathscr{L}_{\mathbf{u}}\,(a)_{3\text{-form}} = (\mathbf{u}\cdot\nabla\mathbf{a} + (\nabla\cdot\mathbf{u})a)_{3\text{-form}}$$

Cartan’s formula in use

- Geometric measure theory
- More about pullback formula
  - Pullback formulae in 3D
  - Lie derivative
  - Intuition behind Cartan’s formula
  - Advection

Cartan formula under integration

Cartan's formula under integration

$$\frac{d}{dt}\int_{\varphi_t(A)} \alpha = \int_A i_{\dot{\varphi}} d\alpha + \oint_{\partial A} i_{\dot{\varphi}} \alpha$$

Translate to vector calculus

- Differentiating integrals

$$\frac{d}{dt}\int_{C_t}\mathbf{f}\cdot\,d\mathbf{l}=\int_{C_t}[(\nabla\times\mathbf{f})\times\dot{C}_t]\cdot d\mathbf{l}+(\mathbf{f}\cdot\dot{C}_t)|_{\partial C_t}$$

rate of change of line integral

$$\frac{d}{dt}\iint_{S_t}\mathbf{f}\cdot\mathbf{n}\,dA=\iint_{S_t}(\nabla\cdot\mathbf{f})\mathbf{u}\cdot\mathbf{n}\,dA+\oint_{\partial S_t}(\mathbf{f}\times\mathbf{u})\cdot d\mathbf{l}$$

rate of change of surface integral

$$\frac{d}{dt}\iiint_{\Omega_t}f\,dV=\oiint_{\partial\Omega_t}f\,\mathbf{u}\cdot\mathbf{n}\,dA$$

rate of change of volume integral (Reynolds' transport theorem)

Advection

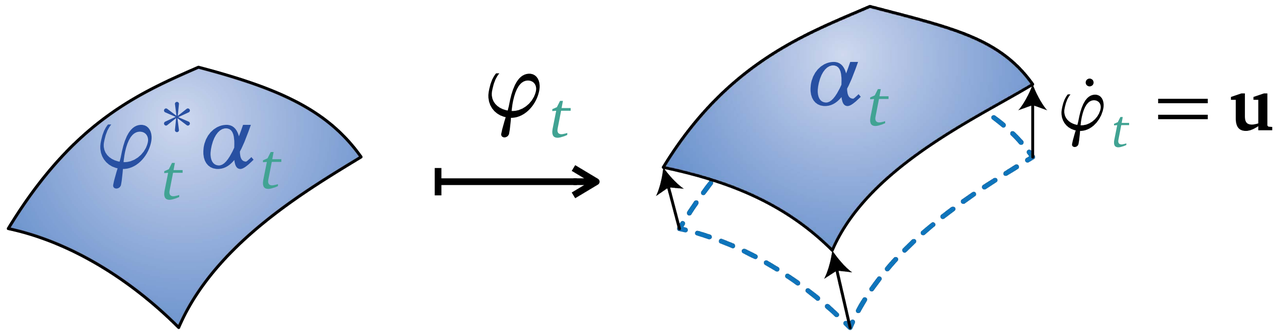
Definition

A time-dependent differential form α_t is said to be **advected by** a vector field $\mathbf{u}$ if

$$\frac{\partial}{\partial t}\alpha_t + \mathscr{L}_{\mathbf{u}}\alpha_t = 0$$

- This is the same as

$$\frac{\partial}{\partial t}(\varphi_t^*\alpha_t) = 0$$
$$\Leftrightarrow \quad \varphi_t^*\alpha_t = \alpha_0$$
$$\Leftrightarrow \quad \alpha_t = \varphi_t^{-*}\alpha_0$$



Advection

$$\frac{\partial}{\partial t}\alpha_t + \mathcal{L}_{\mathbf{u}}\,\alpha_t = 0 \quad \Leftrightarrow \quad \alpha_t = \varphi_t^{-*}\alpha_0$$

$$\Leftrightarrow \quad e^{-t\,\mathcal{L}_{\mathbf{u}}} = \varphi_t^{-*}$$

$$\Leftrightarrow \boxed{e^{t\,\mathcal{L}_{\mathbf{u}}} = \varphi_t^{*}}$$

Generalized Taylor expansion

$$\varphi_t{}^{*} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\,\mathcal{L}_{\mathbf{u}}^{n}$$

- Example: pullback for 2-form

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\Big(\mathbf{u}\cdot\nabla - \nabla\mathbf{u} + (\nabla\cdot\mathbf{u})\mathbf{I}\Big)^n \mathbf{a} = \operatorname{cof}(d\varphi_t)^{\intercal}\mathbf{a}\circ\varphi_t$$

Advections in vector calculus

$$(\frac{\partial}{\partial t} + \mathscr{L}_{\mathbf{u}})(f)_{0\text{-form}} = 0$$

material derivative

$$(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla) f = 0$$

$$(\frac{\partial}{\partial t} + \mathscr{L}_{\mathbf{u}})(\mathbf{a})_{1\text{-form}} = 0$$

Oldroyd's lower-convection

$$(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla)\mathbf{a} = -(\nabla \mathbf{u})^{\intercal}\mathbf{a}$$

$$(\frac{\partial}{\partial t} + \mathscr{L}_{\mathbf{u}})(\mathbf{w})_{2\text{-form}} = 0$$

stretching & compression

$$(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla)\mathbf{w} = \mathbf{w} \cdot \nabla \mathbf{u} - (\nabla \cdot \mathbf{u})\mathbf{w}$$

$$(\frac{\partial}{\partial t} + \mathscr{L}_{\mathbf{u}})(q)_{3\text{-form}} = 0$$

$$(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla) q = -(\nabla \cdot \mathbf{u}) q$$

compression