CSE 270 (FA25)
Vector Field Design

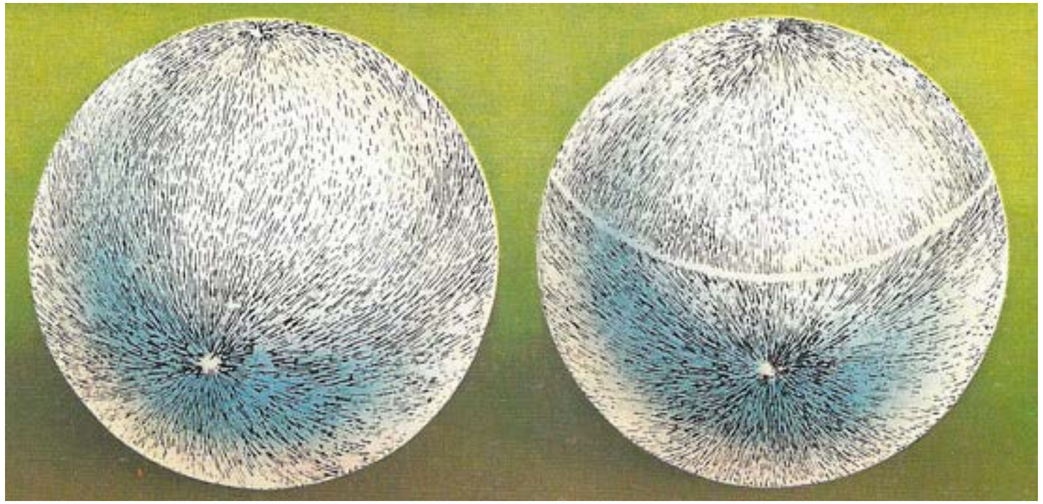
Albert Chern

UC San Diego
JACOBS SCHOOL OF ENGINEERING

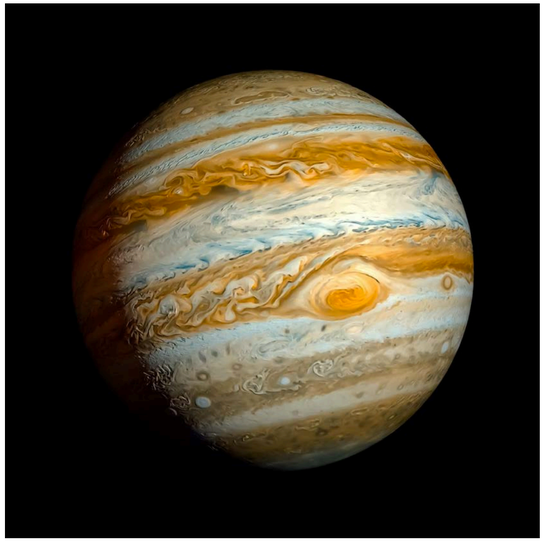
Hairy Ball Theorem

- Hairy ball theorem
- Optimal sections of a bundle
- Curvature and singularities
- Gauge theory in physics

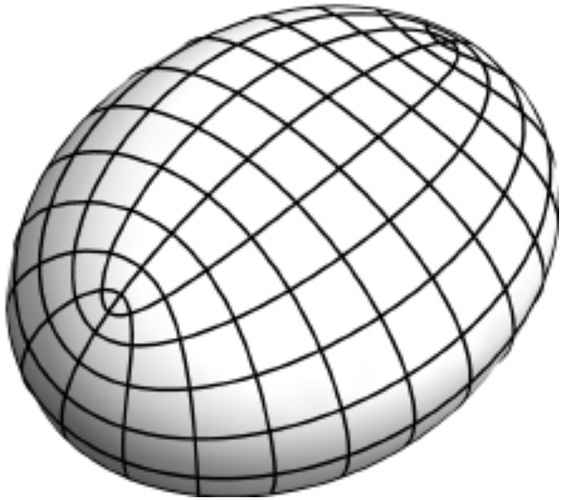
Hairy ball theorem







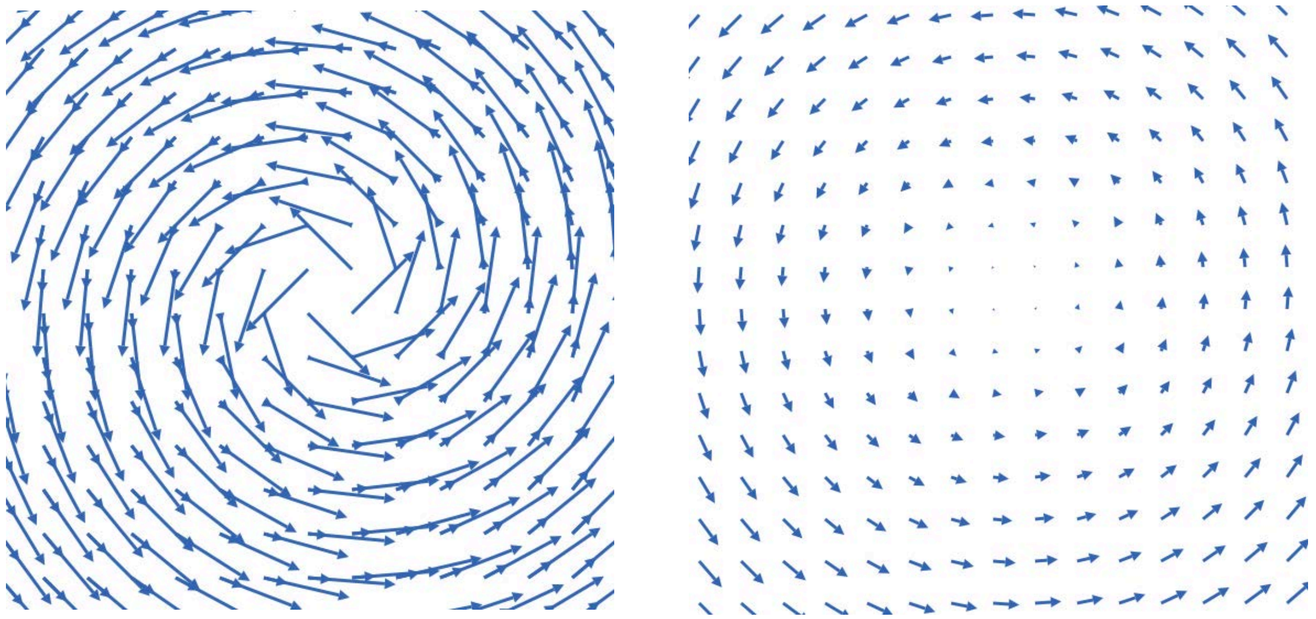




Hairy ball theorem

Theorem (Hairy ball theorem)

Every continuous vector field on a topological sphere must have zeros (a.k.a. singularities).



(singularity in orientation) & (continuity) forces the vector field dipping to zero

Hairy ball theorem

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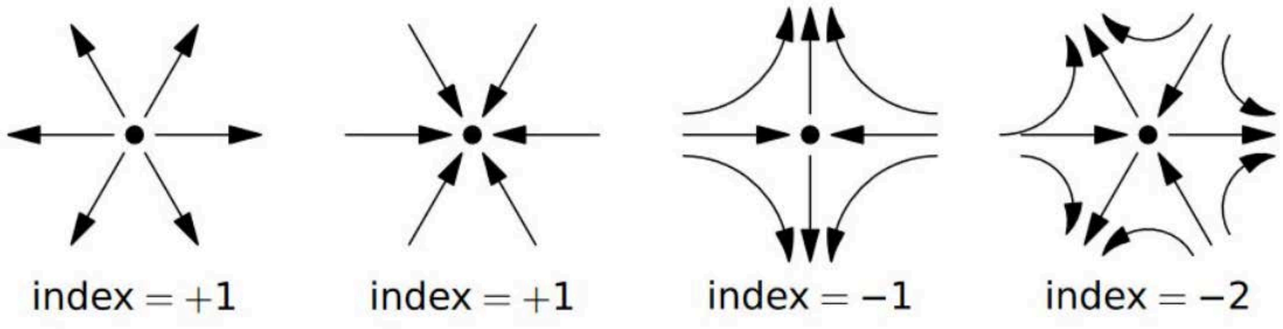
Keenan Crane 2005

Poincaré–Hopf theorem

Theorem (General hairy ball theorem)

Every continuous vector field on a closed surface must have total index equal to the Euler characteristic.

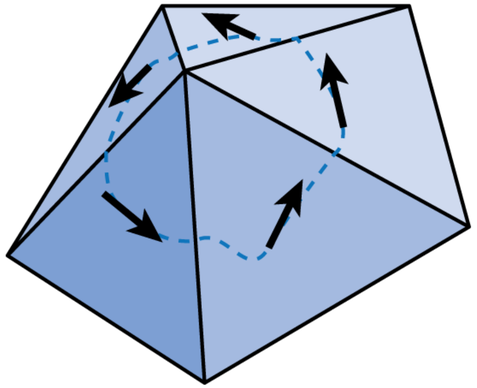
Definition The index of a singularity is the turning number of the vector field around the singularity. That is, it is the number of full (counterclockwise) turns the vector makes when we walk counterclockwise around a small circle around the singularity.

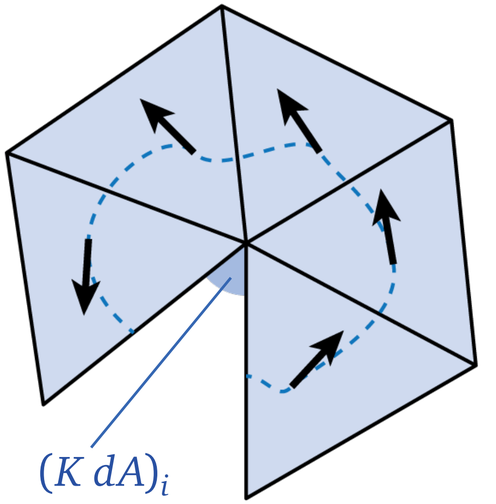


Poincaré index in angle defect

Recall:

$$(K\,dA)_i = 2\pi - \left(\begin{array}{l} \text{turning angle} \\ \text{of a loop around} \\ \text{the point i} \end{array} \right)$$

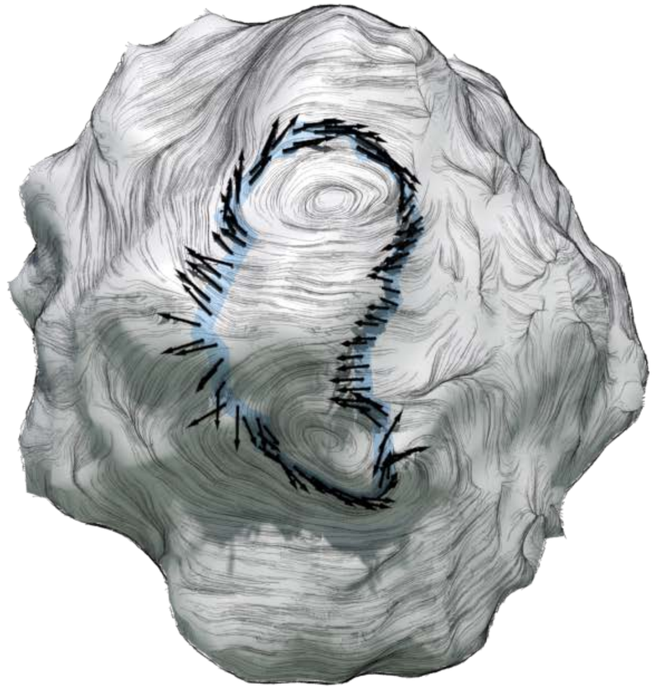




Poincaré index in angle defect

Recall:

$$(K\,dA)_i = 2\pi - \left(\begin{array}{l} \text{turning angle} \\ \text{of a loop around} \\ \text{the point i} \end{array} \right)$$



$$\text{Total turning angle of } \mathbf{v} = 2\pi \sum_{\substack{\text{singularities} \\ \text{enclosed}}} \text{index} - \int_{\substack{\text{region} \\ \text{enclosed}}} K\,dA$$

Index observation from a distance

Proof of Poincaré–Hopf theorem

Theorem (General hairy ball theorem)

Every continuous vector field on a closed surface must have total index equal to the Euler characteristic.

$$\text{Total turning angle of } \mathbf{v} = 2\pi \sum_{\substack{\text{singularities} \\ \text{enclosed}}} \text{index} - \int_{\substack{\text{region} \\ \text{enclosed}}} K \, dA$$

Let the enclosed region be the entire surface. Apply $\int_M K \, dA = 2\pi\chi$

Revising the notion of function

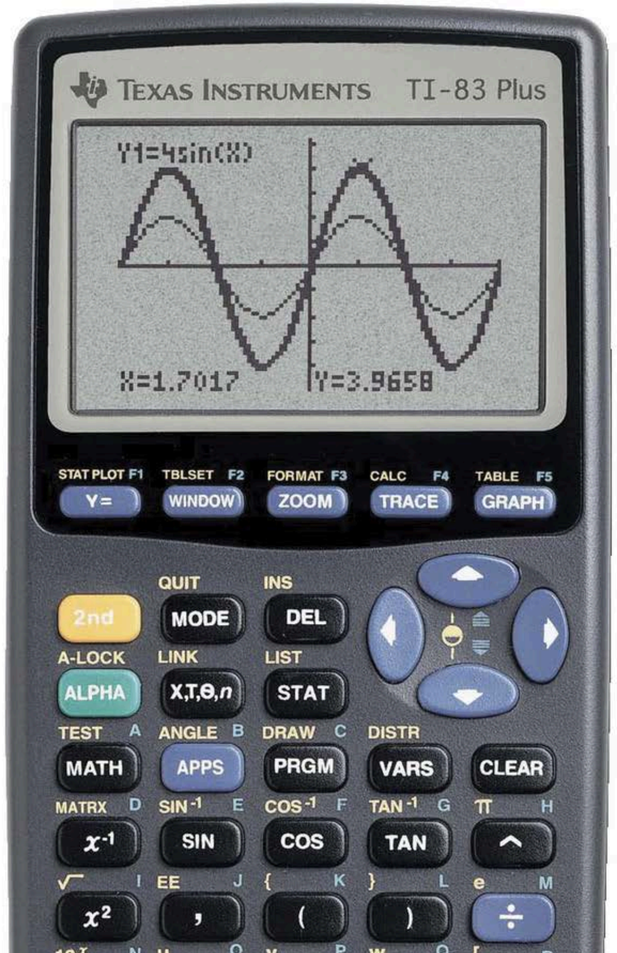
Theorem (General hairy ball theorem)

Every continuous vector field on a closed surface must have total index equal to the Euler characteristic.

- If vector field is like a function over a surface, what prevents us from assigning nonzero value everywhere?
- Unlike a traditional function $f : S_1 \to S_2$
  a vector field $\mathbf{u}$ takes value in different spaces $\mathbf{u}|_{\mathbf{x}} \in T_{\mathbf{x}}M$
- In fact a more unifying picture to talk about "function" is to look at the *function graph*.

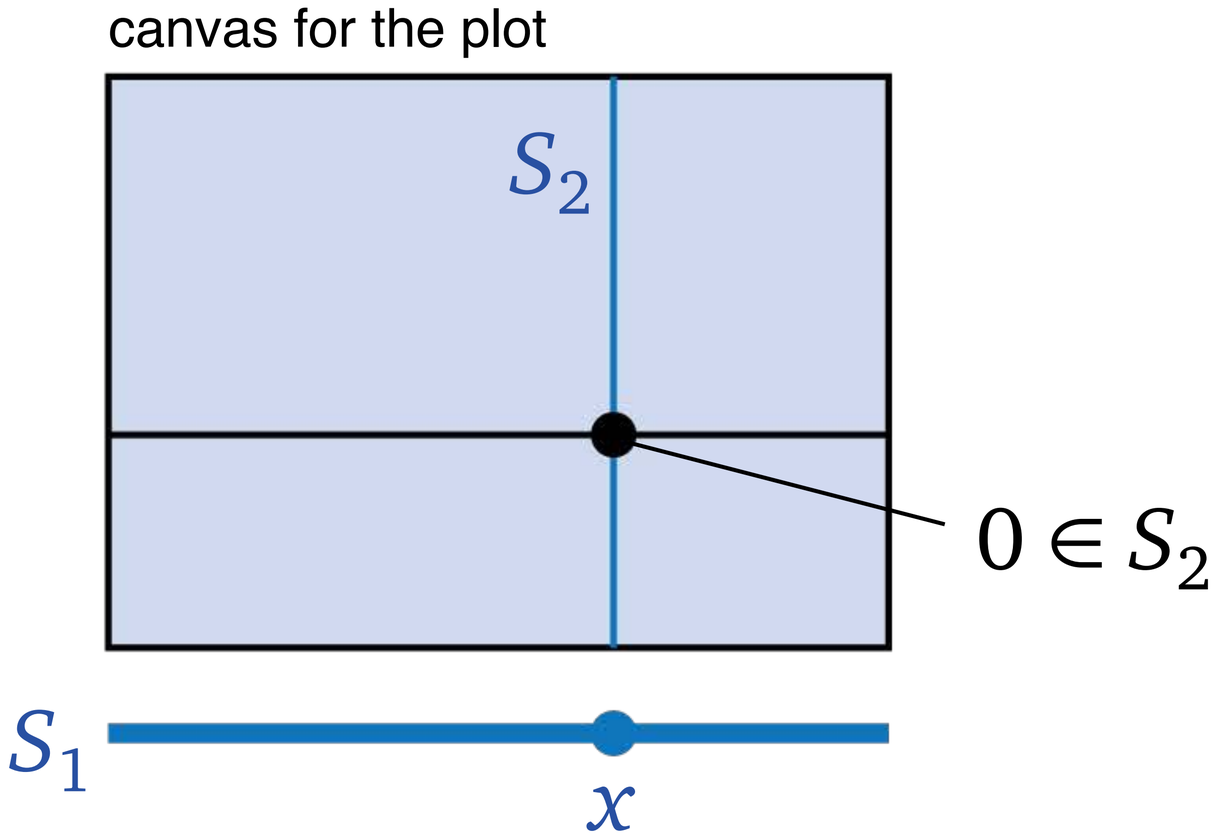
Revising the notion of function

Plotting $f : S_1 \to S_2$



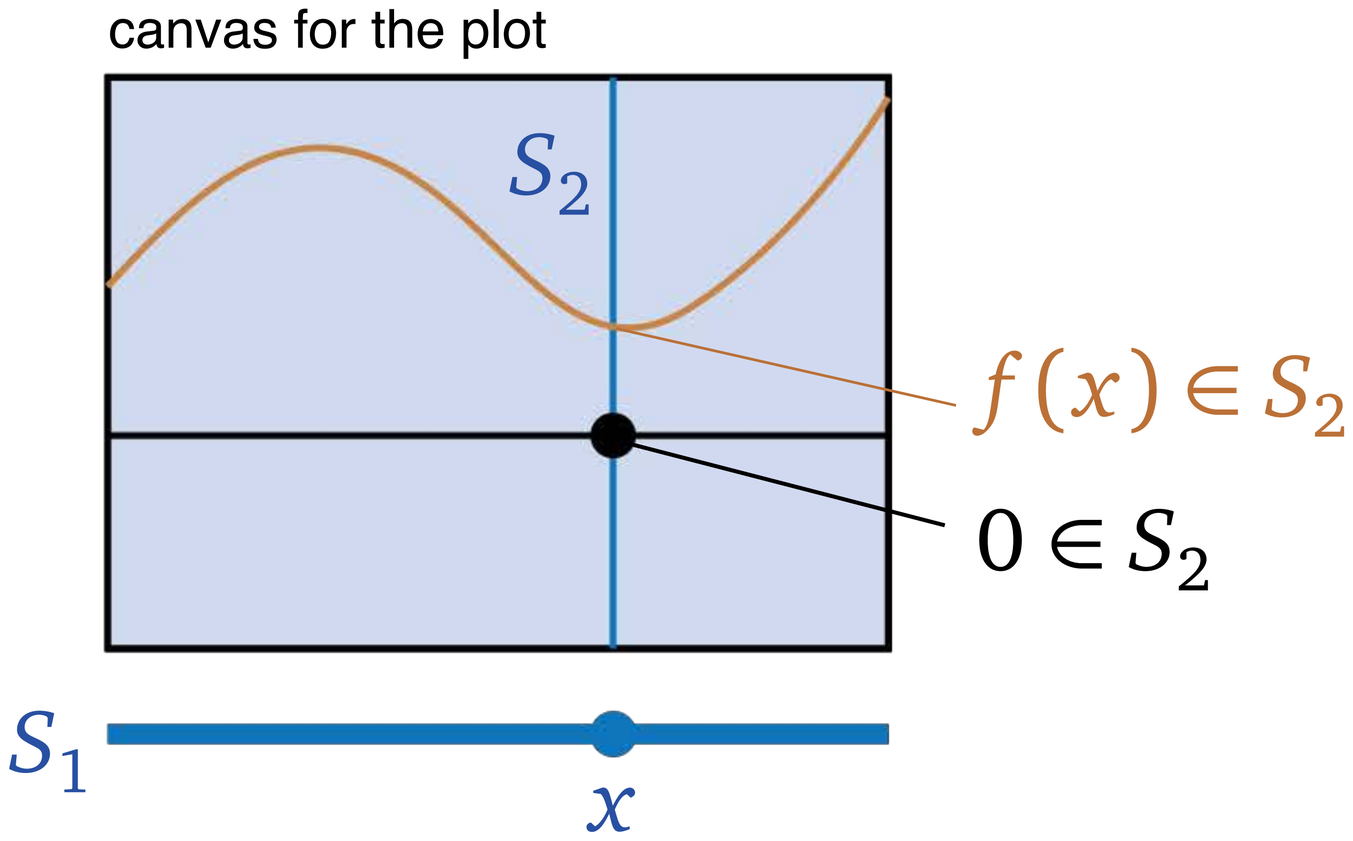
Revising the notion of function

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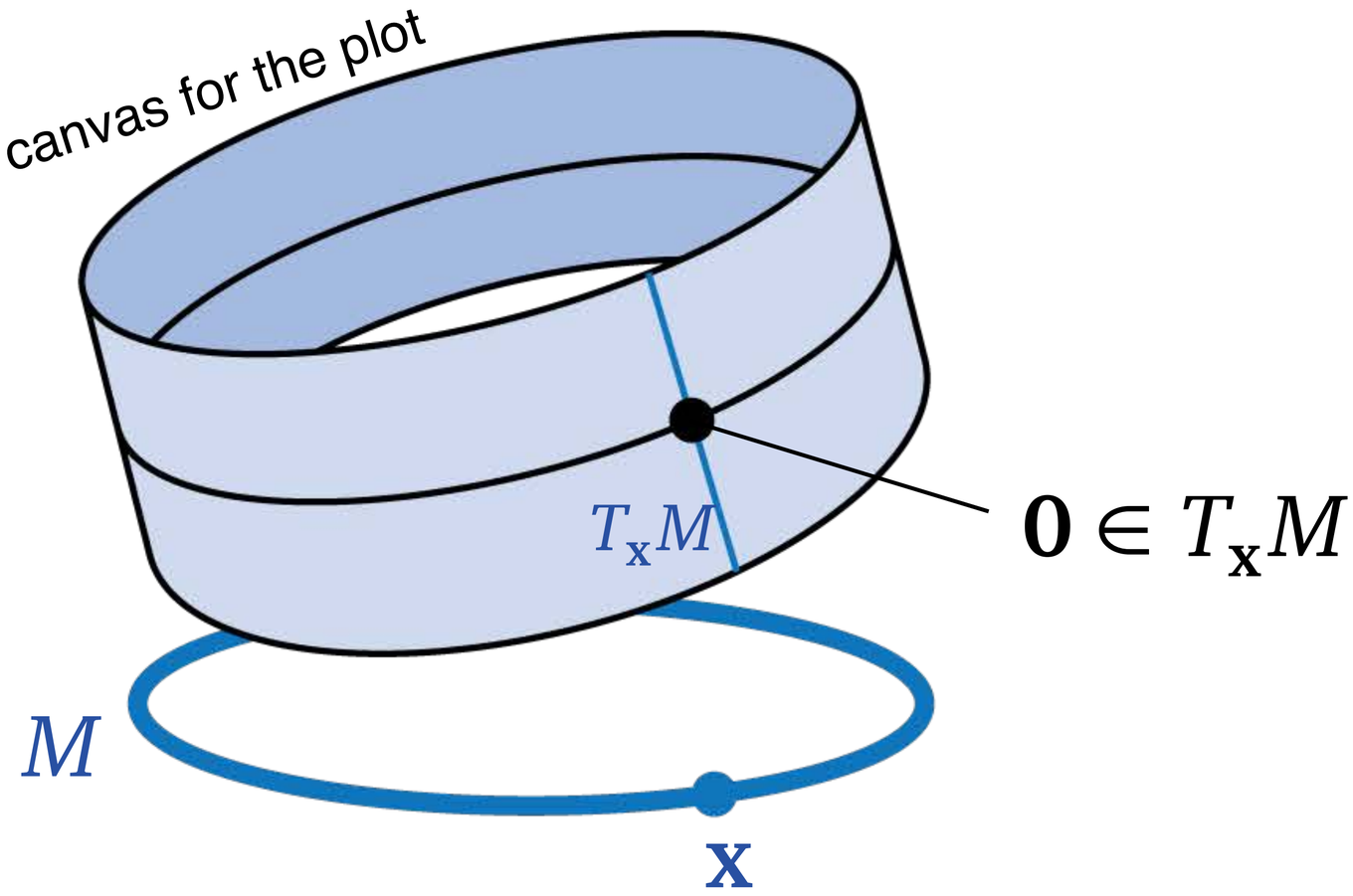
Revising the notion of function

Plotting $f : S_1 \to S_2$



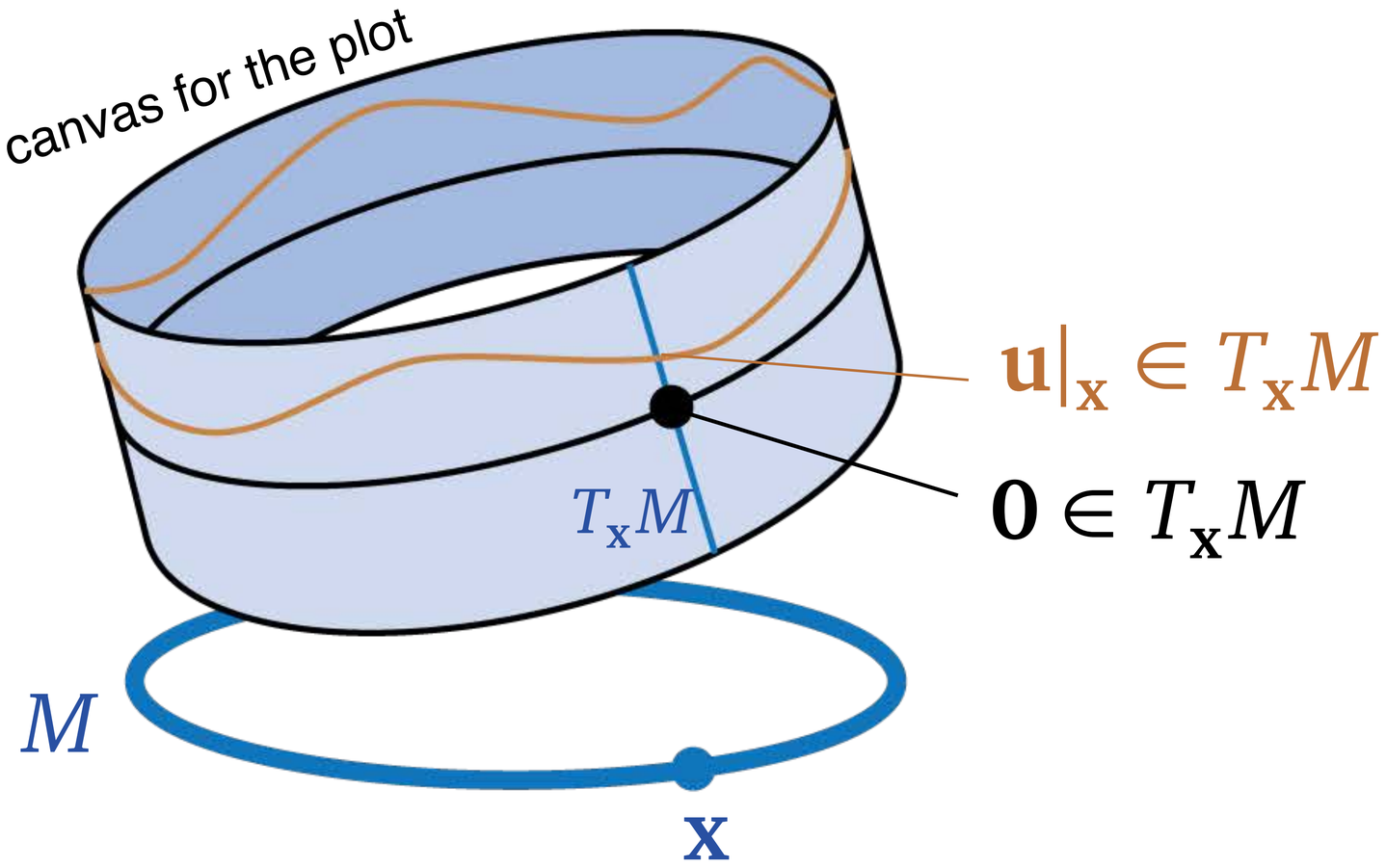
Revising the notion of function

Plotting a tangent vector field $\mathbf{u}$ on M



Revising the notion of function

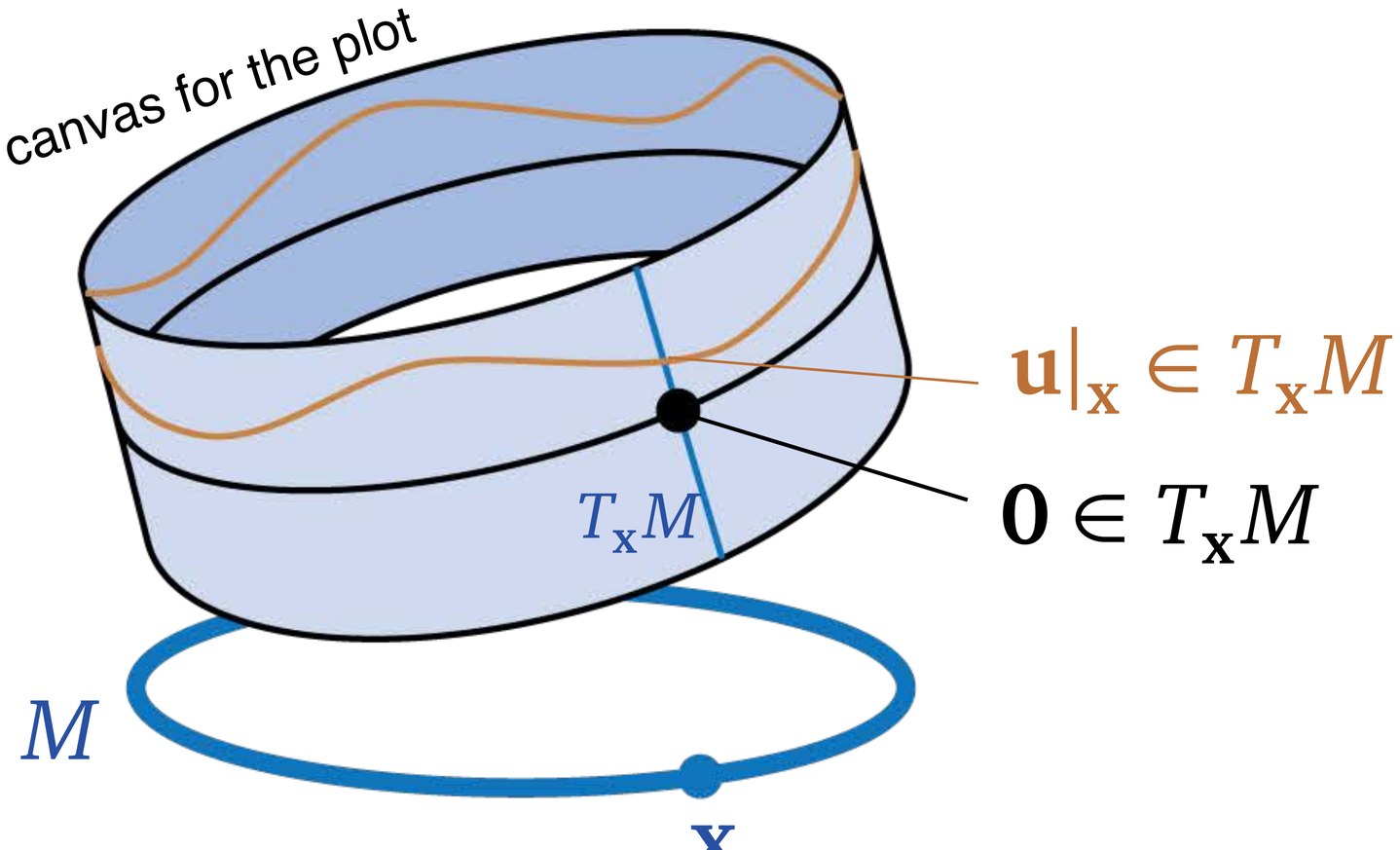
Plotting a tangent vector field $\mathbf{u}$ on M



Revising the notion of function

The “canvas” as a (higher-dimensional) manifold is called a **bundle**. Here we have the tangent bundle TM.

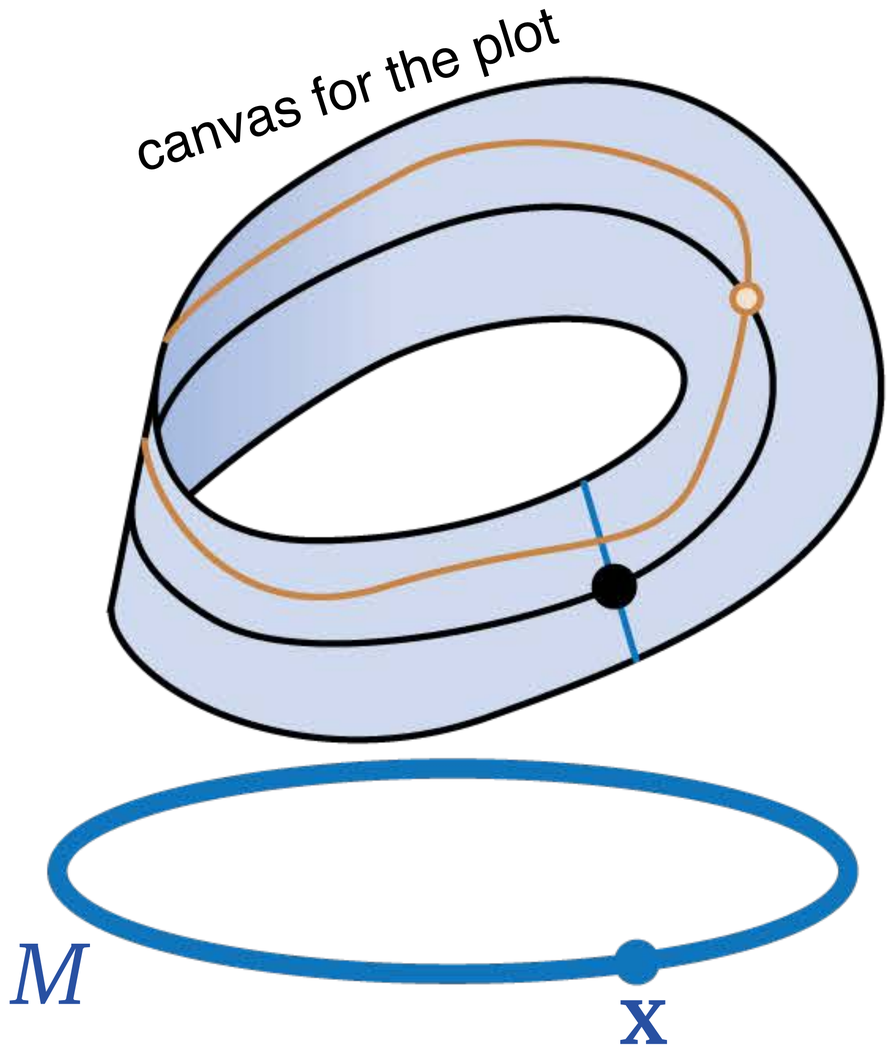
$\mathbf{u} \in \Gamma(TM)$ is called a section of the tangent bundle.



Twistedness of the canvas

Theorem (Hairy ball theorem)

The tangent bundle of a closed surface is twisted in such a way that every pair of sections must have the total signed intersection equals to the Euler characteristic.



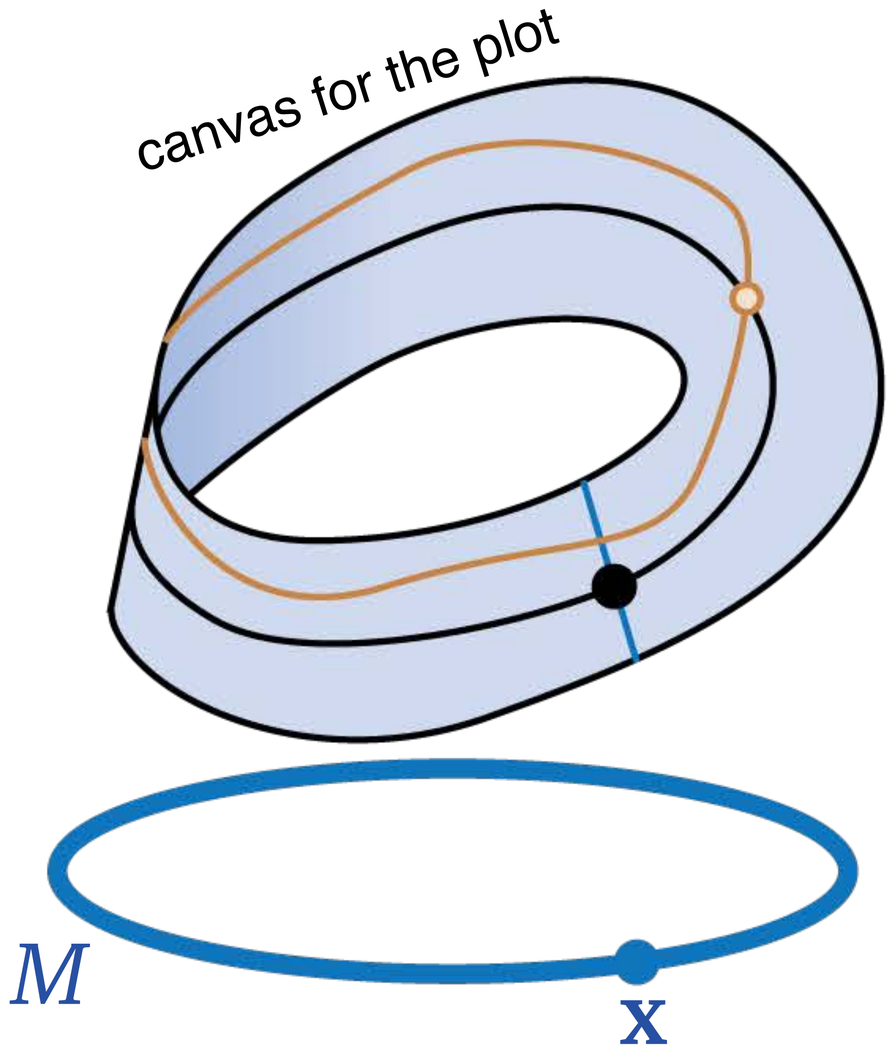
Optimal Sections of a Bundle

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Optimal section

Goal Find the optimal section of a bundle.

- Smoothest vector field
- Patterns best align with prescribed directions
- Automatically finds the optimal singularity locations.



Optimal section

Smoothest tangent vector fields

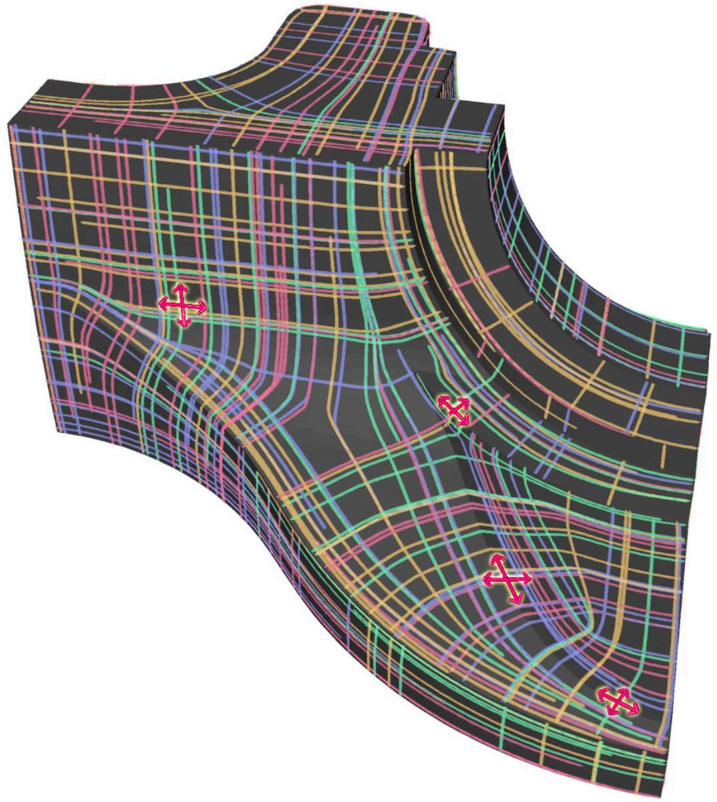


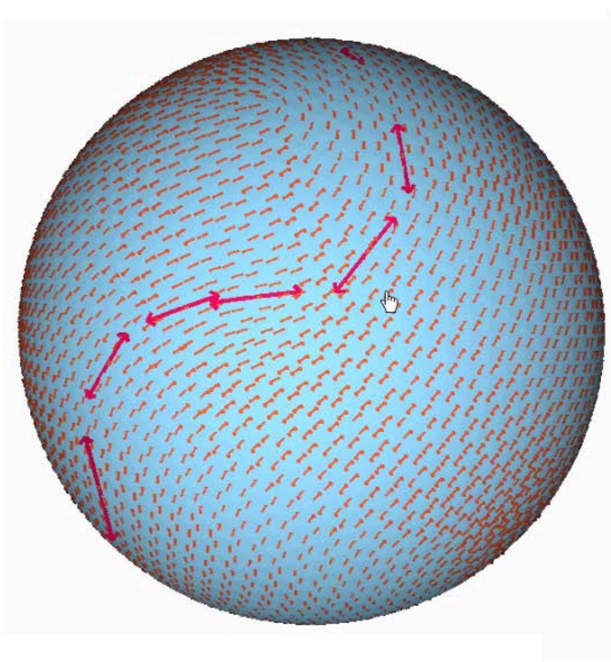


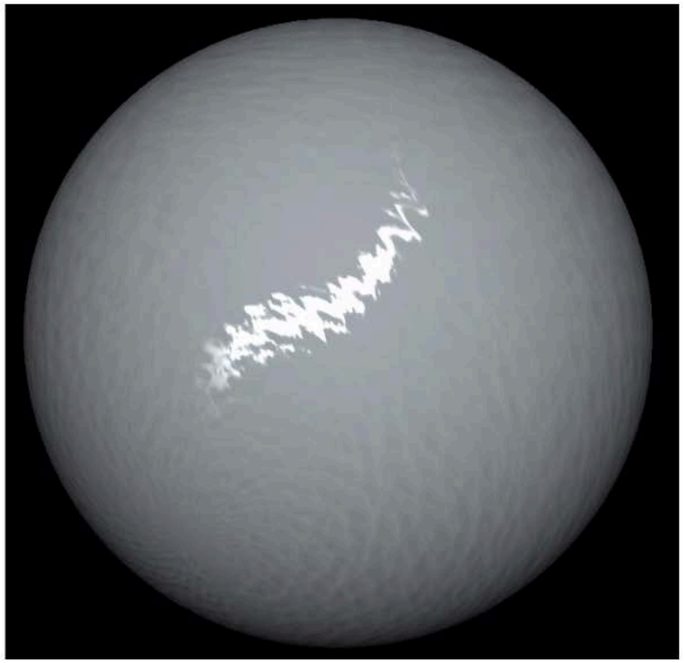
[Knöppel, Crane, Pinkall & Schröder 2013]
Globally optimal direction field

Optimal section

Impainting n-direction field



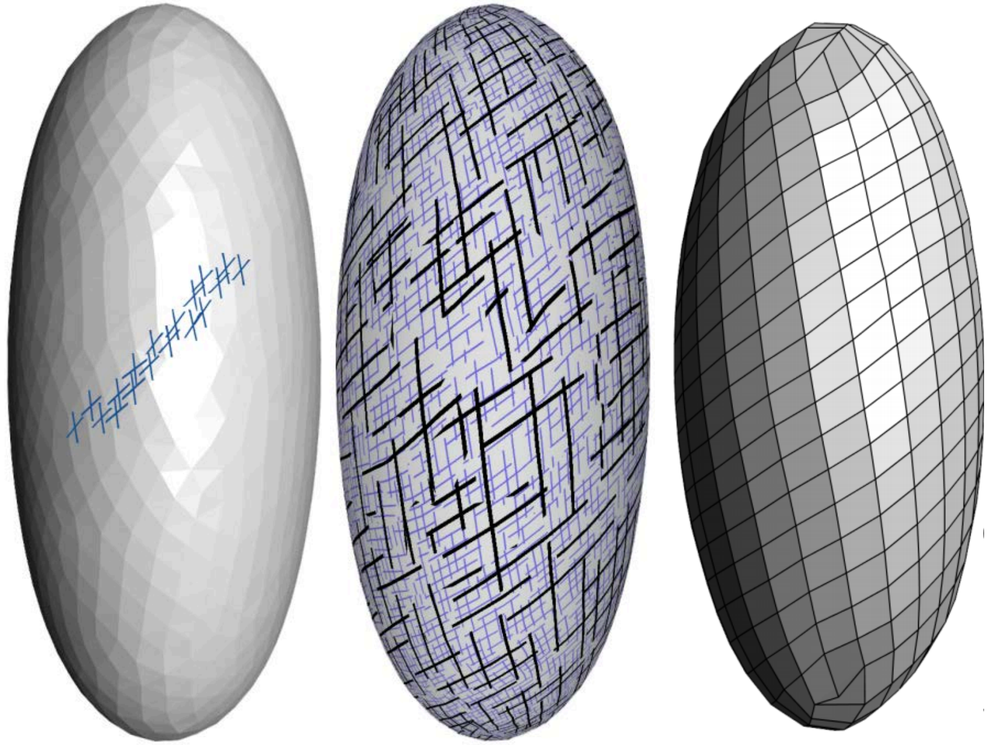




[Brandt, Scandolo, Eisemann & Hildebrandt 2018]
Modeling n-symmetric vector fields using higher-order energies

Optimal section

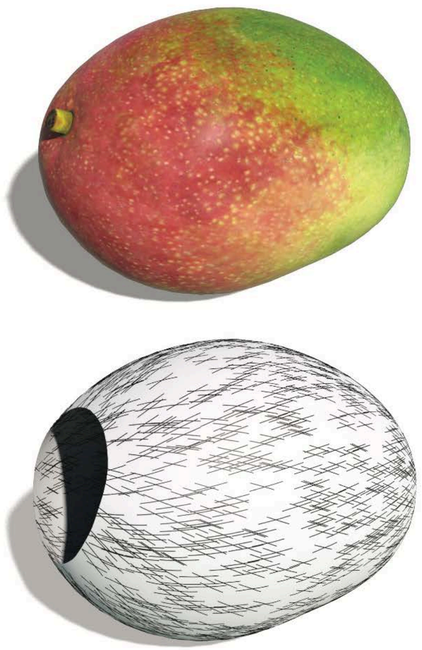
Non-orthogonal cross fields

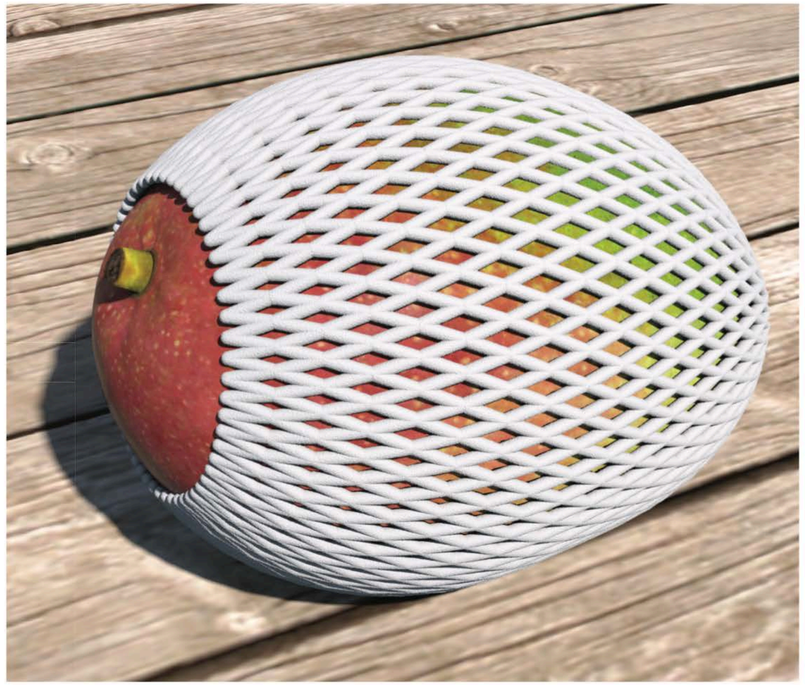


[Diamanti, Vaxman, Panozzo & Sorkine-Hornung 2014]
Designing N-PolyVector fields with complex polynomials

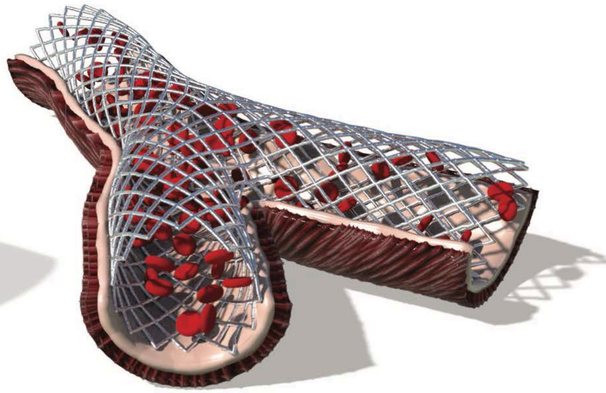
Optimal section

Non-orthogonal cross fields with physical constraints







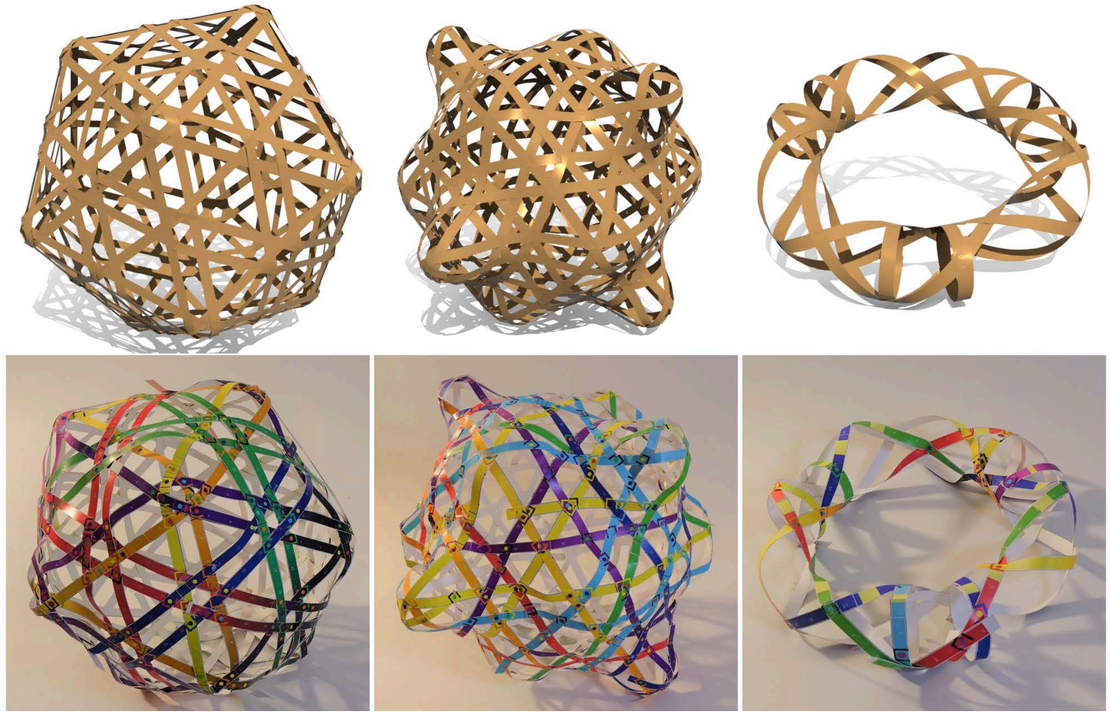


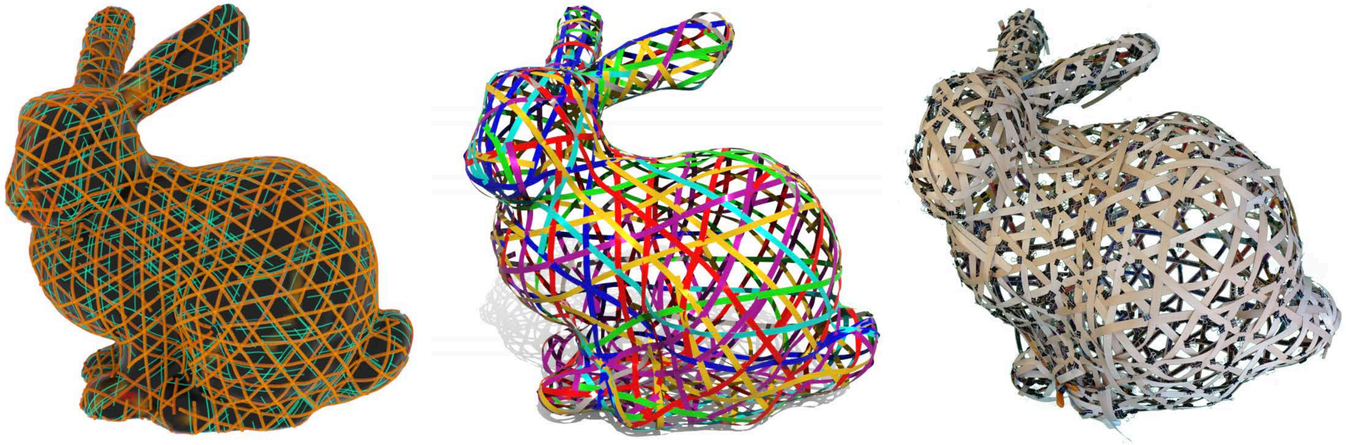


[Sageman-Furnas, Chern, Ben-Chen & Vaxman 2019]
Chebyshev nets from commuting polyvector field

Optimal section

N-direction field with physical constraints





[Vekhter, Zhuo, Fandino, Huang & Vouga 2019]
Weaving geodesic foliations

Optimal section

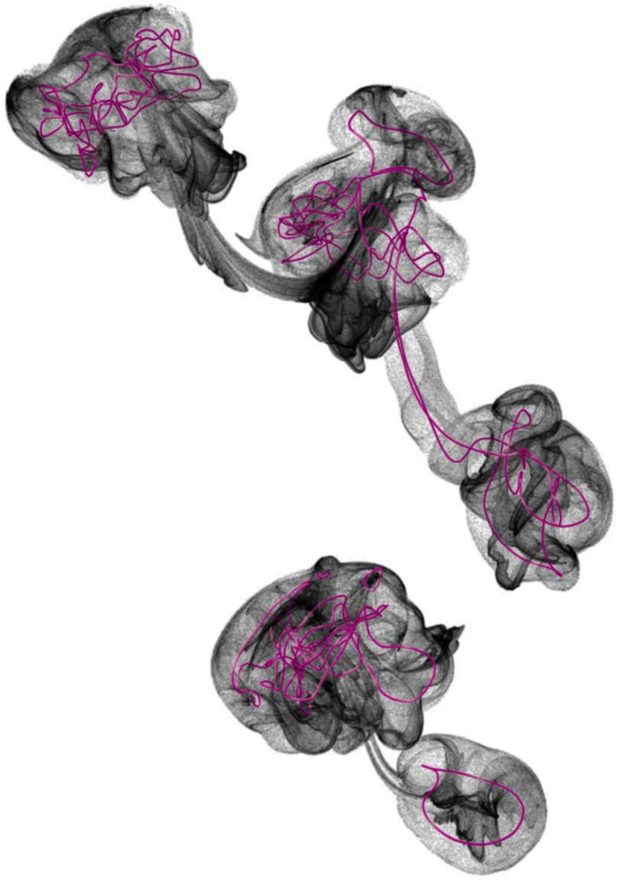
Complex phase field



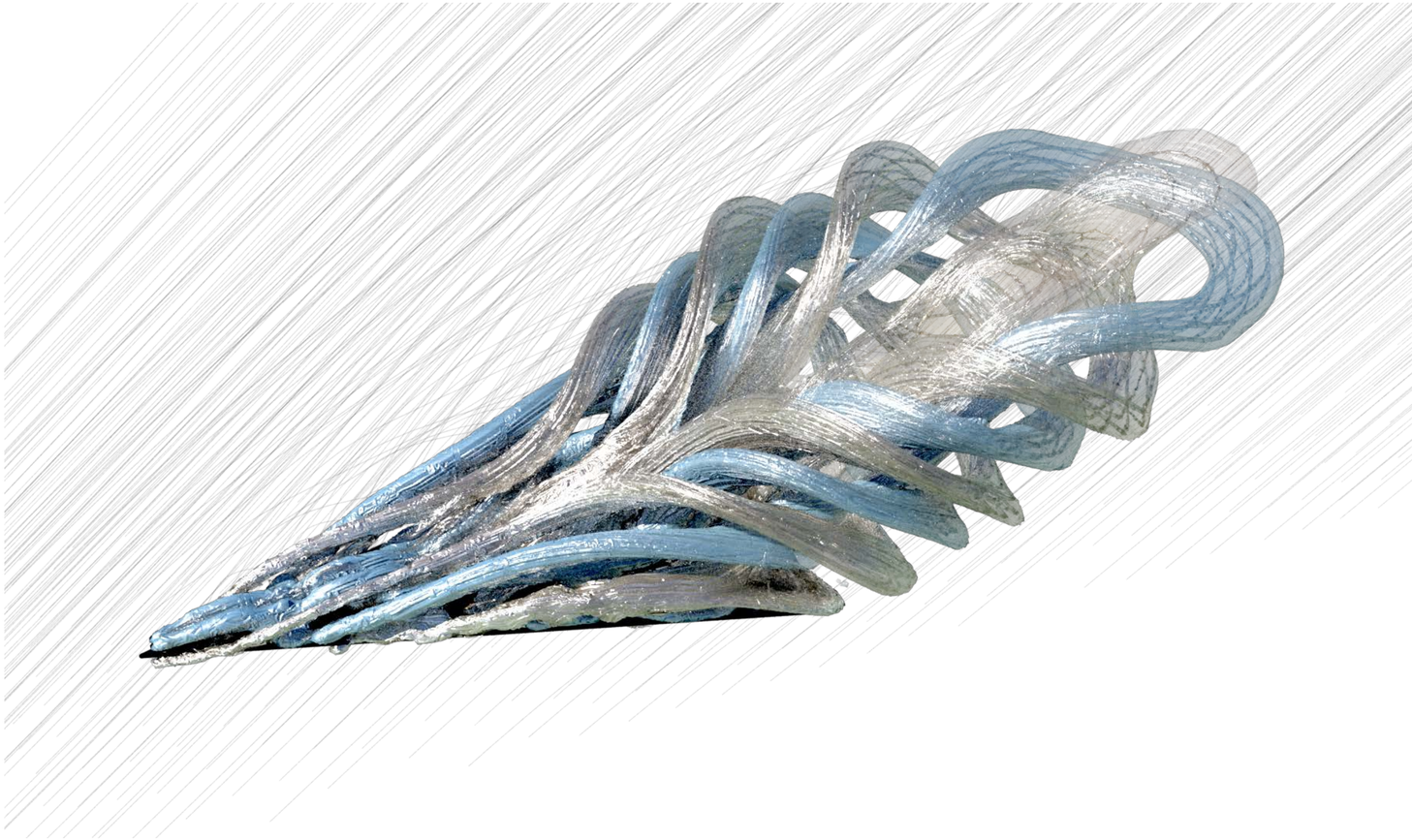
[Knöppel, Crane, Pinkall & Schröder 2015]
Stripe patterns on surfaces

Optimal section

Complex phase field for vortex detection



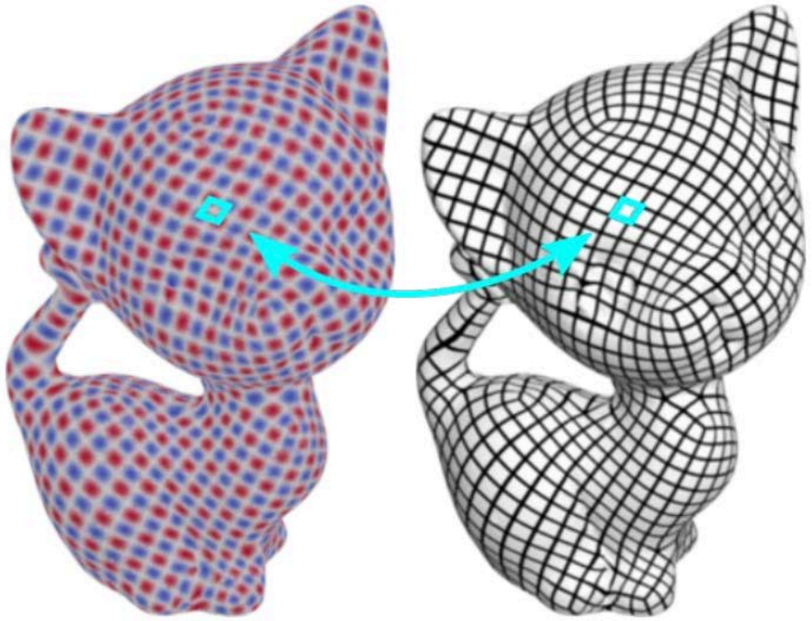
[Weißmann, Pinkall & Schröder 2014]
Smoke rings from smoke

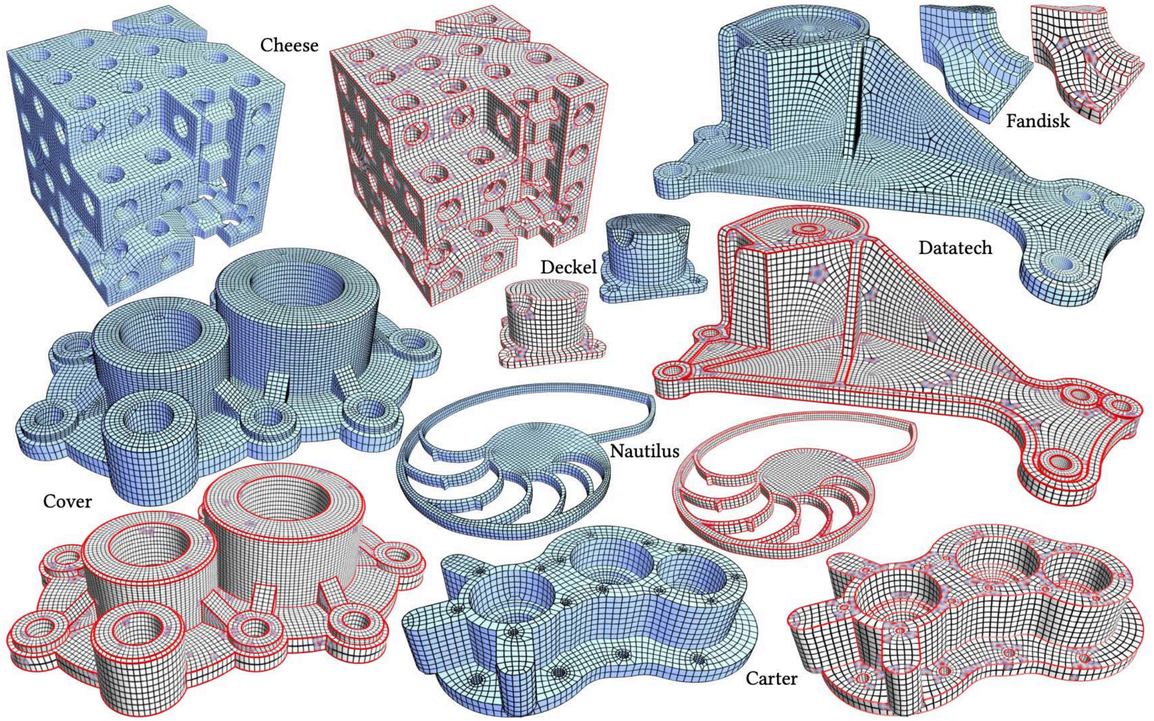


[Chern, Knöppel, Pinkall & Schröder 2017]
Inside fluid: Clebsch maps for visualization and processing

Optimal section

$\mathbb{C}^2$ or $\mathbb{H}$ phase field for quad meshing

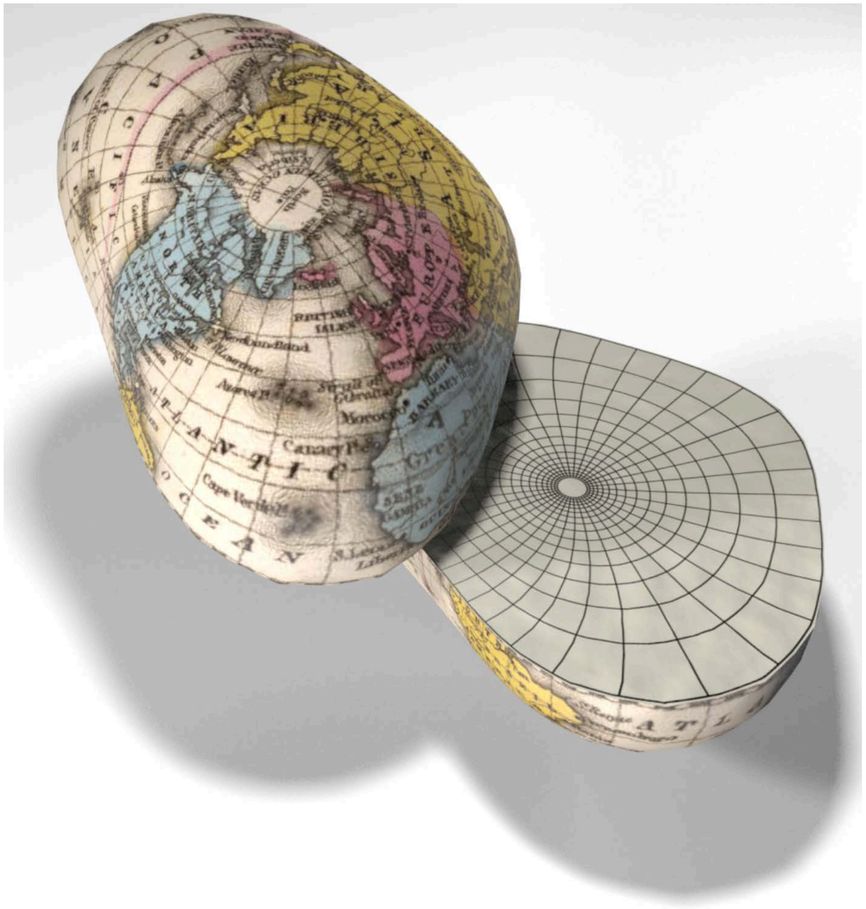


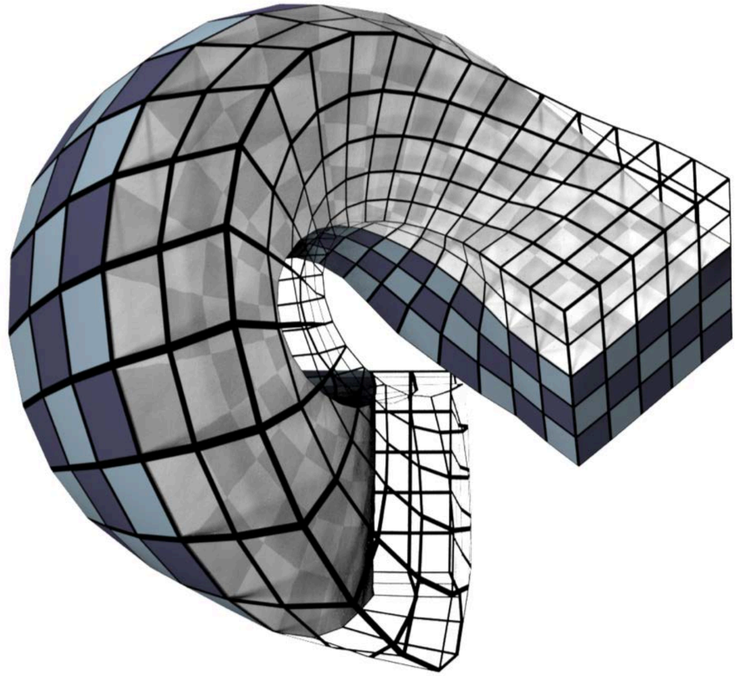


[Fang, Bao, Tong, Desbrun & Huang 2018]
Quadrangulation through Morse-parametrization hybridization

Optimal section

$\mathbb{H}$ phase field for nearly conformal deformations

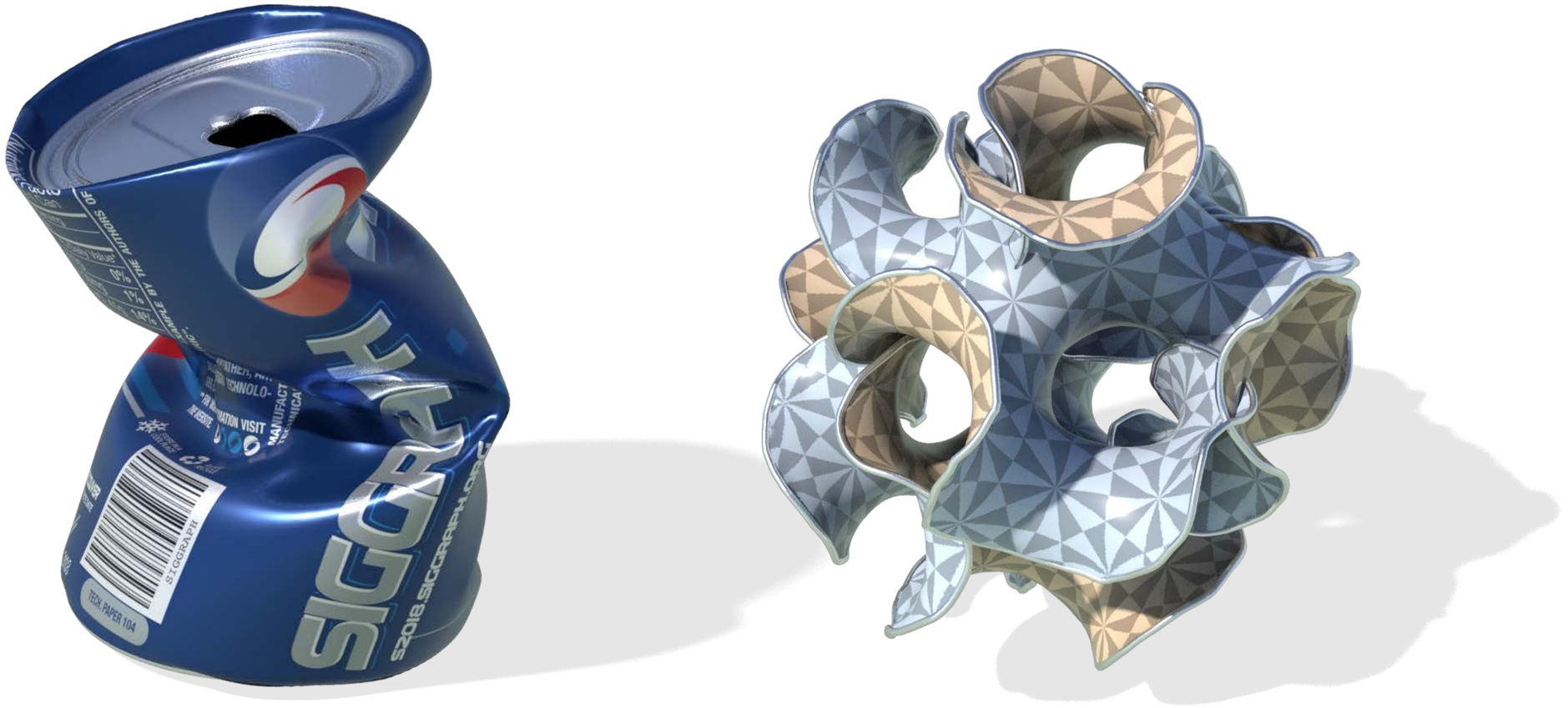




[Chern, Pinkall & Schröder 2015]
Close-to-conformal deformation of volumes

Optimal section

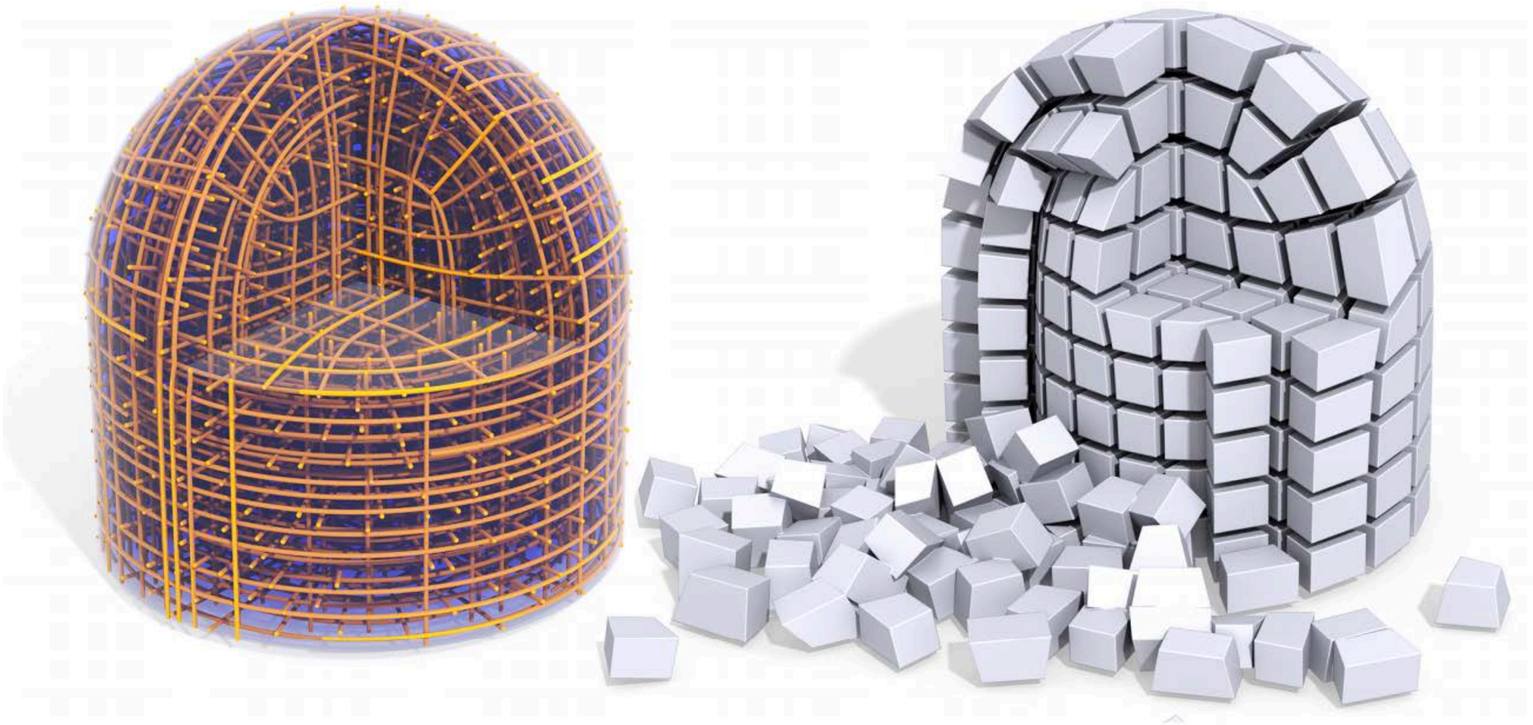
$\mathbb{H}$ spinor field for nearly isometric reconstruction



[Chern, Knöppel, Pinkall & Schröder 2018]
Shape from metric

Optimal section

Frame field for hexahedral meshing

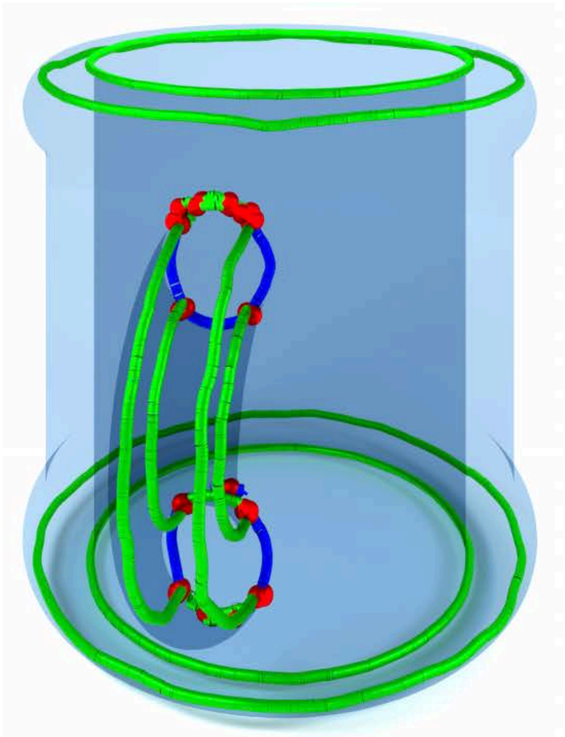


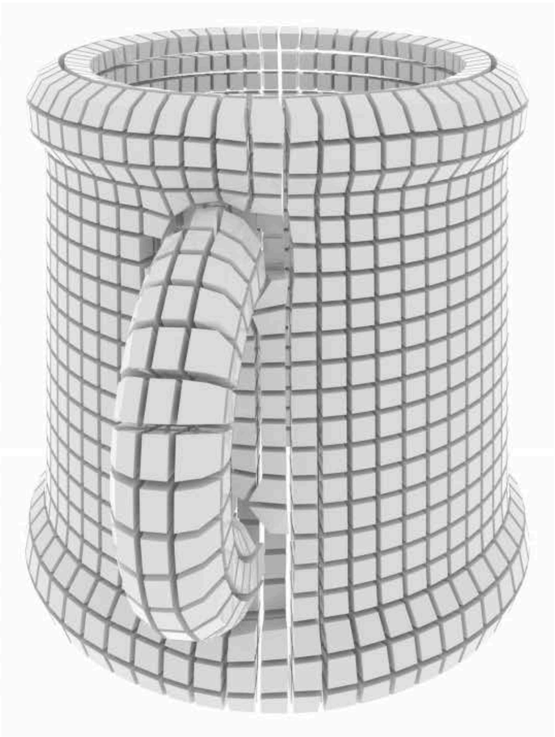
[Corman & Crane 2019]
Symmetric moving frame

Optimal section

Octahedral variety phase field for hexahedral meshing



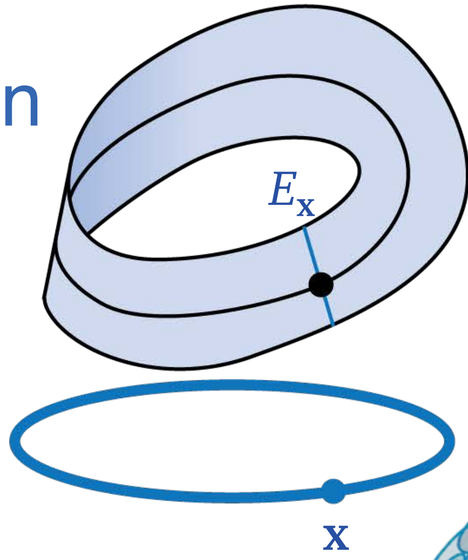


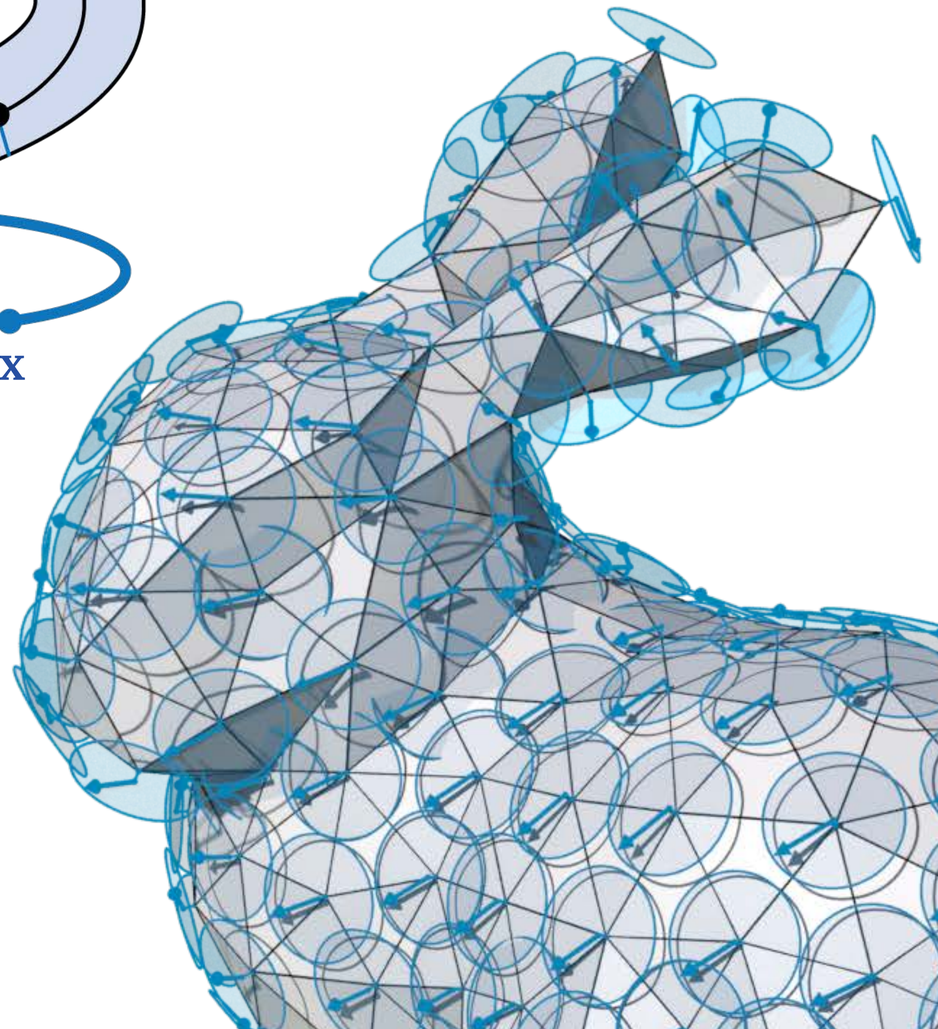


[Palmer, Bommes & Solomon 2020]
Algebraic representations for volumetric frame fields

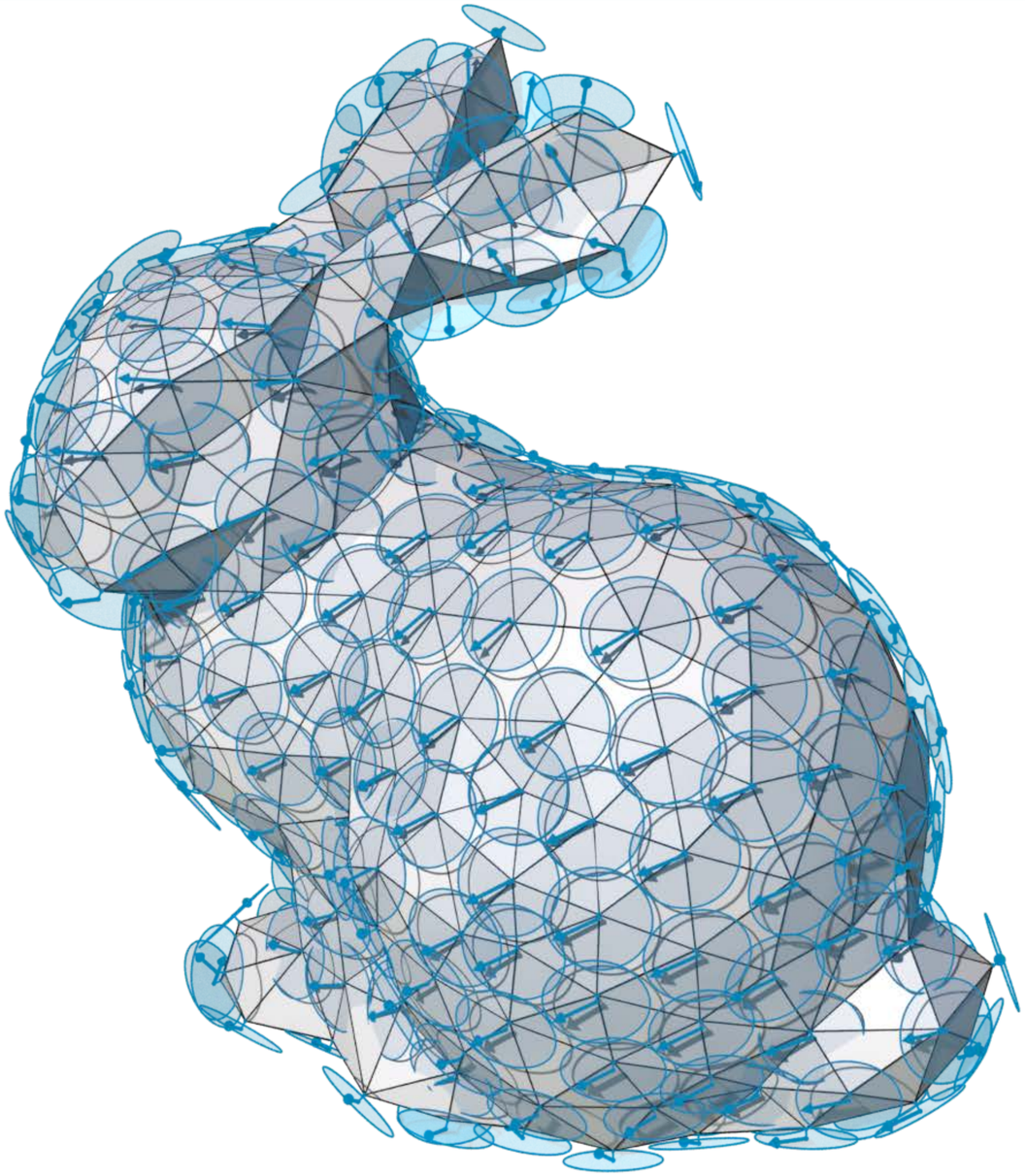
Common ingredient: gauge theory

- Find an algebraic representation of the value space $E_{\mathbf{x}}$ at each $\mathbf{x} \in M$.
- Describe the ***connection (a.k.a. parallel transport)*** between adjacent value spaces
- The derivatives (finite differences) of the field respect the connections.
- Model some energy to minimize.

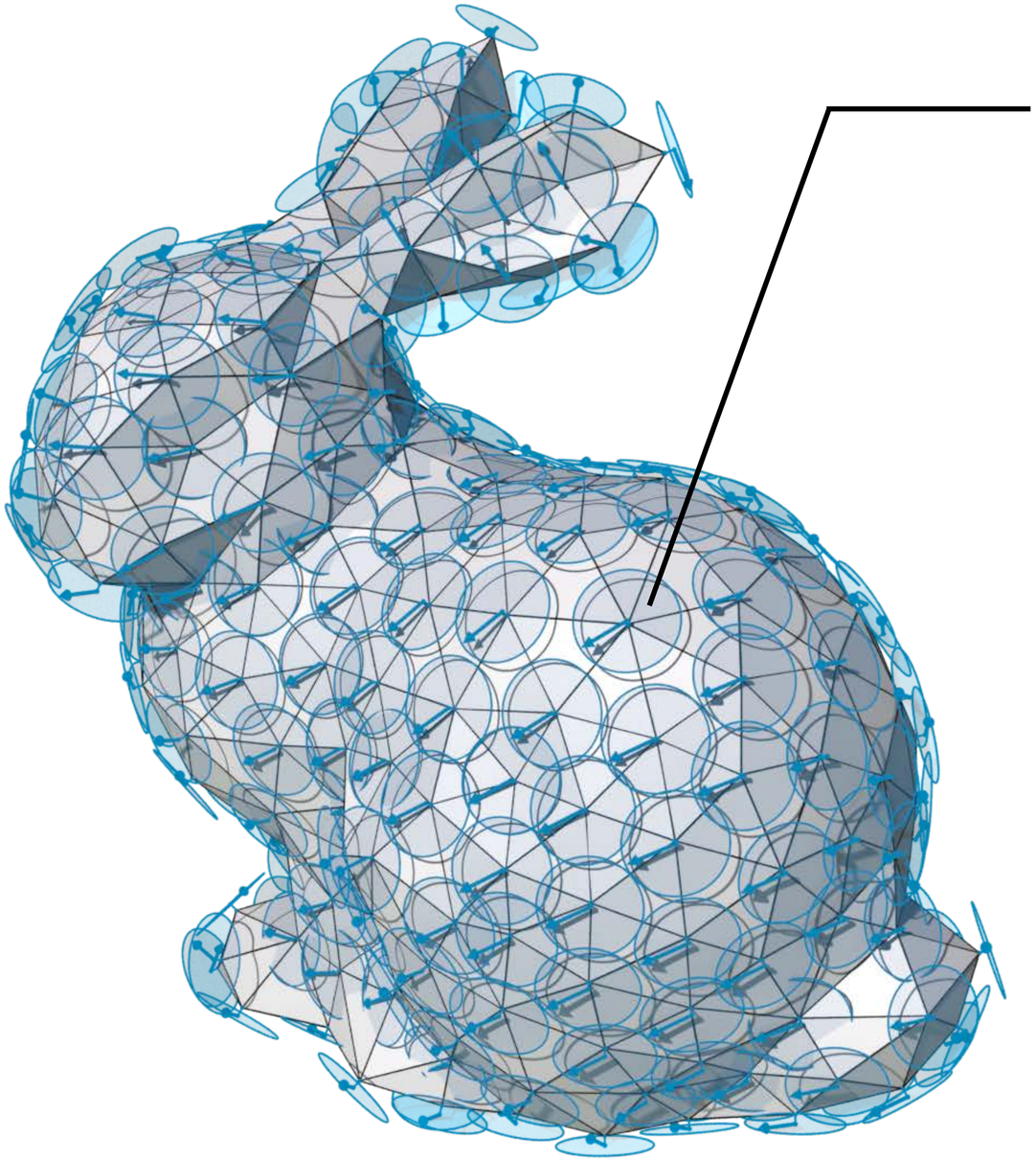




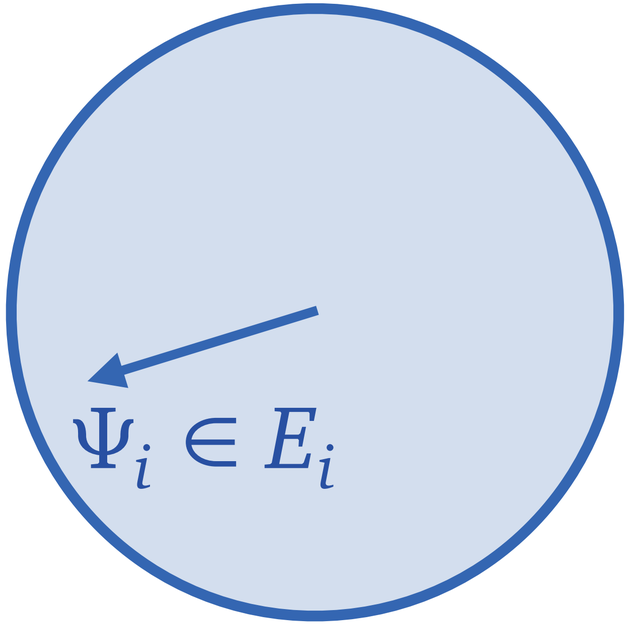
Discrete tangent bundle



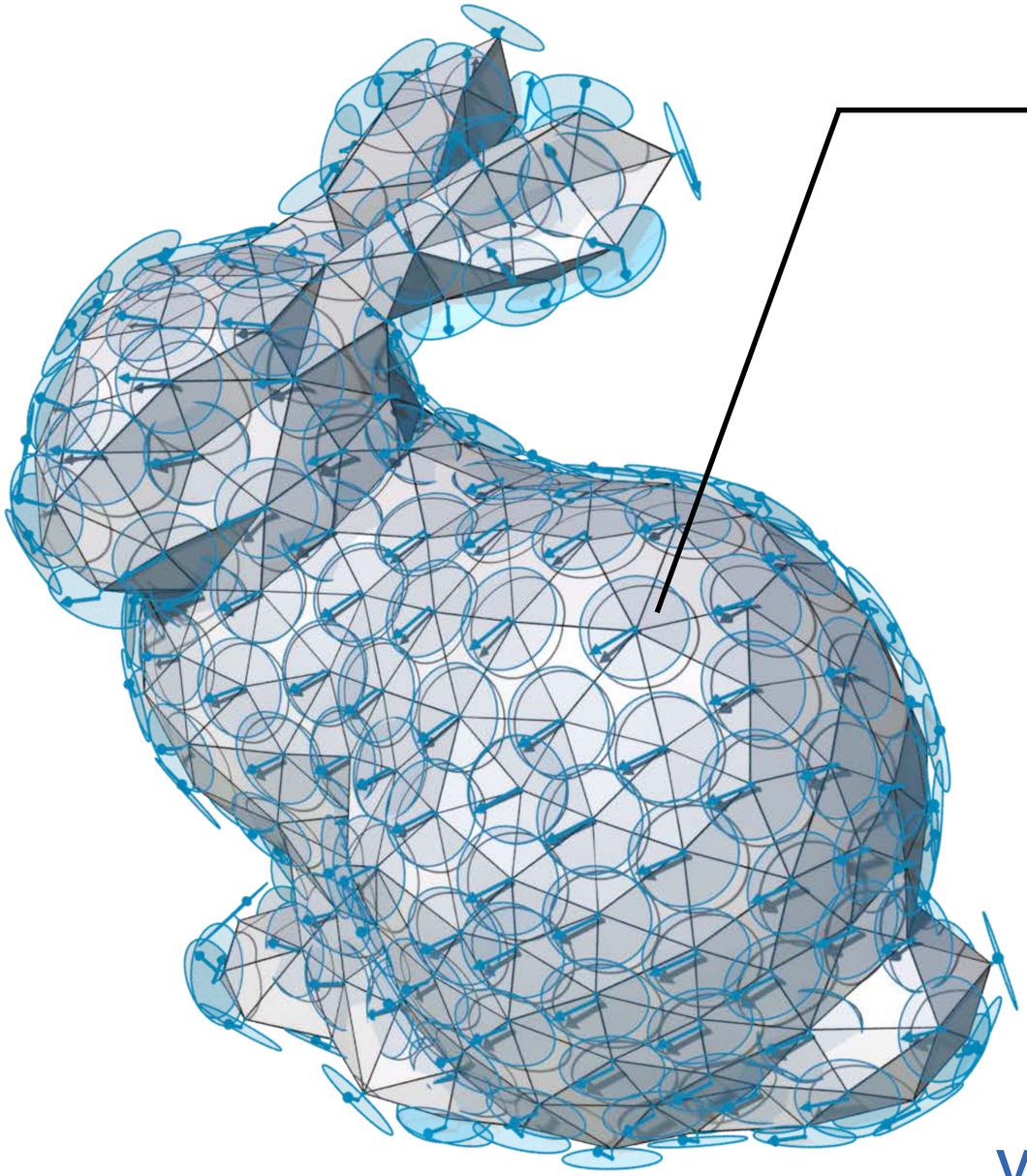
Discrete tangent bundle



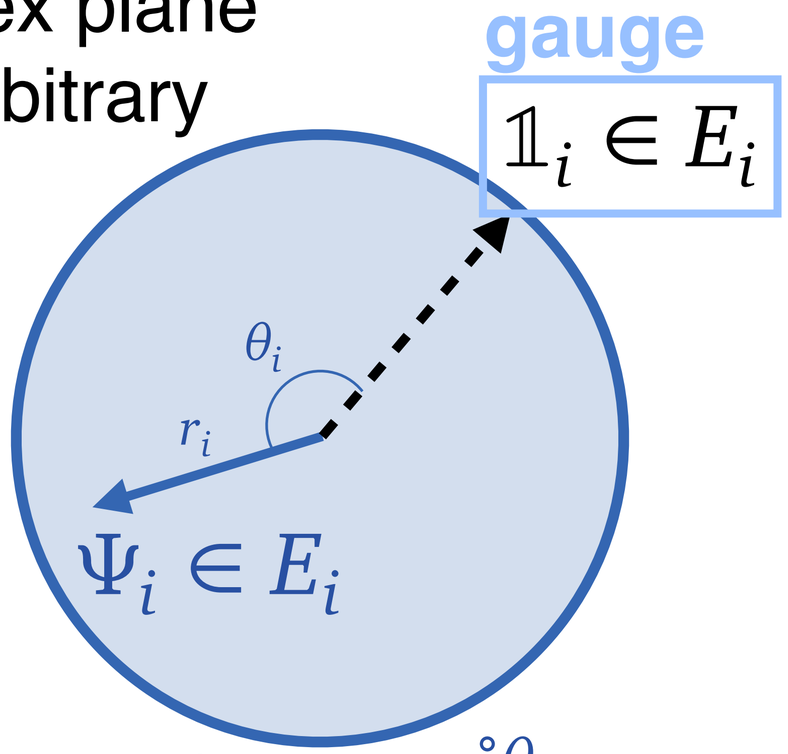
Each value space
E_i



Discrete tangent bundle



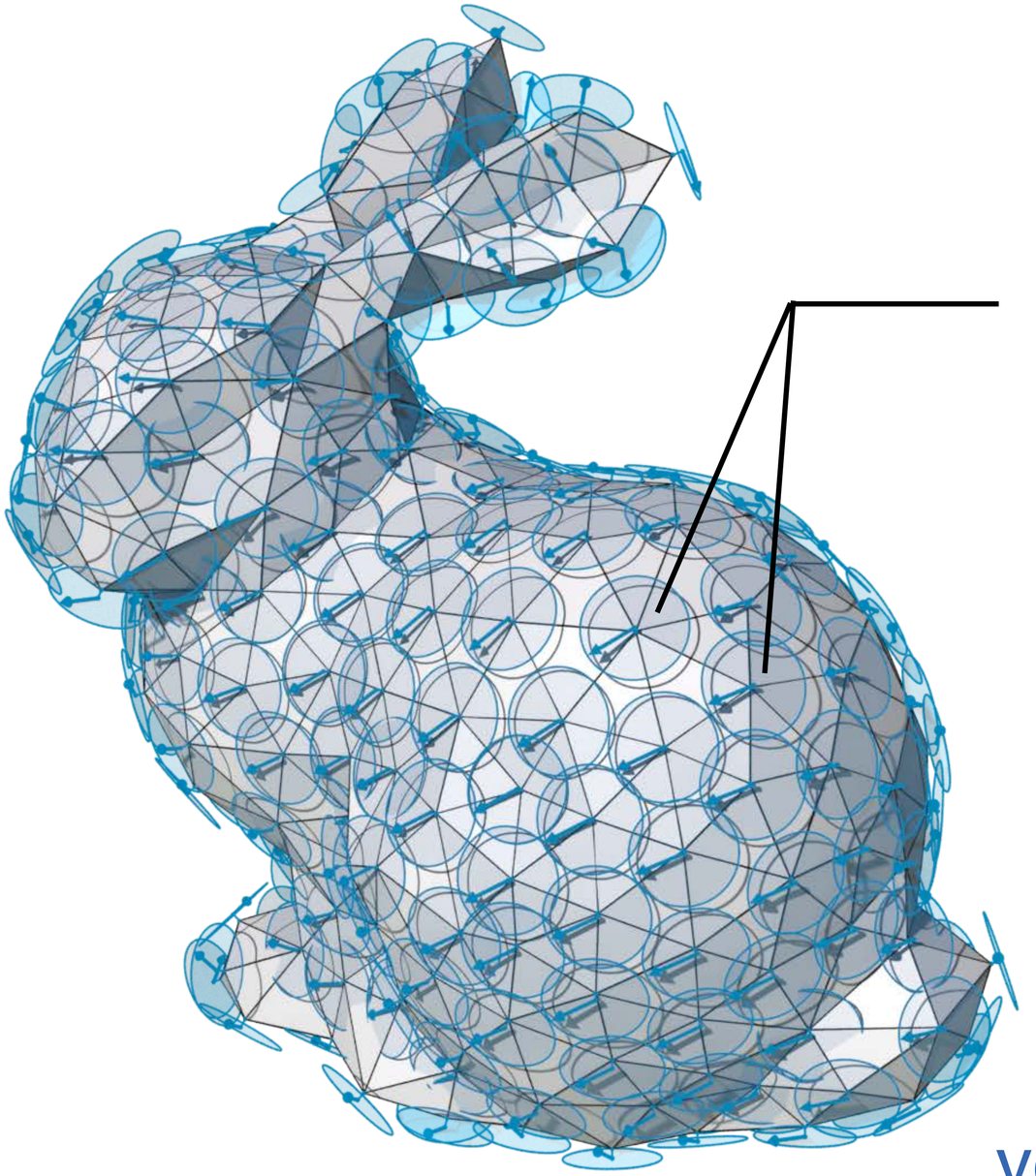
Each value space E_i is viewed as a complex plane via an arbitrary basis.



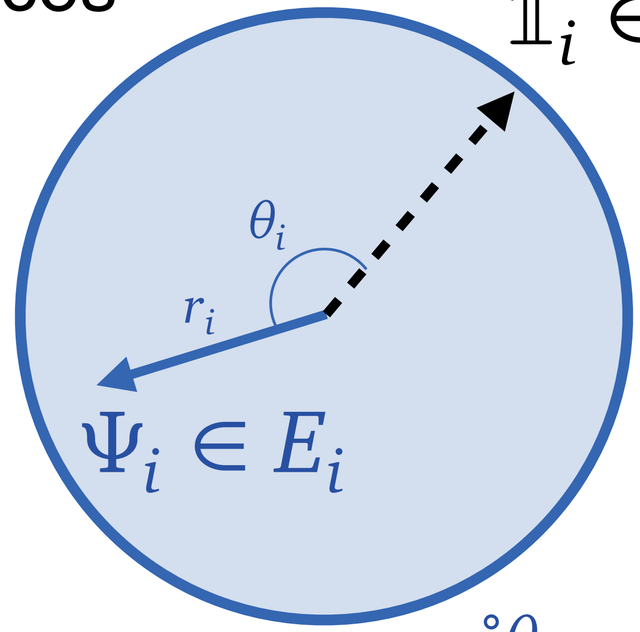
$$\Psi_i \longleftrightarrow \psi_i = r_i e^{\mathbb{i}\theta_i} \in \mathbb{C}$$

vector field $\longleftrightarrow$ $\mathbb{C}$-valued function

Discrete connection



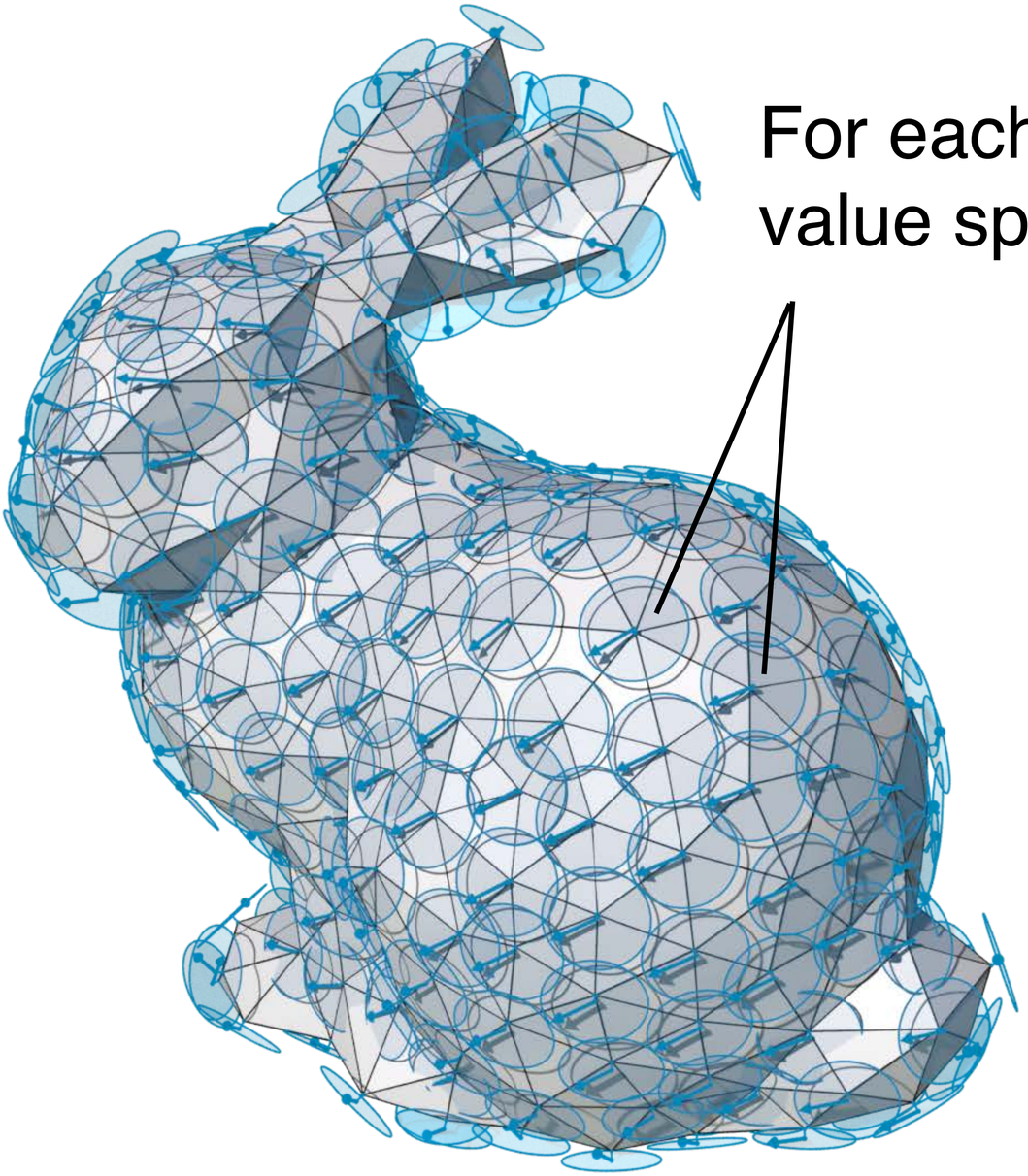
For each adjacent value spaces



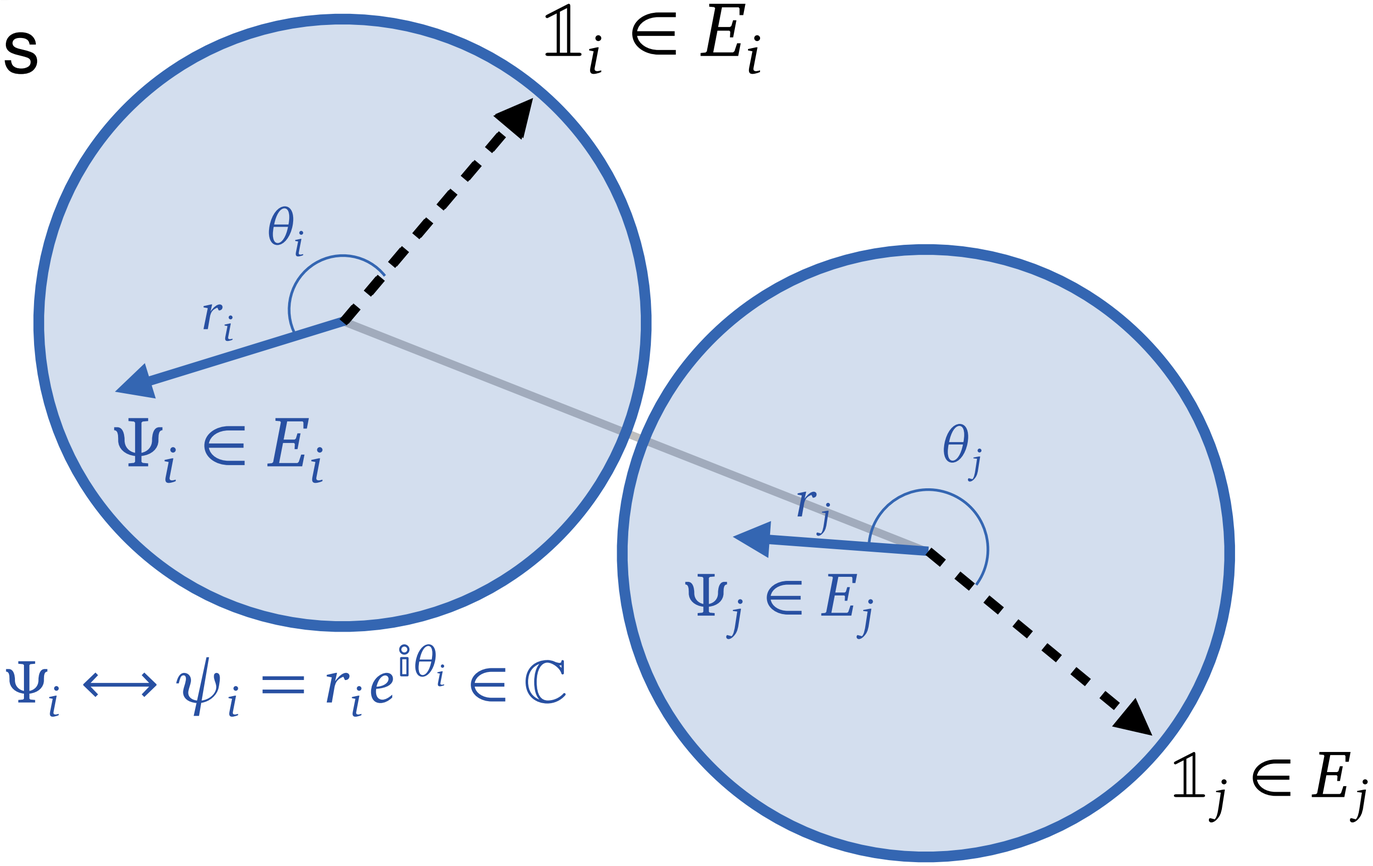
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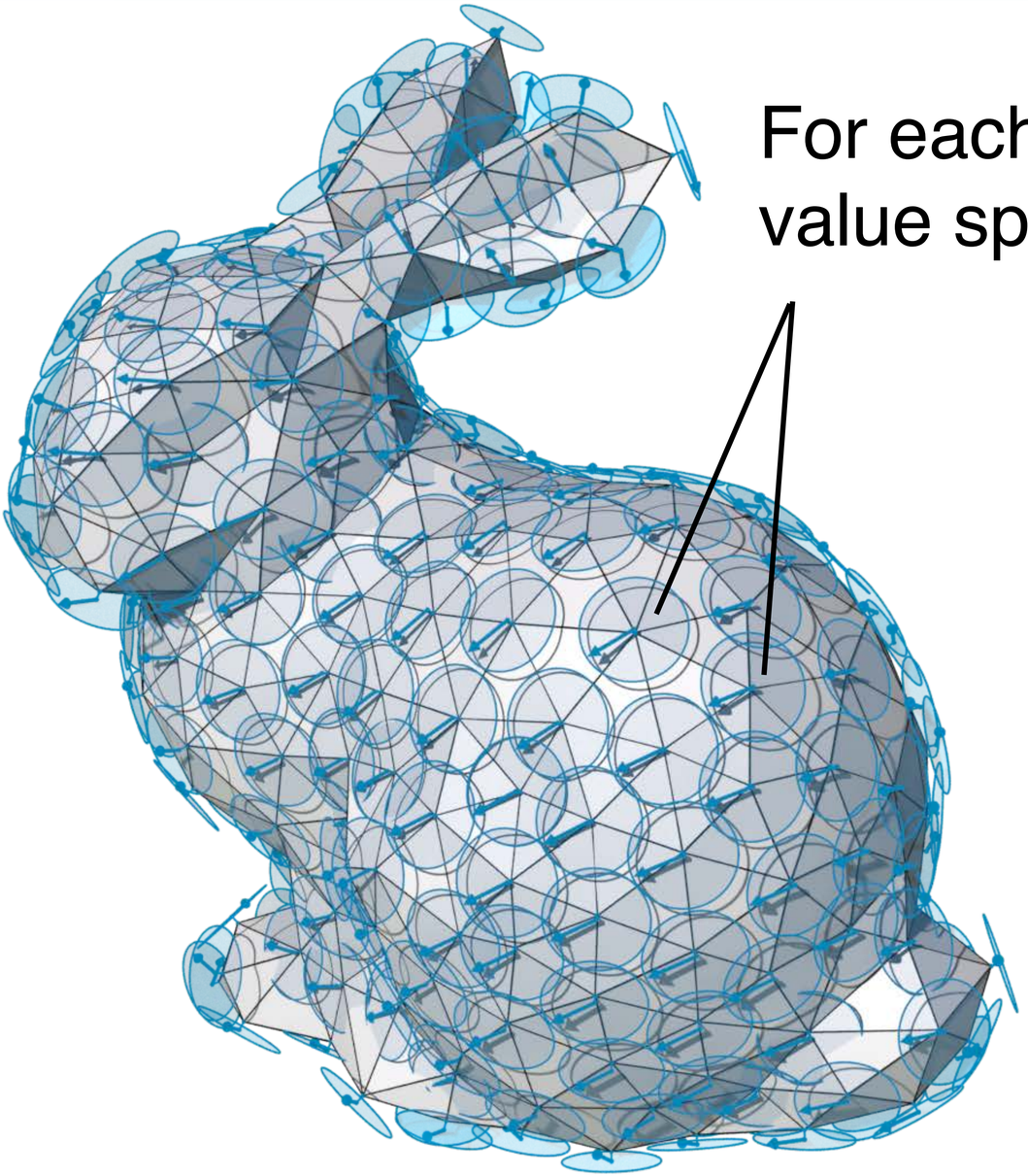
For each adjacent value spaces



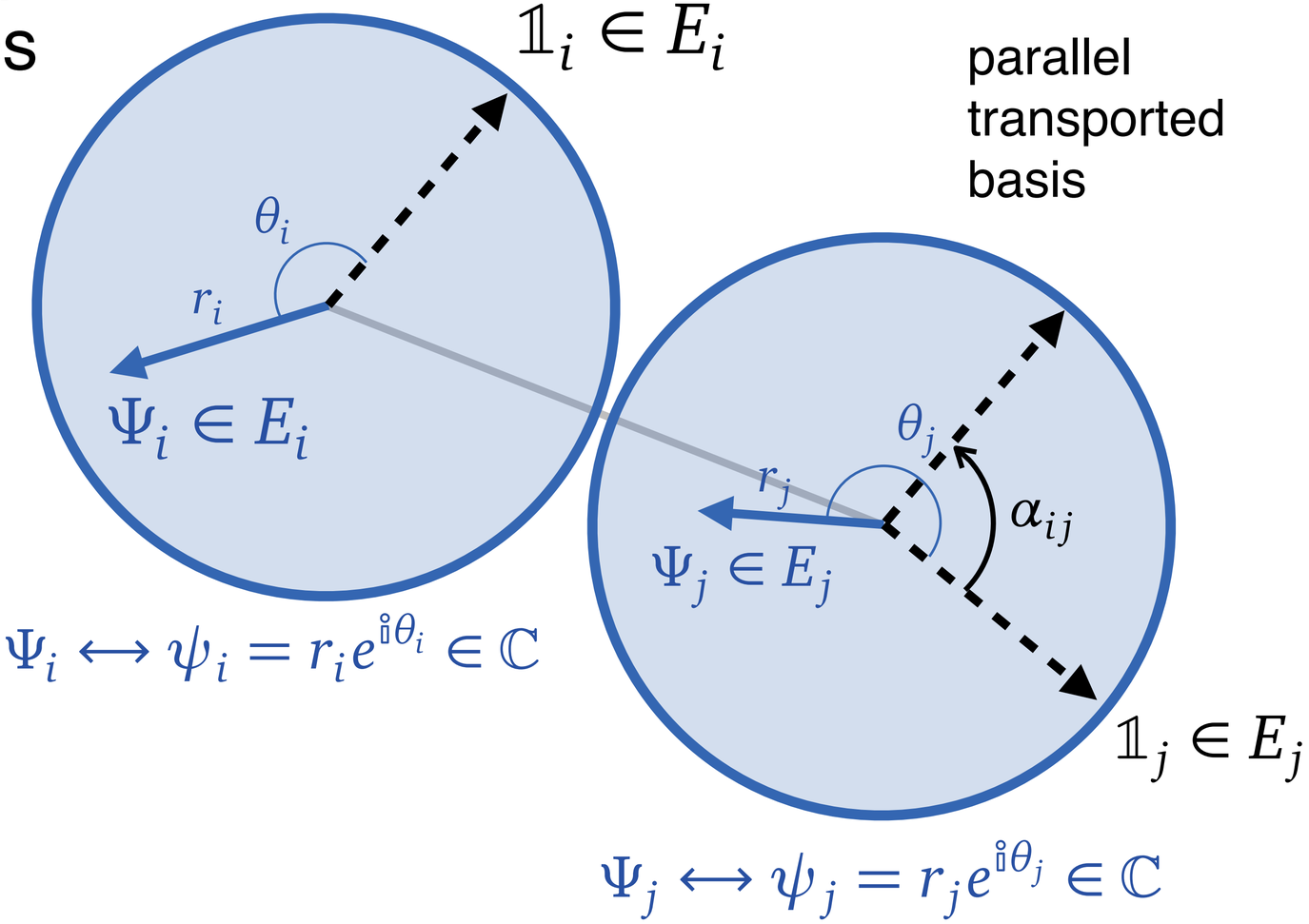
$$\Psi_i \longleftrightarrow \psi_i = r_i e^{\mathbb{i}\theta_i} \in \mathbb{C}$$

$$\Psi_j \longleftrightarrow \psi_j = r_j e^{\mathbb{i}\theta_j} \in \mathbb{C}$$

Discrete connection



For each adjacent value spaces



$$\Psi_i \longleftrightarrow \psi_i = r_i e^{\mathbb{i}\theta_i} \in \mathbb{C}$$

$$\Psi_j \longleftrightarrow \psi_j = r_j e^{\mathbb{i}\theta_j} \in \mathbb{C}$$

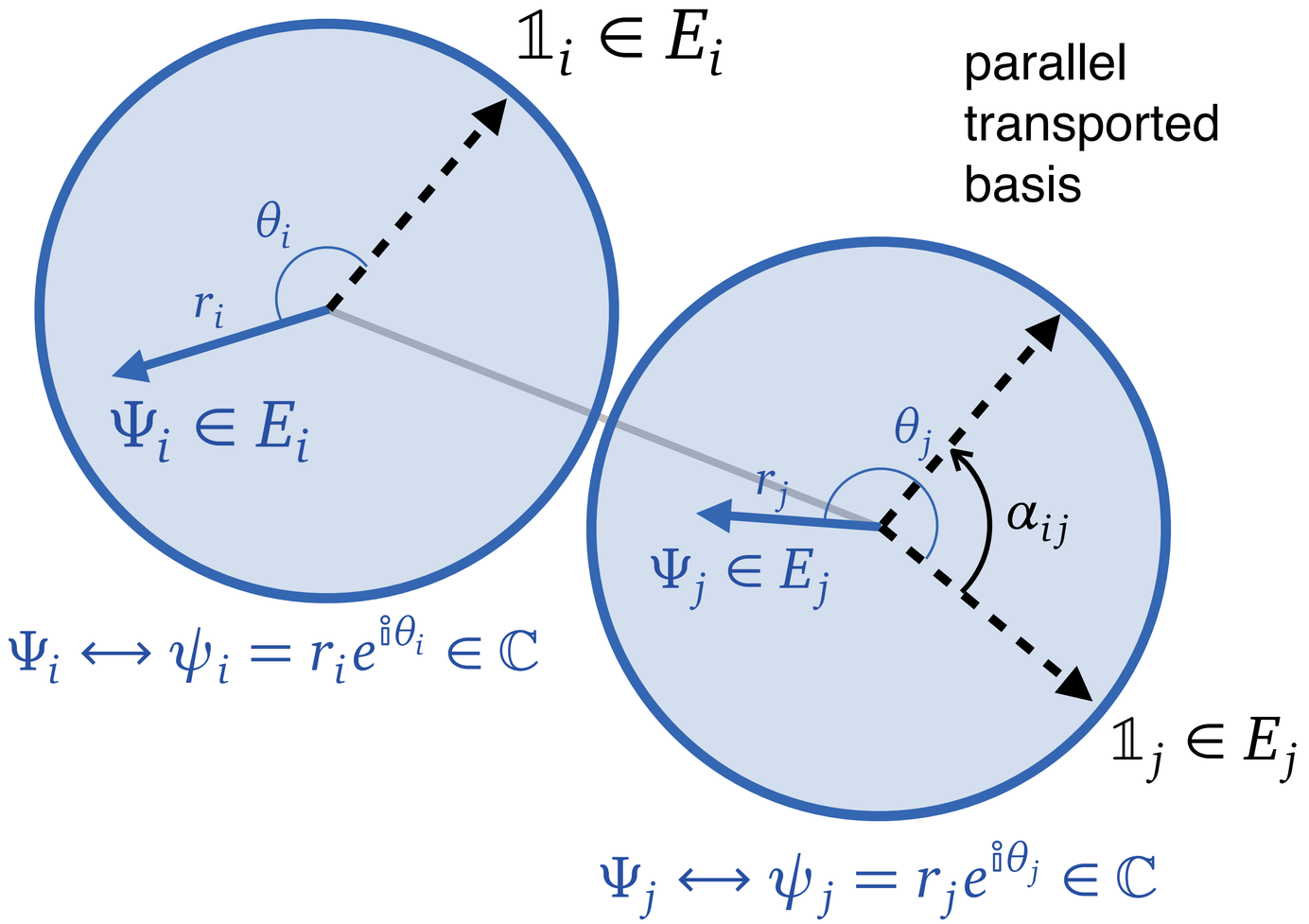
Discrete connection

For each adjacent points i, j encode the parallel transport by an angle α_{ij}, with

$$\alpha_{ij} = -\alpha_{ji} \mod 2\pi$$

i.e. an angle-valued 1-form.

It is called the **connection** as an additional structure for the bundle.



Covariant derivative

The derivative of the section Ψ is denoted by $d^{\nabla}\Psi$.

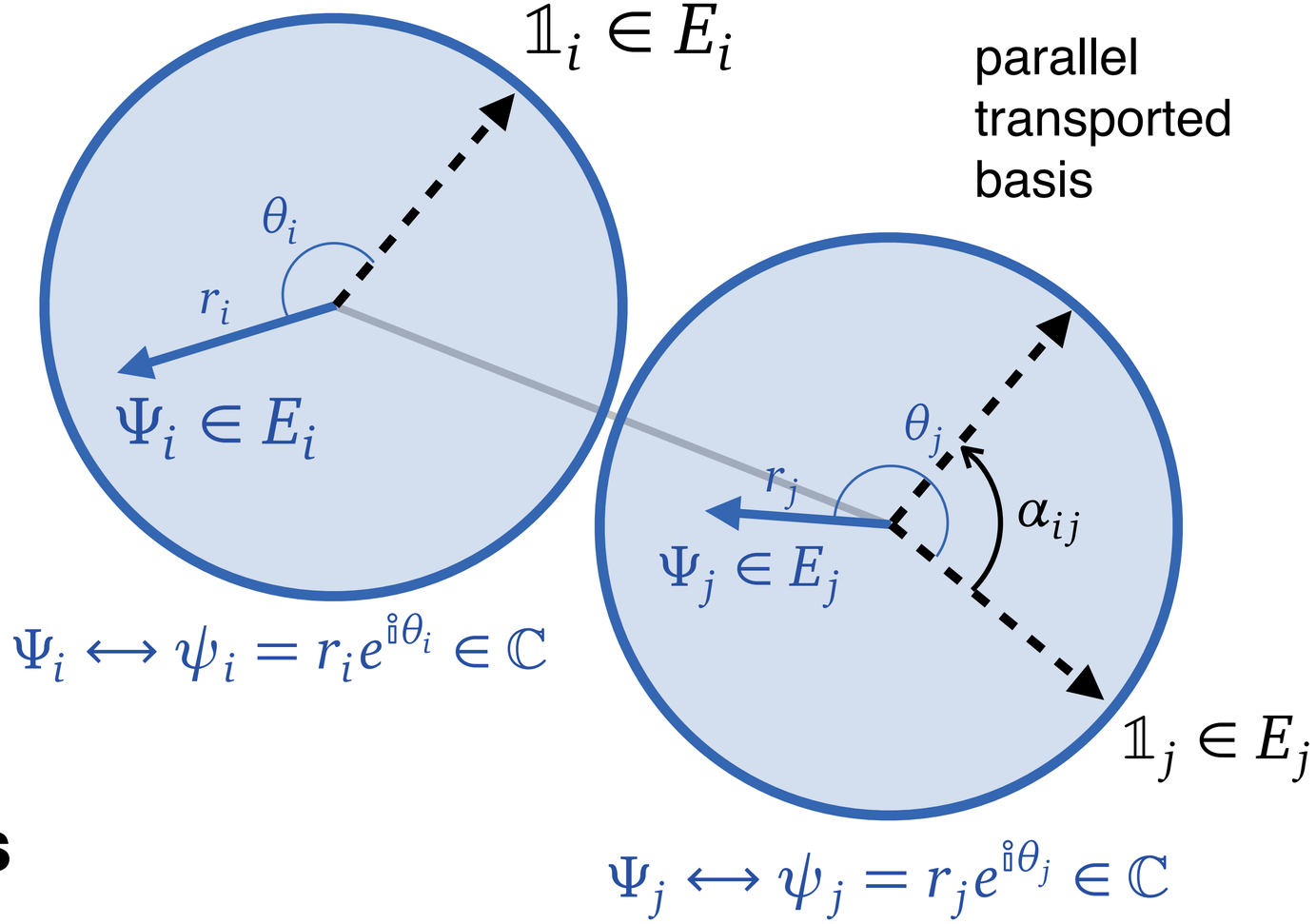
In terms of the basis $\mathbb{1}_i$

$$(d^{\nabla}\Psi)_{ij} \longleftrightarrow e^{-i\alpha_{ij}}\psi_j - \psi_i$$

In terms of the basis $\mathbb{1}_j$

$$(d^{\nabla}\Psi)_{ij} \longleftrightarrow \psi_j - e^{i\alpha_{ij}}\psi_i$$

The magnitude $|(d^{\nabla}\Psi)_{ij}|$ is basis-choice independent.



Covariant derivative

The derivative of the section Ψ is denoted by $d^{\nabla}\Psi$.

In terms of the basis $\mathbb{1}_i$

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In terms of the basis $\mathbb{1}_j$

$$(d^{\nabla}\Psi)_{ij} \longleftrightarrow \psi_j - e^{i\alpha_{ij}}\psi_i$$

The magnitude $|(d^{\nabla}\Psi)_{ij}|$ is basis-choice independent.

traditional derivative

$$d = \vec{ij} \begin{bmatrix} & i & & j & \\ & & \vdots & & \\ & -1 & & 1 & \\ & & \vdots & & \end{bmatrix}$$

connection derivative

$$d^{\nabla} = \vec{ij} \begin{bmatrix} & i & & j & \\ & & \vdots & & \\ & -1 & & e^{-i\alpha_{ij}} & \\ & & \vdots & & \end{bmatrix}$$

Dirichlet energy

The magnitude $|(d^{\nabla}\Psi)_{ij}|$ is basis-choice independent.

$$\mathcal{E}[\Psi] = \sum_{ij \in \text{half edges}} w_{ij} |(d^{\nabla}\Psi)_{ij}|^2$$

cotan weights

connection derivative

$$d^{\nabla} = \begin{matrix} \\ \vec{ij} \end{matrix} \begin{bmatrix} i & j \\ \vdots & \\ -1 & e^{-\mathbb{i}\alpha_{ij}} \\ \vdots & \end{bmatrix}$$

Dirichlet energy

The magnitude $|(d^{\nabla}\Psi)_{ij}|$ is basis-choice independent.

$$\mathcal{E}[\Psi] = \sum_{ij \in \substack{\text{half} \\ \text{edges}}} w_{ij} |(d^{\nabla}\Psi)_{ij}|^2$$

$$= \overline{\psi}^{\mathsf{T}} \underbrace{\overline{[d^{\nabla}]}^{\mathsf{T}} [\star_1] [d^{\nabla}]}_{L^{\nabla}} \psi$$

L^{∇} connection Laplacian

connection derivative

$$d^{\nabla} = \begin{matrix} \\ \vec{ij} \end{matrix} \begin{bmatrix} i & & j \\ & \vdots & \\ -1 & & e^{-\mathbb{i}\alpha_{ij}} \\ & \vdots & \end{bmatrix}$$

Smoothest section

The Dirichlet energy $\mathcal{E}[\Psi] = \overline{\psi}^{\intercal} L^{\nabla} \psi$ measures the smoothness of the section Ψ

(the smaller, the smoother with respect to the parallel transport)

Ground state problem

$$\min_{\psi} \overline{\psi}^{\intercal} [L^{\nabla}] \psi$$

$$\text{s.t.} \sum_i \left(\frac{\text{point}}{\text{area}}\right)_i |\psi_i|^2 = 1$$

$$\Leftrightarrow \quad \min_{\psi} \frac{\overline{\psi}^{\intercal} [L^{\nabla}] \psi}{\overline{\psi}^{\intercal} [\star_0] \psi}$$

smallest eigenvalue problem

$$\Leftrightarrow \quad L^{\nabla} \psi = \lambda \star_0 \psi$$

Smoothest section

The Dirichlet energy $\mathcal{E}[\Psi] = \overline{\psi}^{\mathsf{T}} L^{\nabla} \psi$ measures the smoothness of the section Ψ

(the smaller, the smoother with respect to the parallel transport)

Ground state with pointwise norm=1 constraint

$$\min_{\psi} \overline{\psi}^{\mathsf{T}} [L^{\nabla}] \psi$$

$$\text{s.t. } |\psi_i|^2 = 1 \quad \forall\, i$$

Ginzburg–Landau model

$$\min_{\psi} \quad \overline{\psi}^{\mathsf{T}} L^{\nabla} \psi + \frac{1}{\varepsilon^2} \sum_i \begin{pmatrix} \text{point} \\ \text{weight} \end{pmatrix}_i (|\psi_i|^2 - 1)^2$$

Complex line bundle setup summary

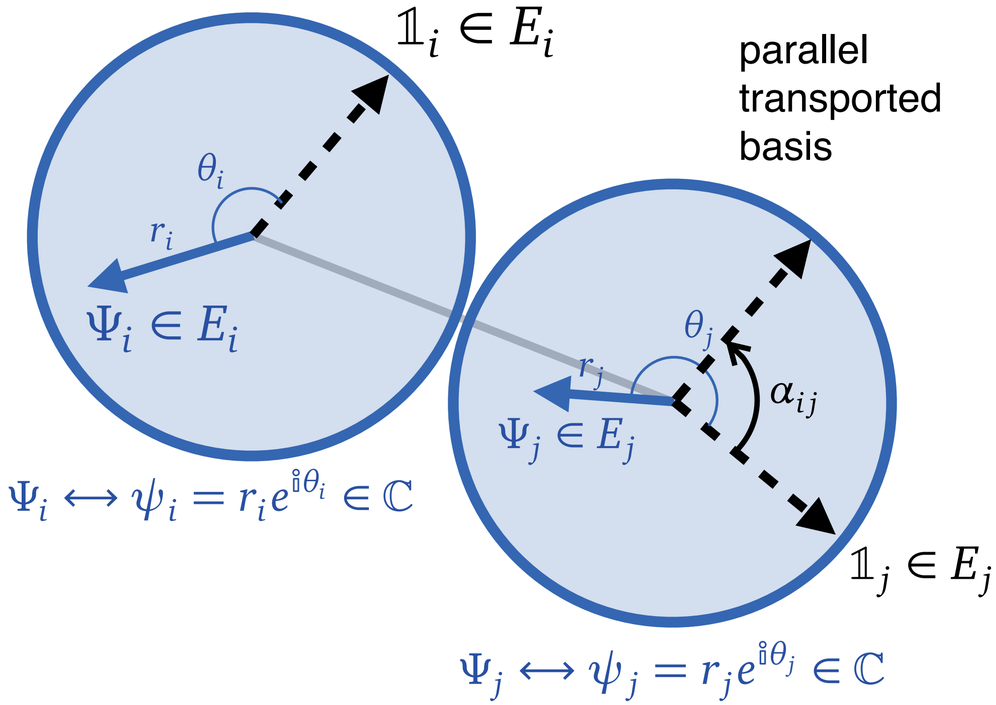
- The $\mathbb{C}$-valued function $\psi : \{\text{points}\} \to \mathbb{C}$ describes the geometric object called a section $\Psi \in \Gamma(E)$

- We need to provide an angle-valued 1-form α

- The Laplacian for ψ is

$$L^{\nabla} = \overline{d^{\nabla}}^{\mathsf{T}} \star_1 d^{\nabla}$$

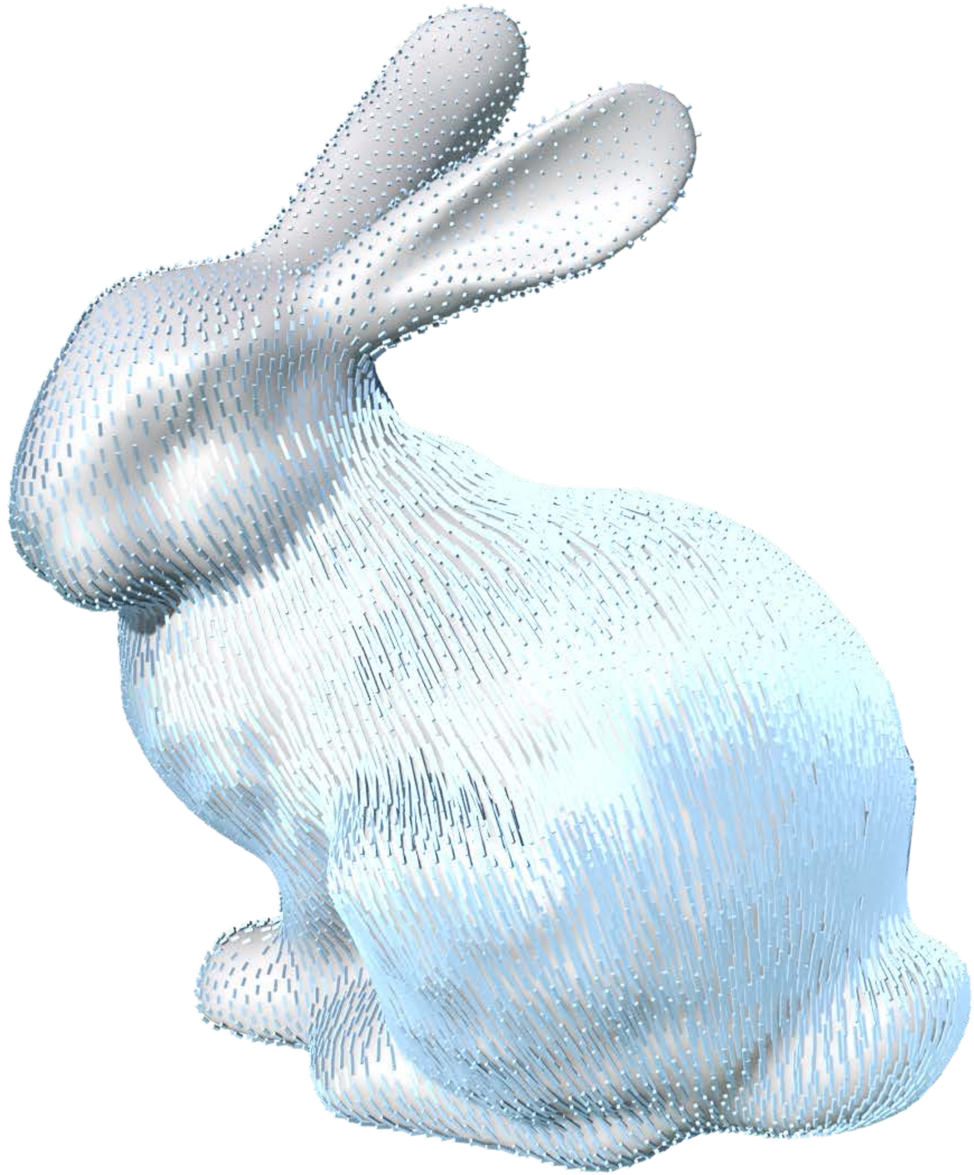
where $d^{\nabla} = \vec{ij}$ $\begin{bmatrix} & i & & j & \\ & & \vdots & & \\ & -1 & & e^{-\mathbb{i}\alpha_{ij}} & \\ & & \vdots & & \end{bmatrix}$



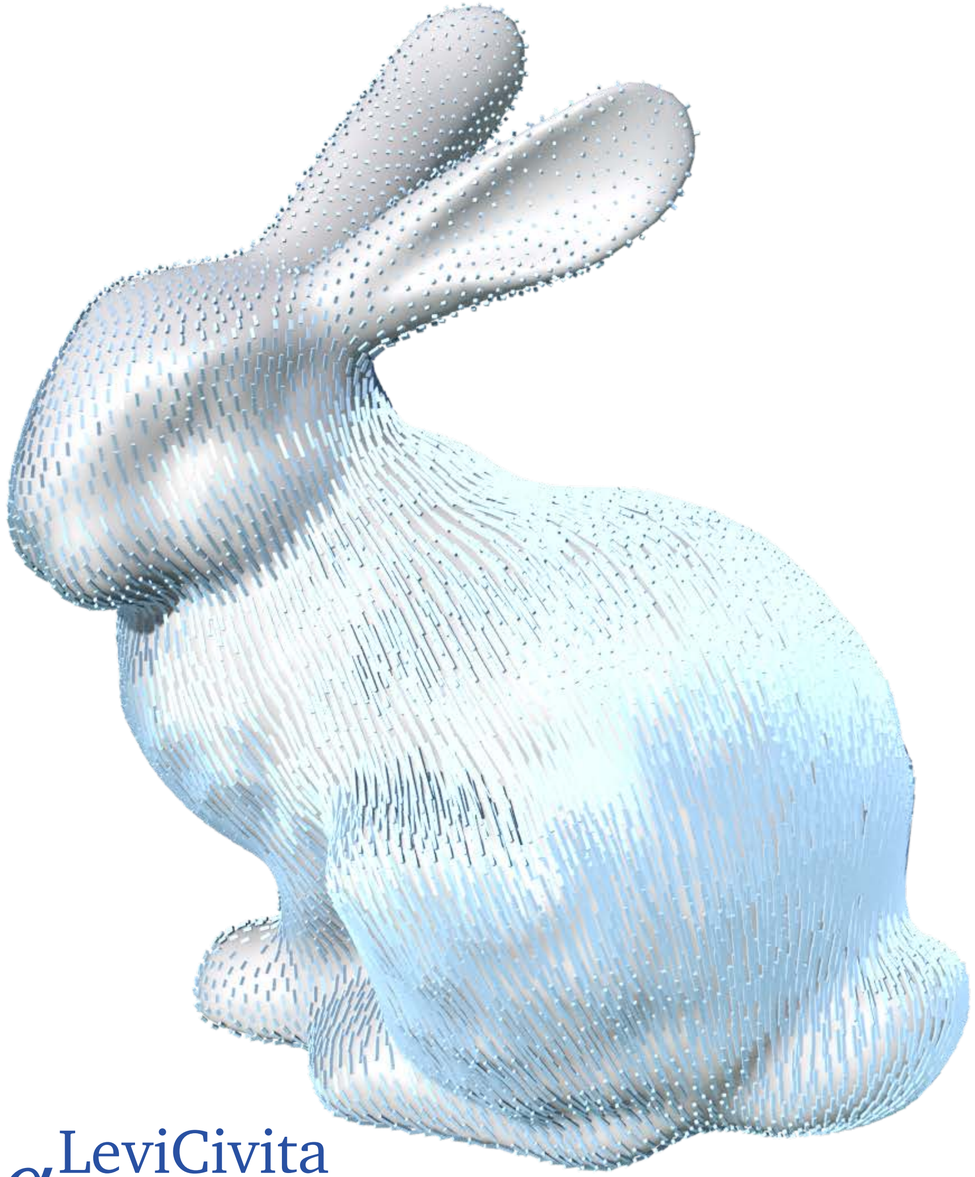
Effect of connection

If the complex line bundle is the tangent bundle,

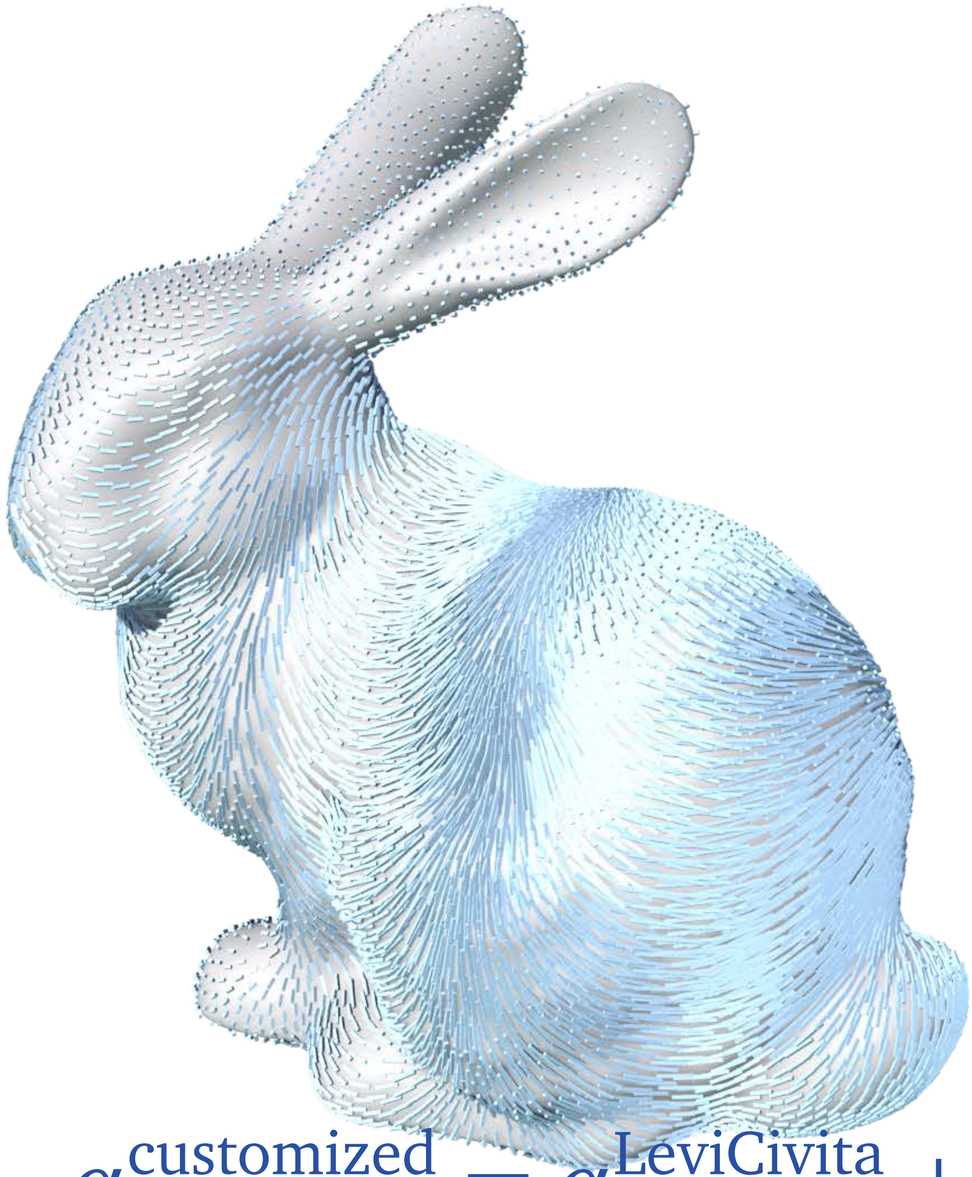
then there is a canonical α, known as the **Levi-Civita connection**, that describes the metric-induced parallel transport.



Effect of connection

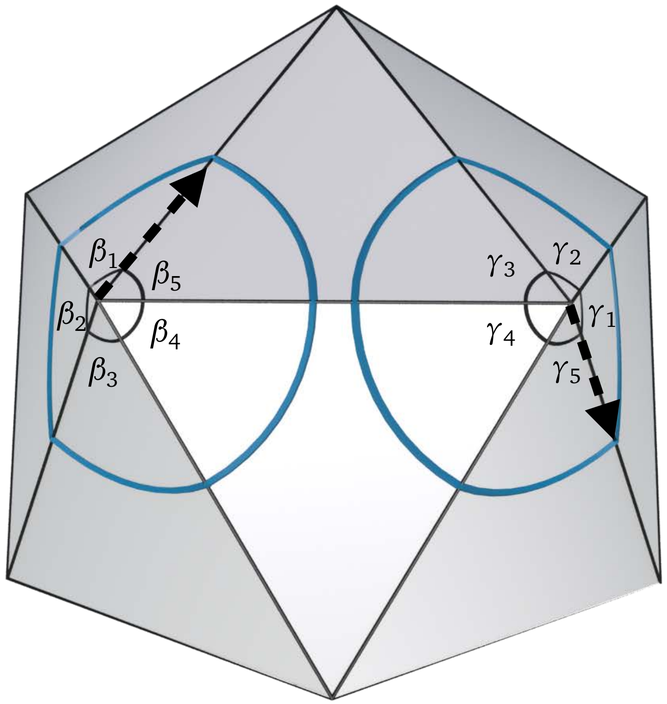


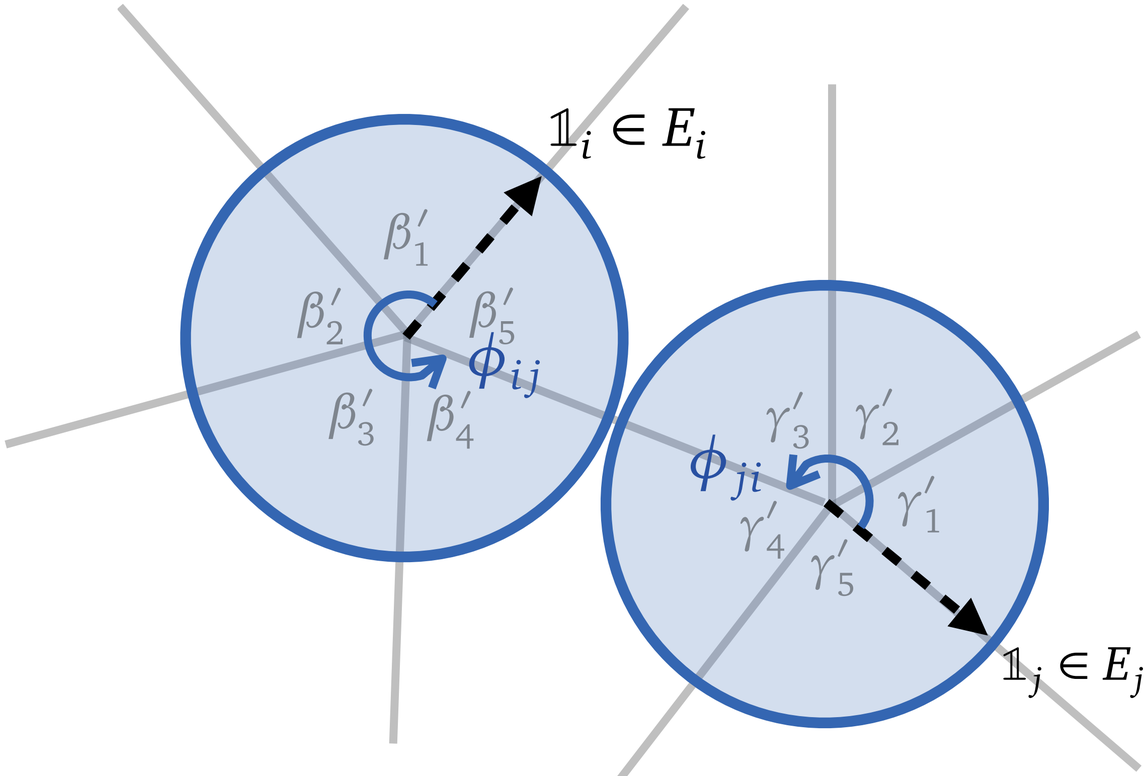
$\alpha^{\text{LeviCivita}}$



$\alpha^{\text{customized}} = \alpha^{\text{LeviCivita}} + \mathbf{v}^{\flat}$

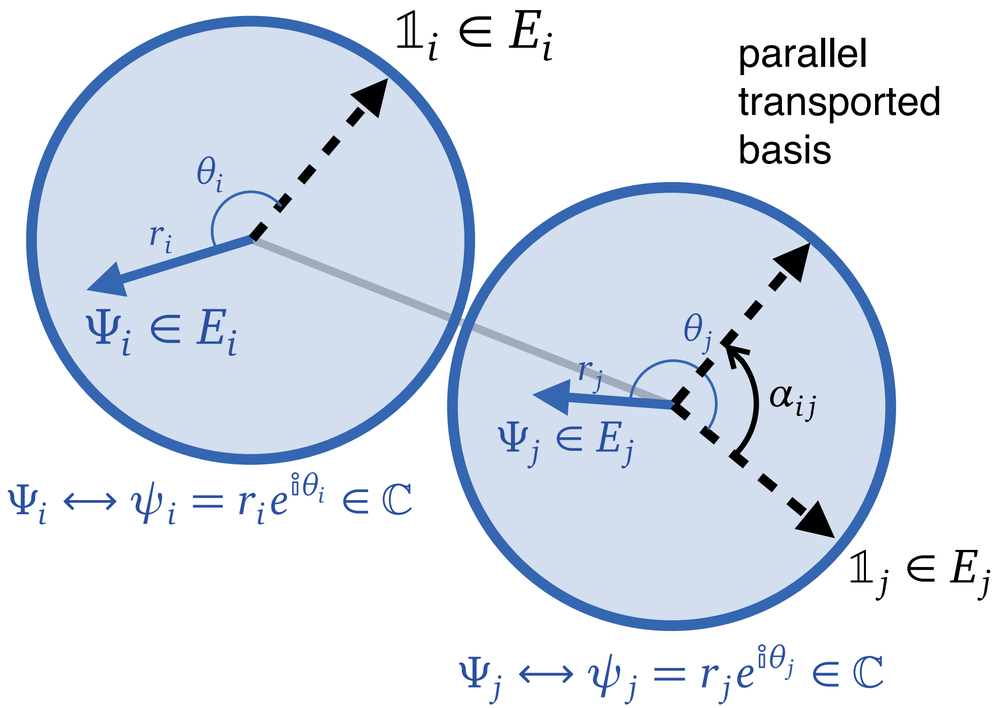
Levi-Civita connection





- Assign the polar angle ϕ_{ij} of each half-edge
  - In the picture $\phi_{ij} = \sum_{k=1}^{4} \beta'_k$ where $\beta'_k = \frac{\beta_k}{\sum_\ell \beta_\ell} 2\pi$
- The Levi-Civita connection is $\alpha_{ij} = \phi_{ji} - \phi_{ij} - \pi \mod 2\pi$

Gauge transformation

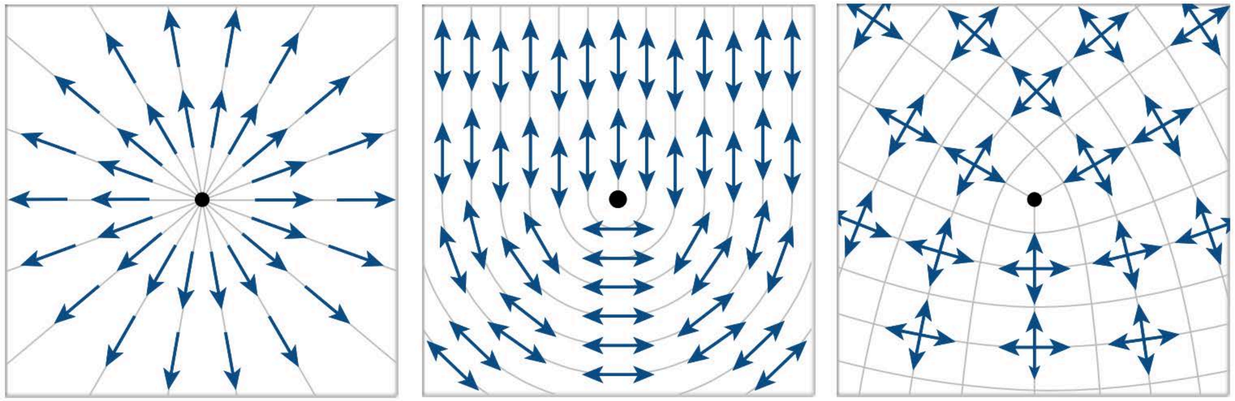


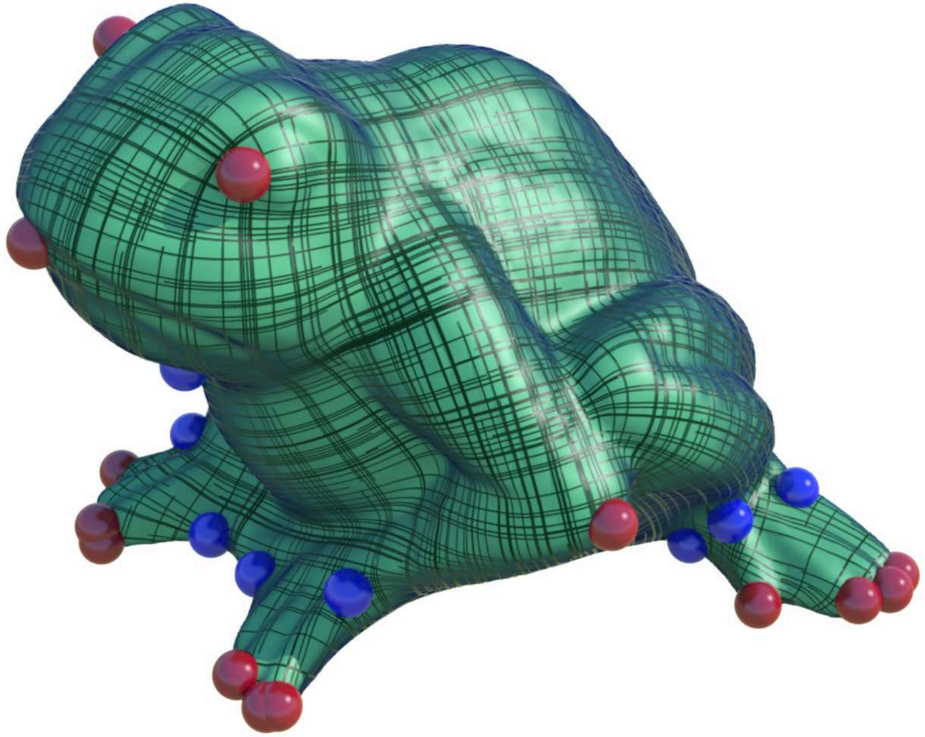
$\varphi_i \in \mathbb{R}$ (not angle-valued, but actual real-valued)

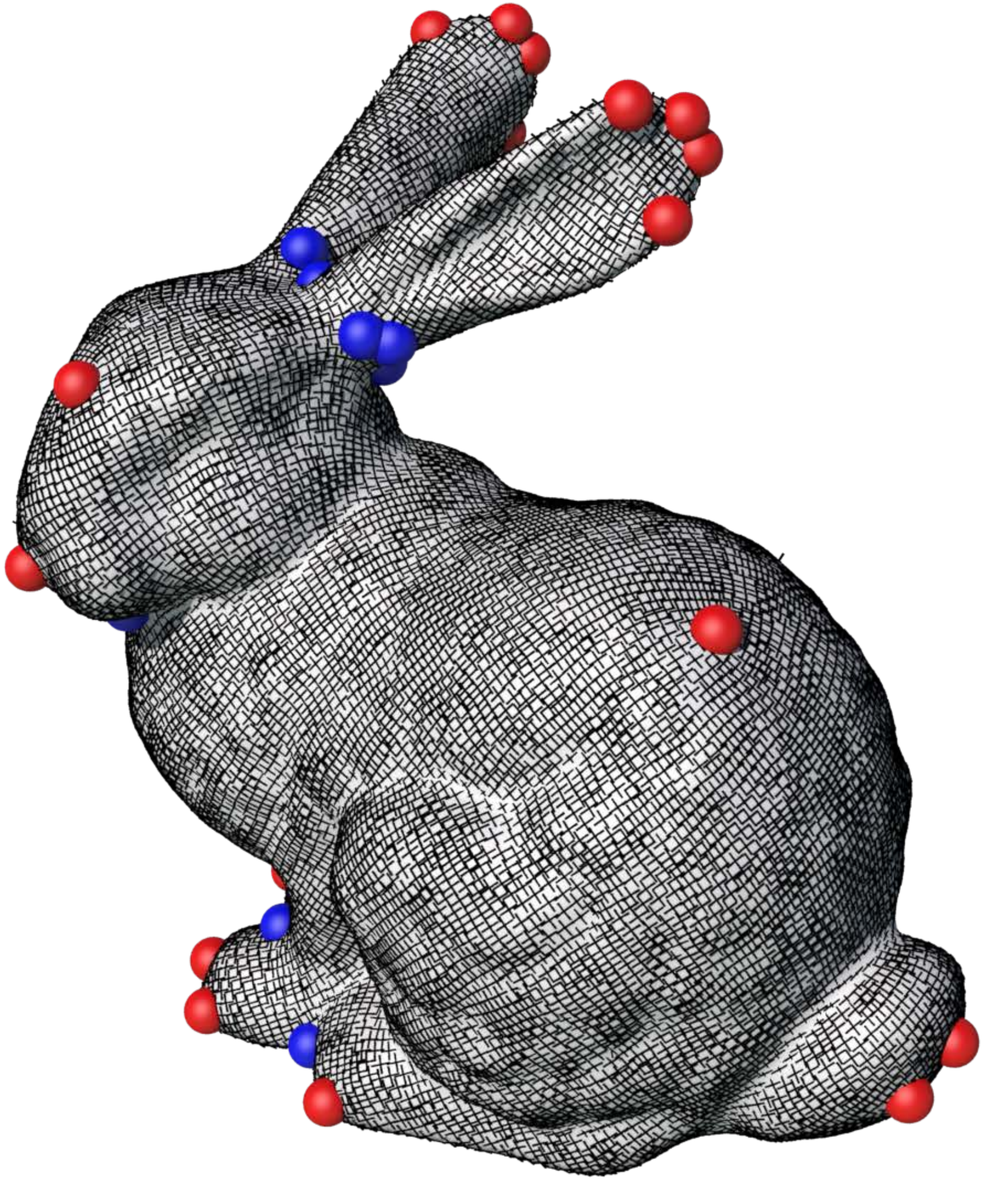
$$\widetilde{\mathbb{1}}_i = e^{-\mathbb{i}\varphi_i}\mathbb{1}_i \qquad \widetilde{\psi}_i = e^{\mathbb{i}\varphi_i}\psi_i \qquad \widetilde{\alpha}_{ij} = \alpha_{ij} + (d\varphi)_{ij}$$

$$= \alpha_{ij} + \varphi_j - \varphi_i$$

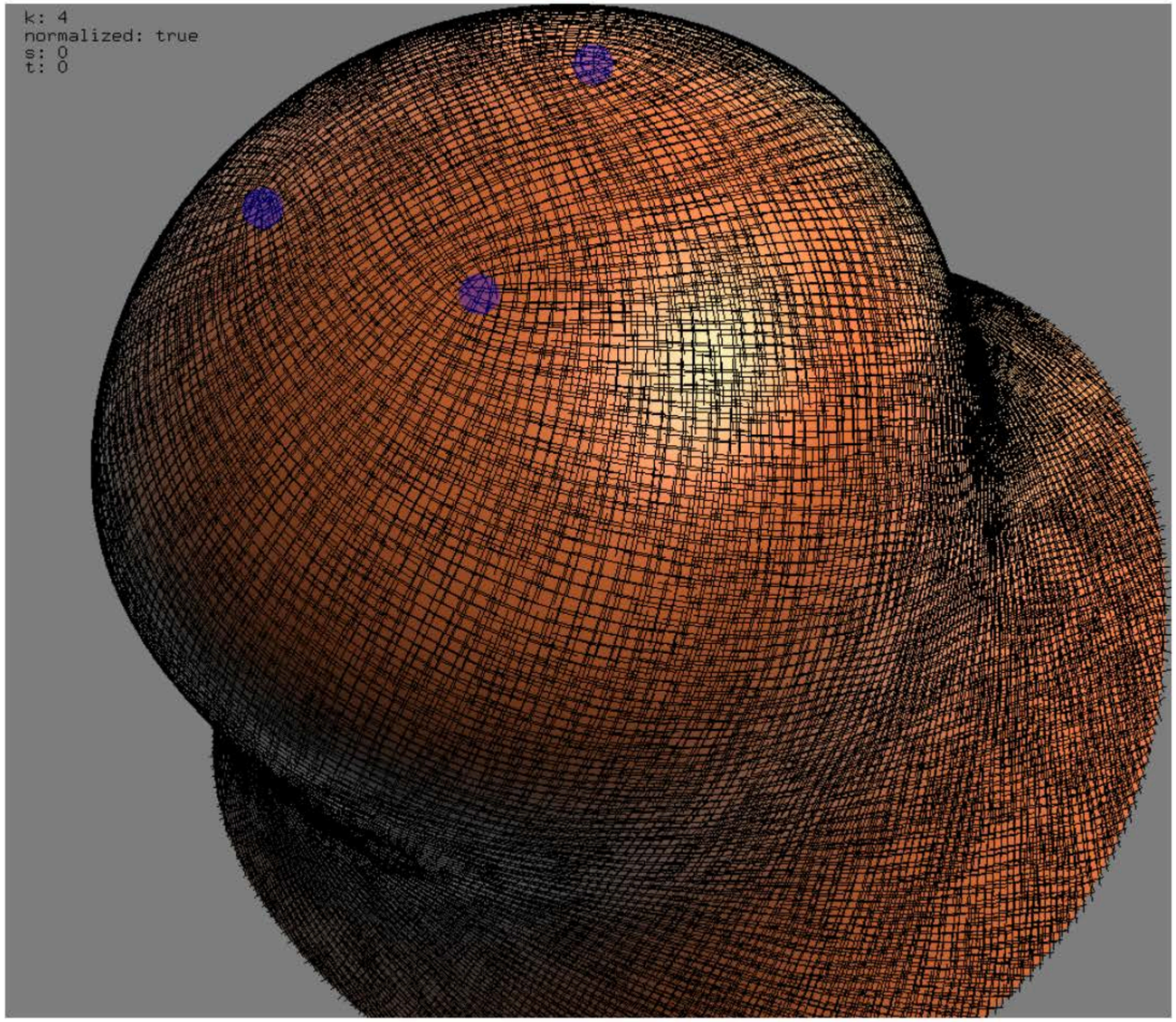
n-direction field







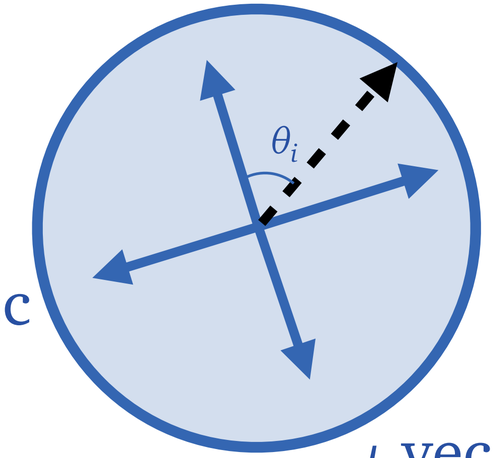
n-direction field





n-direction field

$$\psi_i^{\text{vec}} = r_i e^{\mathbb{i}\theta_i}$$

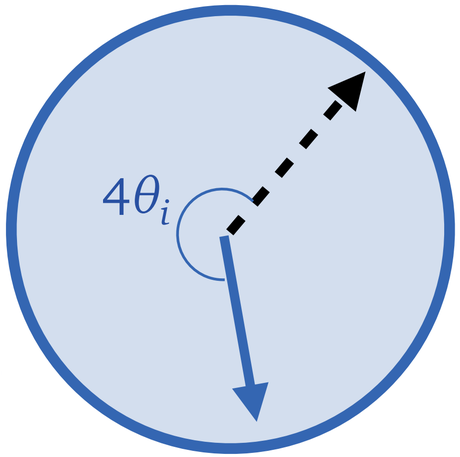


$-\mathbb{i}\psi_i^{\text{vec}}$

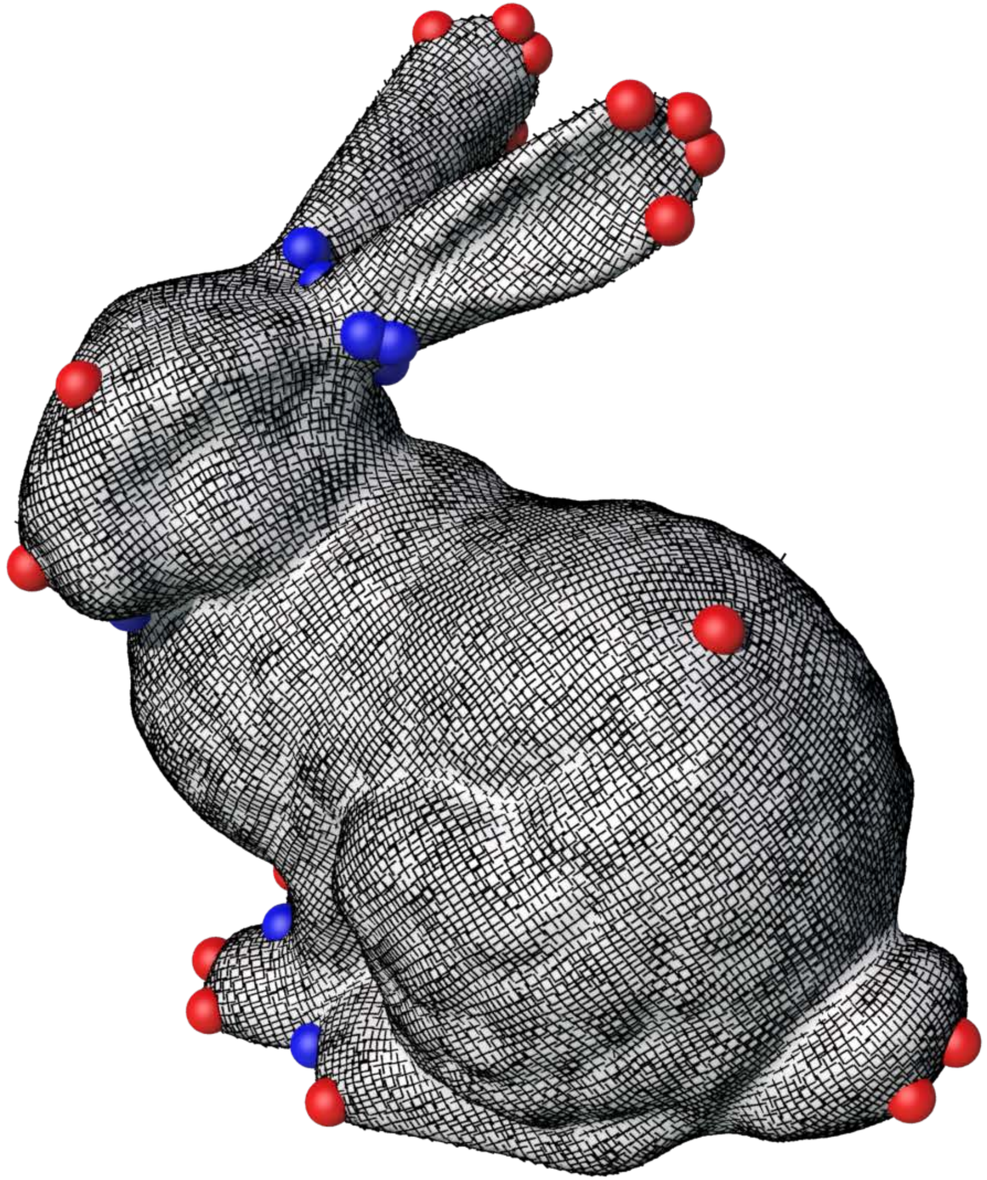
$\mathbb{i}\psi_i^{\text{vec}}$

$-\psi_i^{\text{vec}}$

$z \mapsto z^4$



$$\psi_i^{\text{cross}} = (\psi_i^{\text{vec}})^4$$



n-direction field

$$\psi_i^{\text{cross}} = (\psi_i^{\text{vec}})^4$$

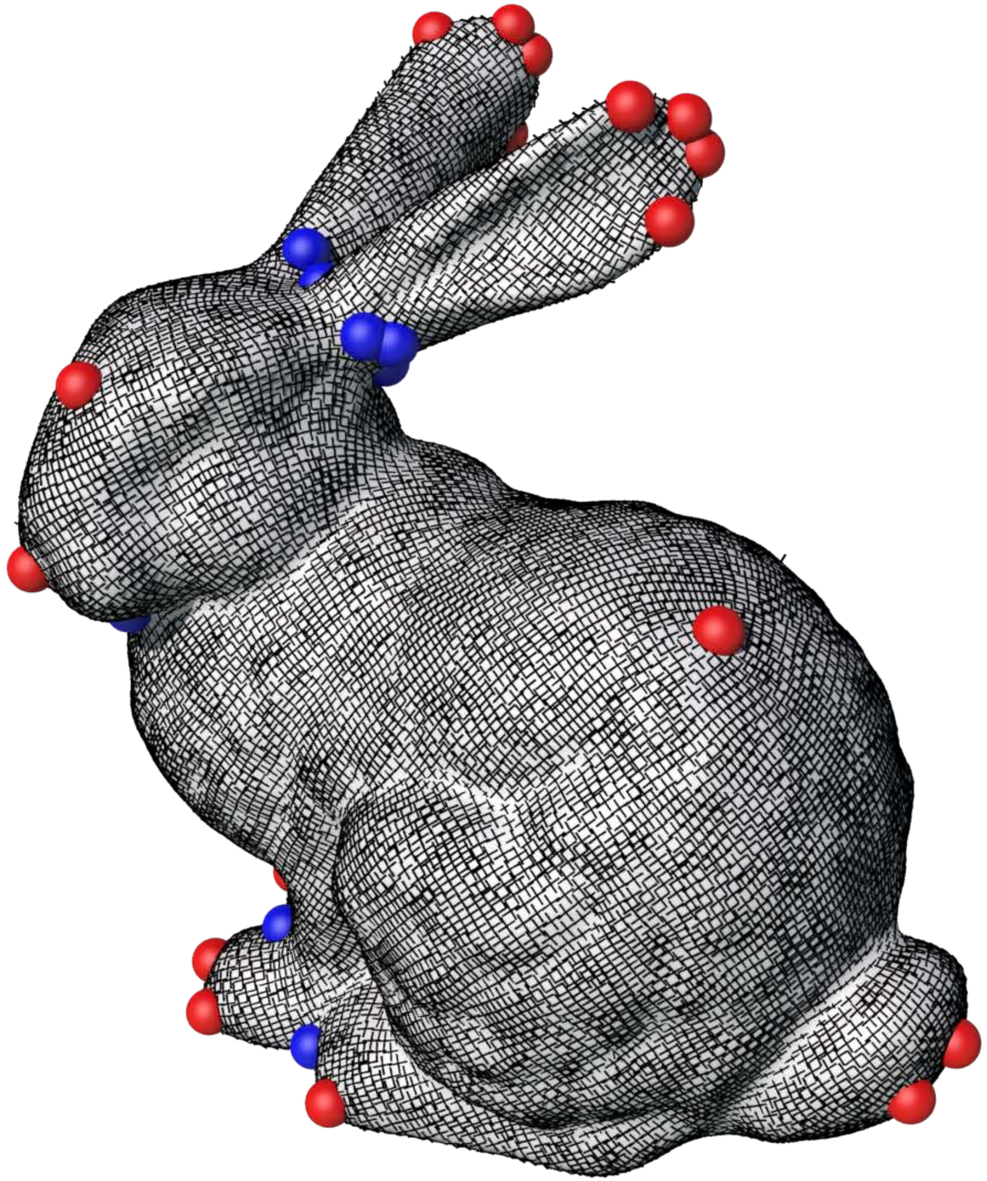
parallel transport for ψ_i^{vec}

$$e^{\mathbb{i}\alpha_{ij}^{\text{vec}}}\psi_i^{\text{vec}}$$

parallel transport for ψ_i^{cross}

$$e^{\mathbb{i}\alpha_{ij}^{\text{cross}}}\psi_i^{\text{cross}} = e^{\mathbb{i}4\alpha_{ij}^{\text{vec}}}(\psi_i^{\text{vec}})^4$$

Therefore $\alpha_{ij}^{\text{cross}} = 4\alpha_{ij}^{\text{vec}}$



n-direction field

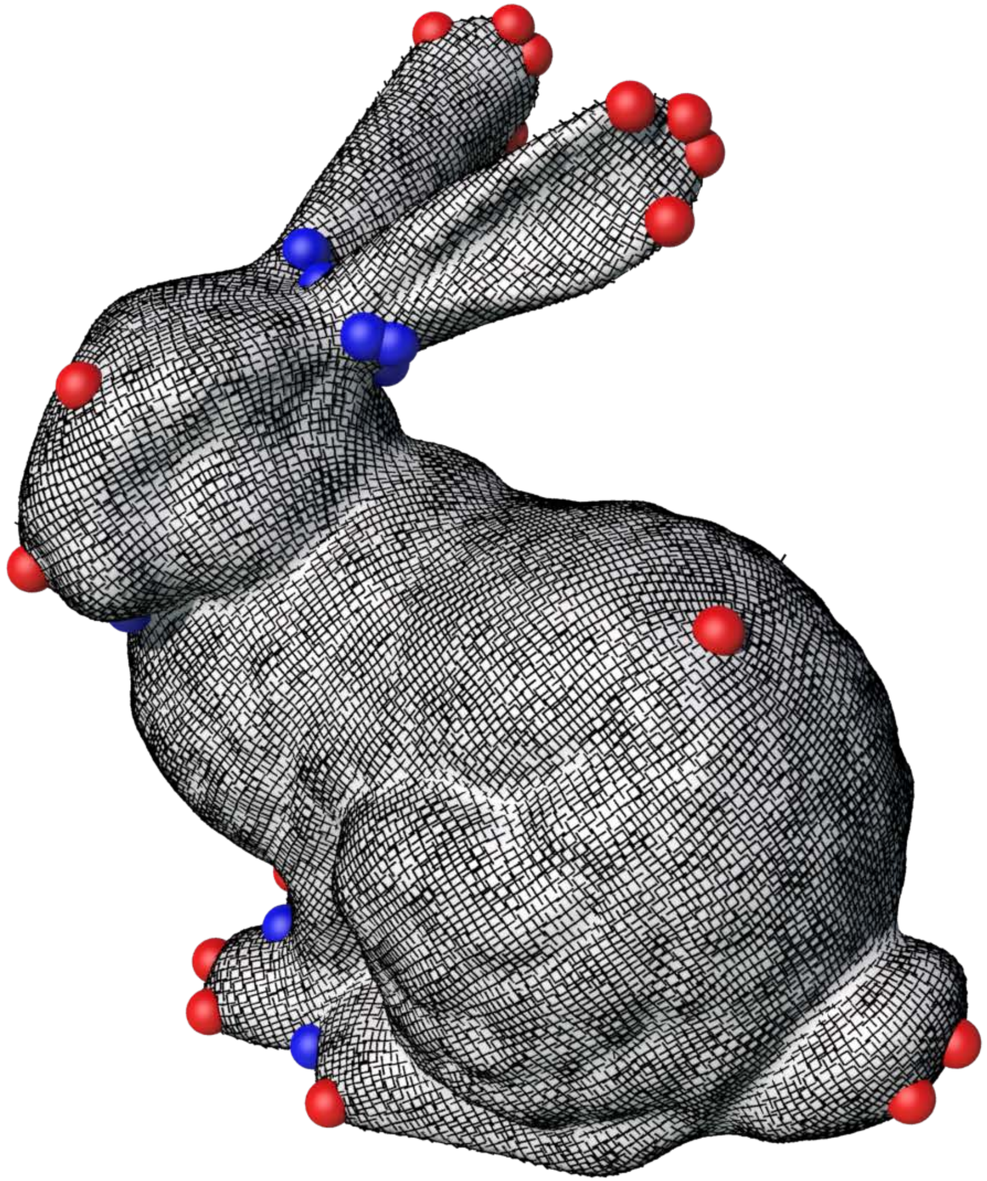
$$\alpha_{ij}^{\text{cross}} = 4\alpha_{ij}^{\text{vec}}$$

Build $L^{\nabla} = \overline{d^{\nabla}}^{\mathsf{T}} \star_1 d^{\nabla}$

where $d^{\nabla} = \begin{matrix} \\ \vec{ij} \end{matrix} \begin{bmatrix} i & j \\ \vdots & \\ -1 & e^{-\mathbb{i}\alpha_{ij}^{\text{cross}}} \\ \vdots & \end{bmatrix}$

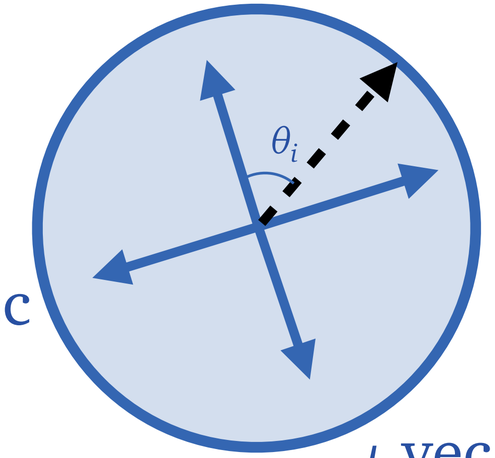
Solve smallest eigenvalue problem

$$L^{\nabla}\psi^{\text{cross}} = \lambda \star_0 \psi^{\text{cross}}$$



n-direction field

$$\psi_i^{\text{vec}} = r_i e^{\mathbb{i}\theta_i}$$

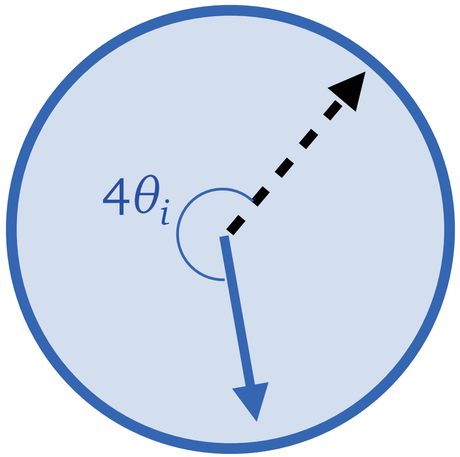


$-\mathbb{i}\psi_i^{\text{vec}}$

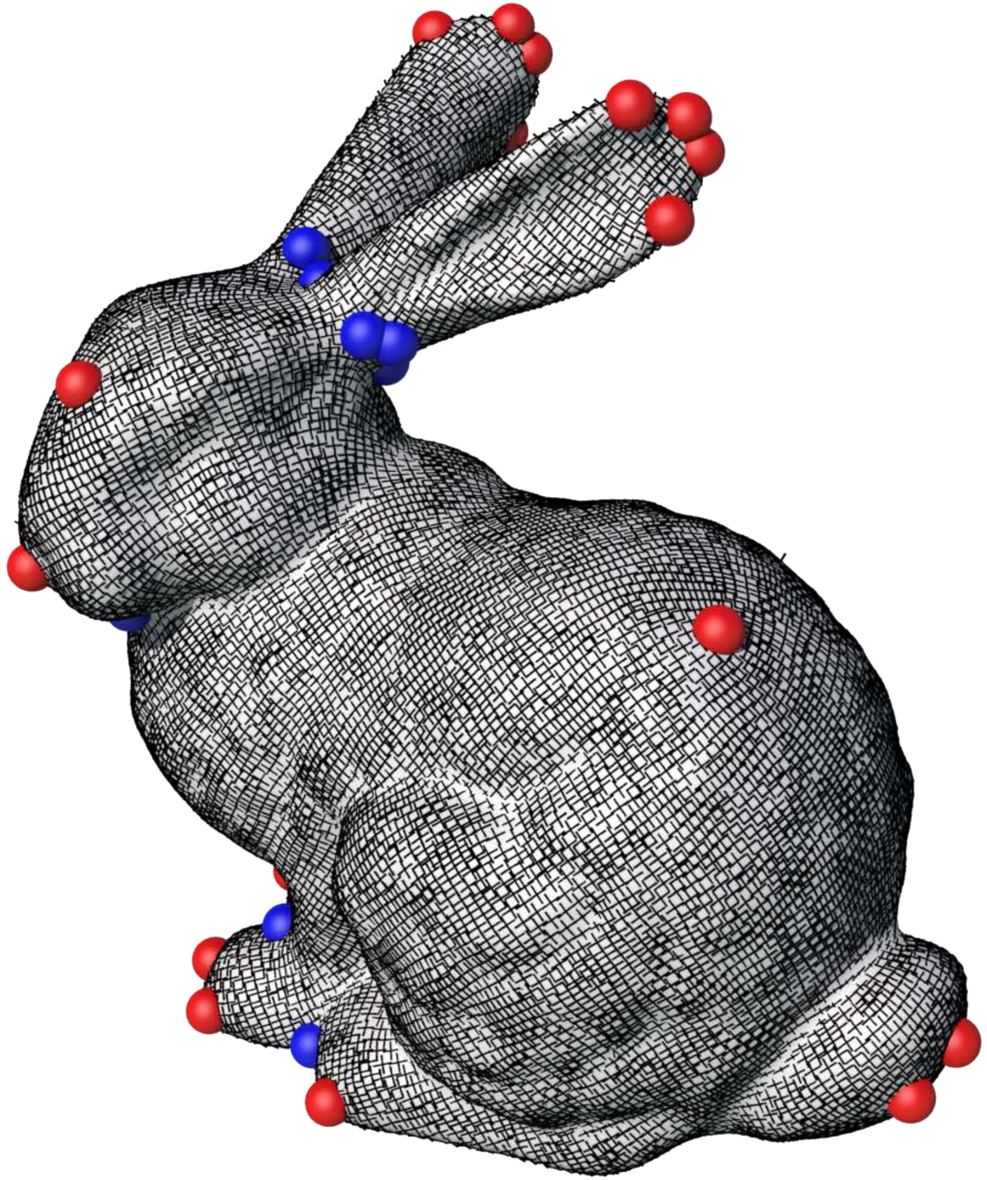
$\mathbb{i}\psi_i^{\text{vec}}$

$-\psi_i^{\text{vec}}$

$z \longmapsto z^{1/4}$



$$\psi_i^{\text{cross}} = (\psi_i^{\text{vec}})^4$$



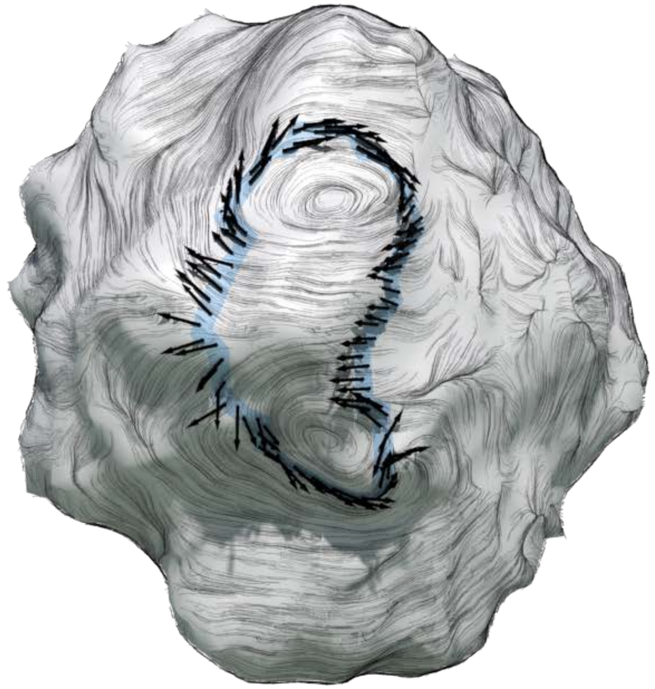
Turning angle, curvature, singularities

- Hairy ball theorem
- Optimal sections of a bundle
- Curvature and singularities
- Gauge theory in physics

Poincaré index in angle defect

Recall:

$$(K\,dA)_i = 2\pi - \left(\begin{array}{l} \text{turning angle} \\ \text{of a loop around} \\ \text{the point i} \end{array} \right)$$



$$\text{Total turning angle of } \mathbf{v} = 2\pi \sum_{\substack{\text{singularities} \\ \text{enclosed}}} \text{index} - \int_{\substack{\text{region} \\ \text{enclosed}}} K\,dA$$

Index observation from a distance

Poincaré index in angle defect

$$\text{Total turning angle of } \mathbf{v} = 2\pi \sum_{\substack{\text{singularities} \\ \text{enclosed}}} \text{index} - \int_{\substack{\text{region} \\ \text{enclosed}}} K \, dA$$

For a general complex line bundle

- Turning angle from point i to point j $\arg\left(\frac{\psi_j}{e^{\mathbb{i}\alpha_{ij}}\psi_i}\right)$
- Total turning angle after a round trip

$$\arg\left(\frac{\psi_{i_2}}{e^{\mathbb{i}\alpha_{i_1,i_2}}\psi_{i_1}}\right) + \cdots + \arg\left(\frac{\psi_{i_1}}{e^{\mathbb{i}\alpha_{i_n,i_1}}\psi_{i_n}}\right)$$

Poincaré index in angle defect

$$\text{Total turning angle of } \mathbf{v} = 2\pi \sum_{\substack{\text{singularities} \\ \text{enclosed}}} \text{index} - \int_{\substack{\text{region} \\ \text{enclosed}}} K \, dA$$

For a general complex line bundle

- Turning angle from point i to point j $\arg\left(\dfrac{\psi_j}{e^{\mathbb{i}\alpha_{ij}}\psi_i}\right)$
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$$\arg\left(\frac{\psi_{i_2}}{e^{\mathbb{i}\alpha_{i_1,i_2}}\psi_{i_1}}\right) + \cdots + \arg\left(\frac{\psi_{i_1}}{e^{\mathbb{i}\alpha_{i_n,i_1}}\psi_{i_n}}\right) \overset{\text{mod } 2\pi}{=} -\mathbb{i}(\alpha_{i_1,i_2} + \cdots + \alpha_{i_n,i_1})$$

Poincaré index in angle defect

$$\text{Total turning angle of } \mathbf{v} = 2\pi \sum_{\substack{\text{singularities} \\ \text{enclosed}}} \text{index} - \int_{\substack{\text{region} \\ \text{enclosed}}} K\, dA$$

$$\arg\left(\frac{\psi_{i_2}}{e^{\mathbb{i}\alpha_{i_1,i_2}}\psi_{i_1}}\right) + \cdots + \arg\left(\frac{\psi_{i_1}}{e^{\mathbb{i}\alpha_{i_n,i_1}}\psi_{i_n}}\right) \overset{\text{mod } 2\pi}{=} -\mathbb{i}(\alpha_{i_1,i_2} + \cdots + \alpha_{i_n,i_1})$$

The **curvature** of the bundle enclosed by the loop is

$$\int K\, dA = \beta = \sum_{\substack{\text{around the} \\ \text{loop}}} \alpha_{i_k,i_{k+1}} \mod 2\pi$$

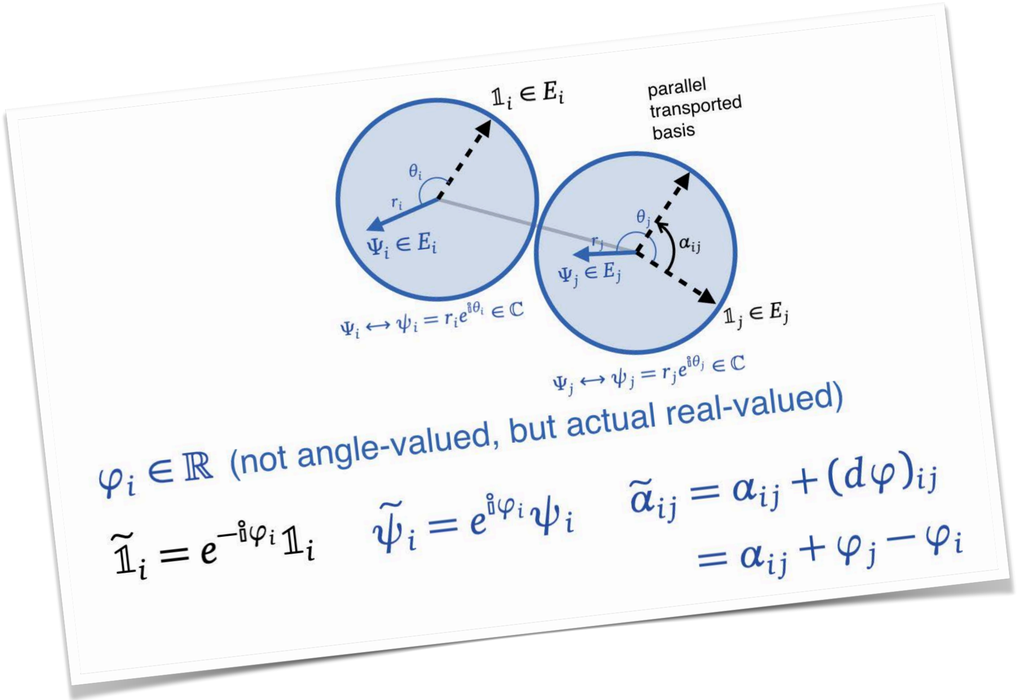
Knowing which 2 pi branch of curvature for every face can give us Poincaré index.

Poincaré index in angle defect

The **curvature** of the bundle enclosed by the loop is

$$\int K\,dA = \beta = \sum_{\text{around the loop}} \alpha_{i_k, i_{k+1}} \mod 2\pi$$

Knowing which 2 pi branch of curvature for every face can give us Poincaré index.



Curvature is gauge invariant

Complex line bundle

A complex line bundle over a triangle mesh is equipped with

- angle-valued 1-form α
- real-valued 2-form β so that $d\alpha = \beta \mod 2\pi$

A complex-valued 0-form ψ and the connection α
are relative to an arbitrary reference basis. Under basis transformation,

$$\psi \mapsto e^{\mathring{\imath}\varphi}\psi \qquad \alpha \mapsto \alpha + d\varphi$$

The turning angle of ψ along an edge is $\arg\left(\frac{\psi_j}{e^{\mathring{\imath}\alpha_{ij}}\psi_i}\right)$

The index formula: (Total turning angle)= $2\pi \sum \text{index} - \int \beta$

Complex line bundle

The index formula: (Total turning angle)= $2\pi \sum \text{index} - \int \beta$

Hairball theorem for complex line bundle

Over the entire closed surface

$$2\pi \sum \text{index} = \int_M \beta$$

Definition. The **degree** of a complex line bundle is the integer

$$\frac{1}{2\pi} \int_M \beta$$

Complex line bundle

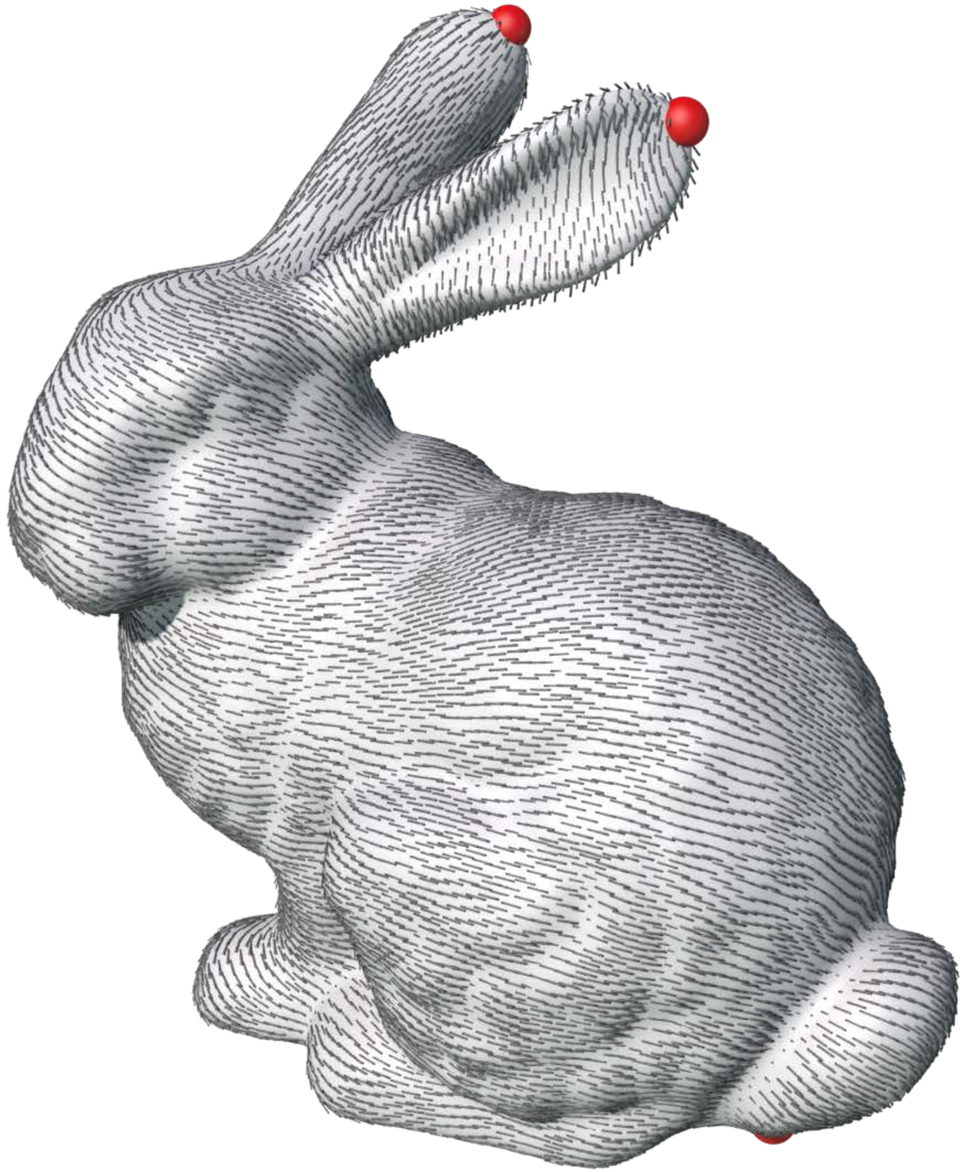
Hairball theorem for complex line bundle

Over the entire closed surface

$$2\pi \sum \text{index} = \int_M \beta$$

Paint some 2-form total integral being 2pi integer, can you build the rest of the ingredients? (i.e 1-form α)

Singularities in optimal n-direction field



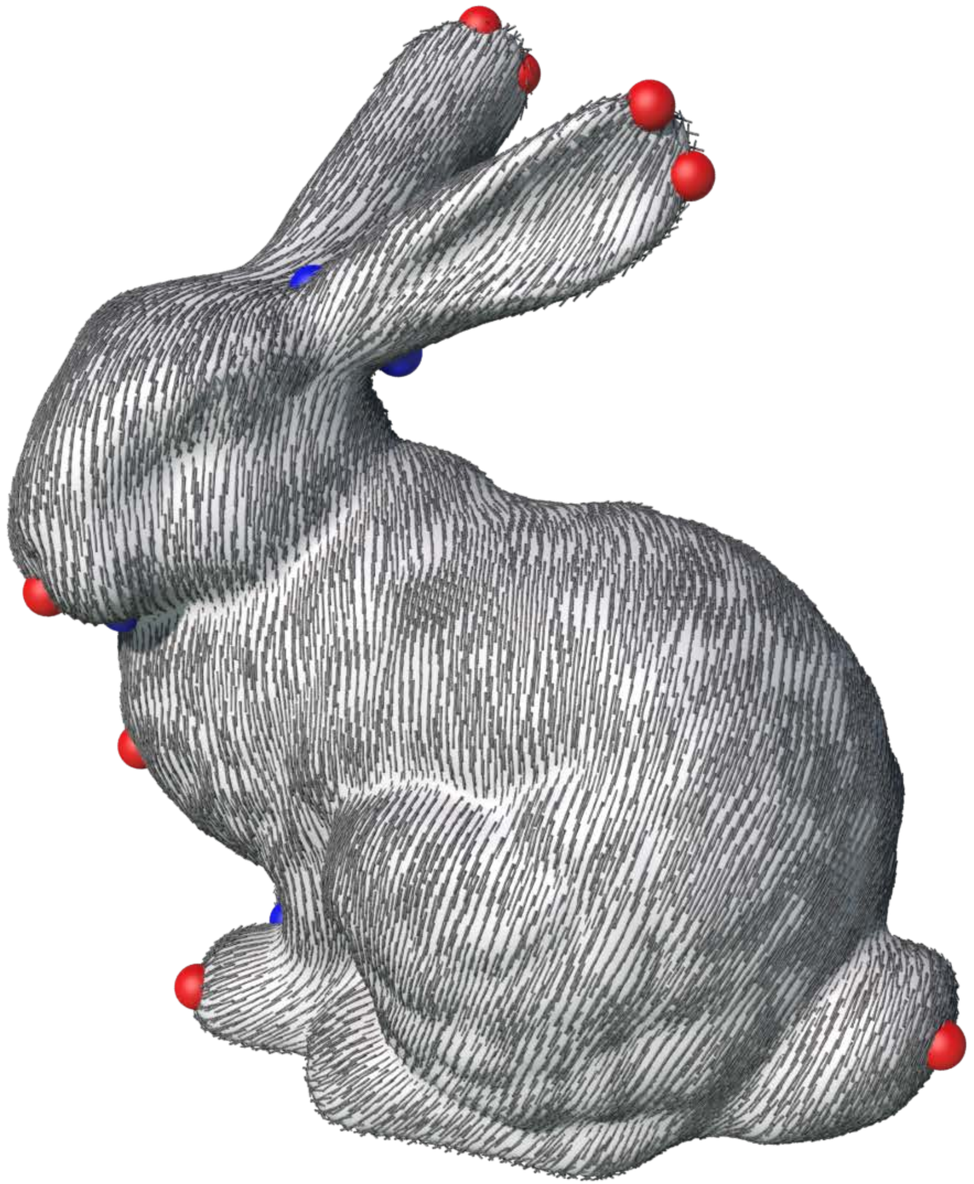
n=1

$$\alpha = \alpha^{\mathrm{LC}}$$

$$\beta = K\, dA$$

degree = 2

Singularities in optimal n-direction field



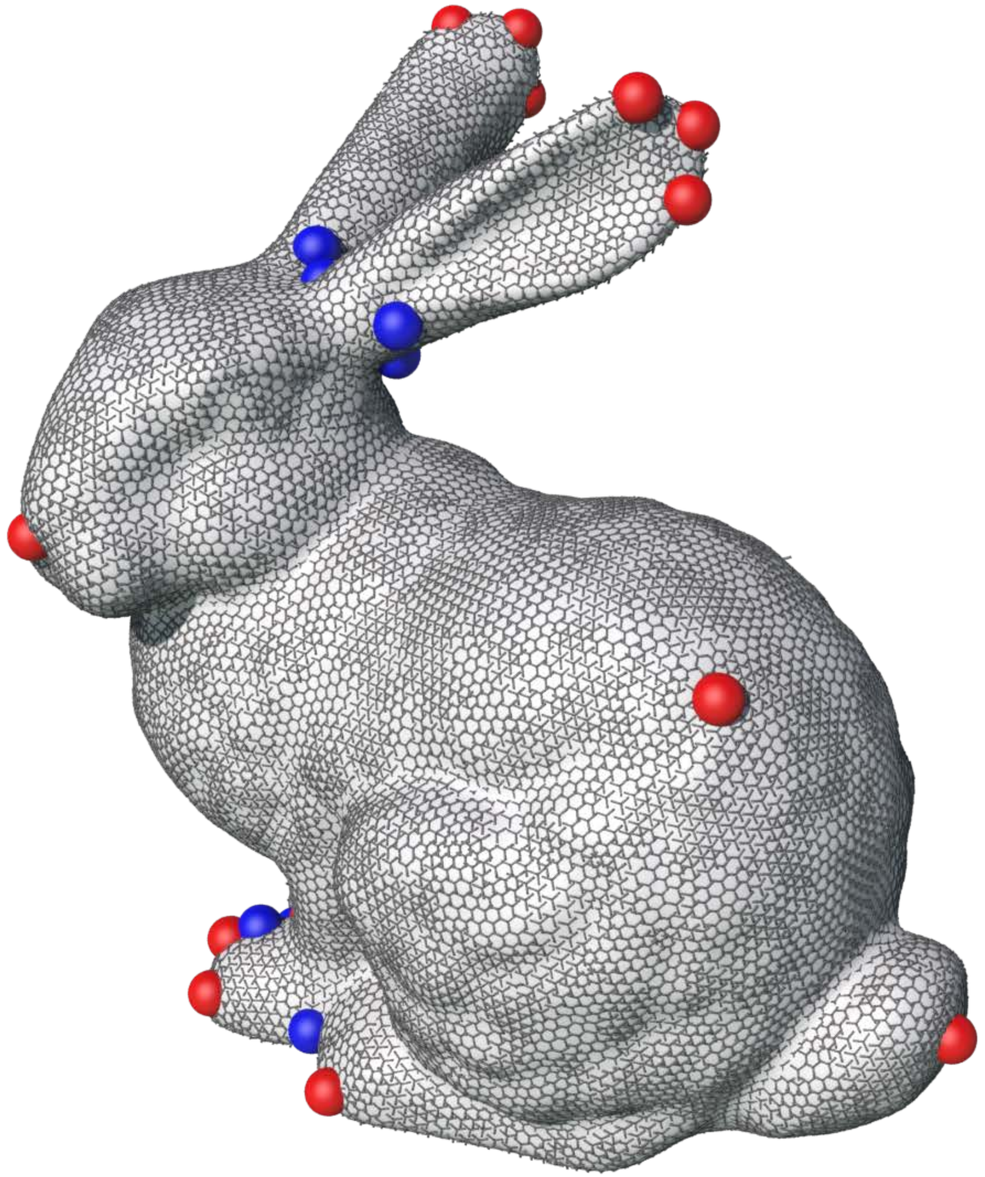
n=2

$$\alpha = 2\alpha^{\mathrm{LC}}$$

$$\beta = 2K\,dA$$

degree = 4

Singularities in optimal n-direction field



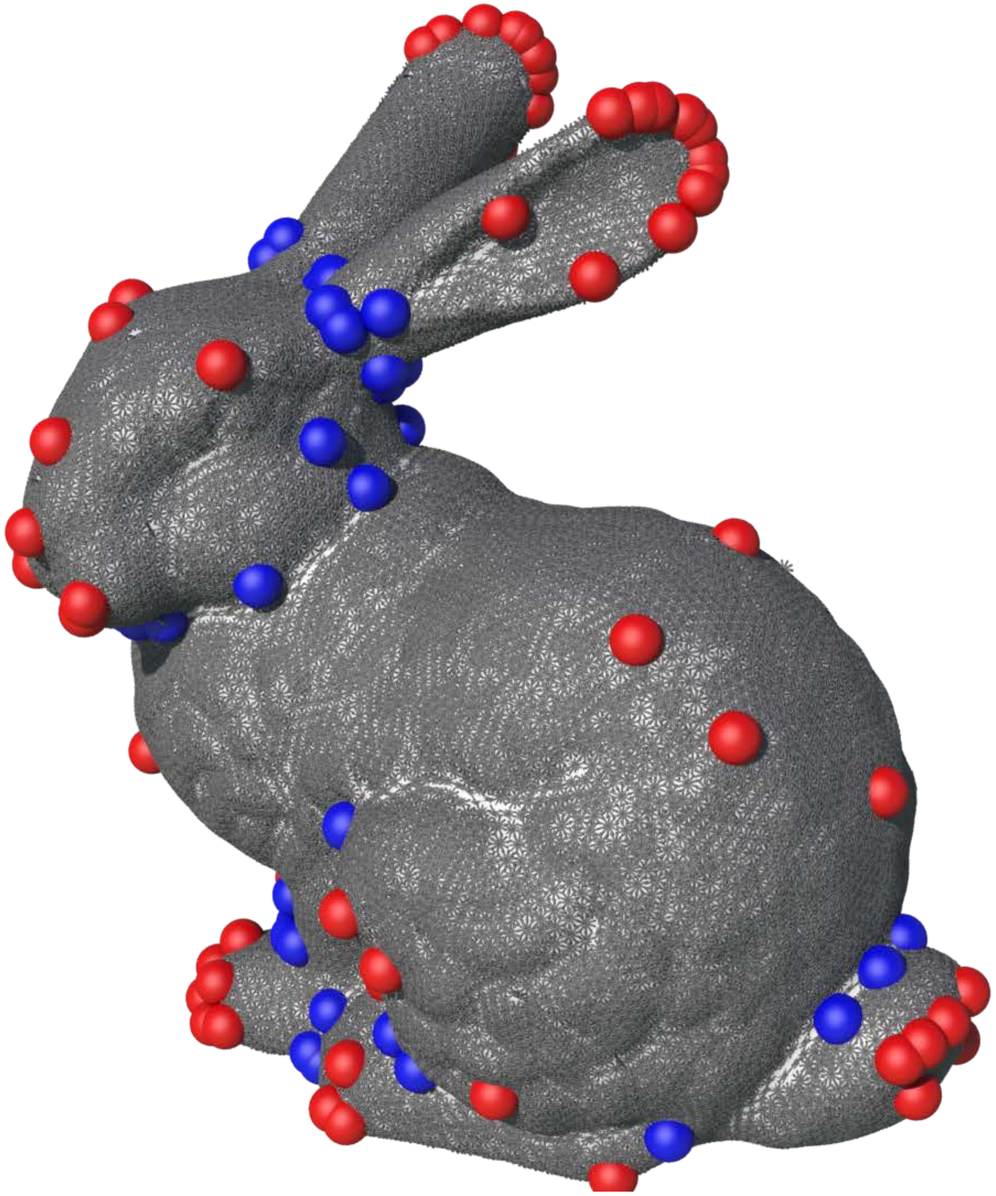
n=3

$$\alpha = 3\alpha^{\mathrm{LC}}$$

$$\beta = 3K\,dA$$

degree = 6

Singularities in optimal n-direction field



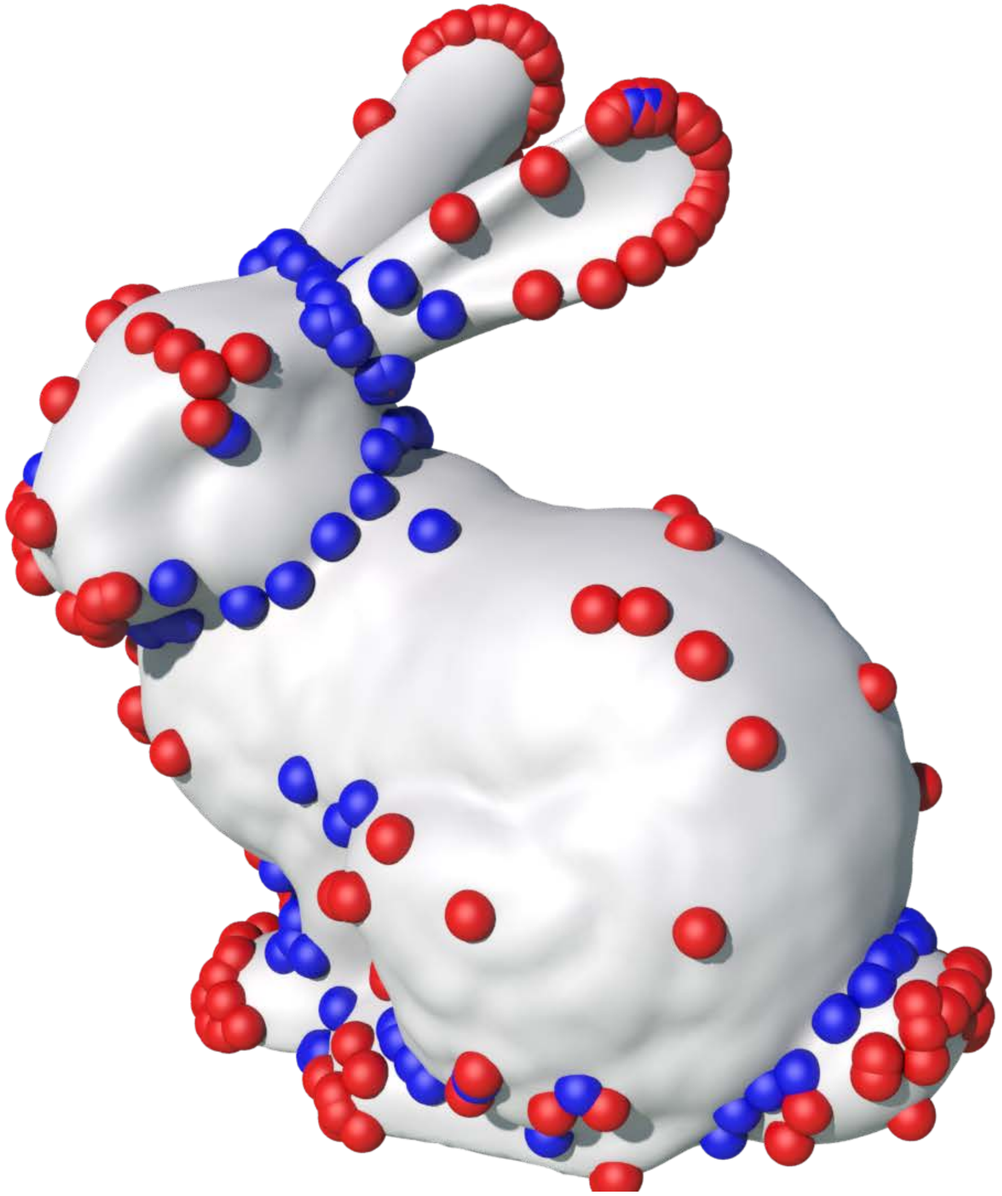
n=10

$$\alpha = 10\alpha^{\mathrm{LC}}$$

$$\beta = 10K\,dA$$

degree = 20

Singularities in optimal n-direction field



n=20

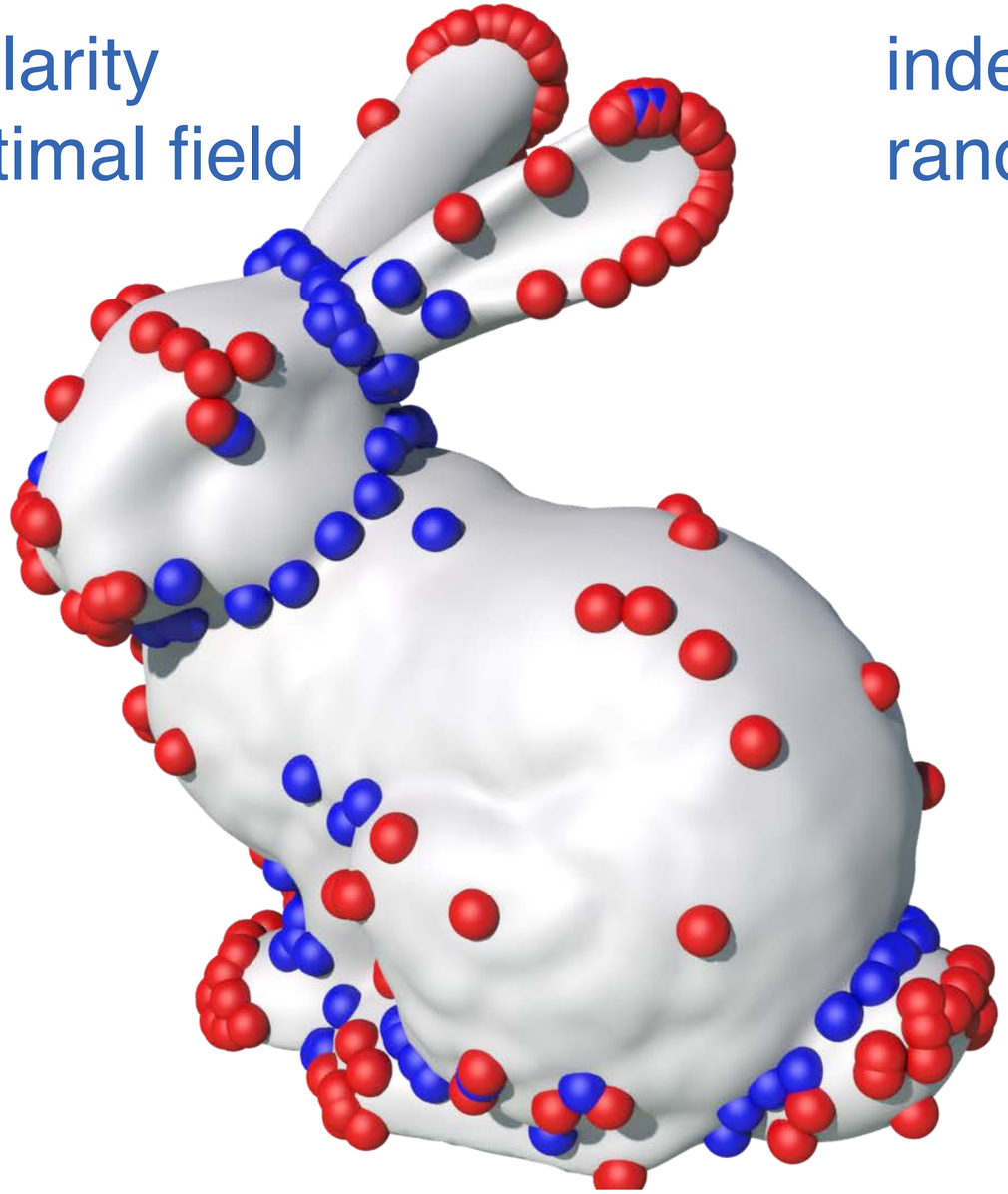
$$\alpha = 20\alpha^{\mathrm{LC}}$$

$$\beta = 20K\,dA$$

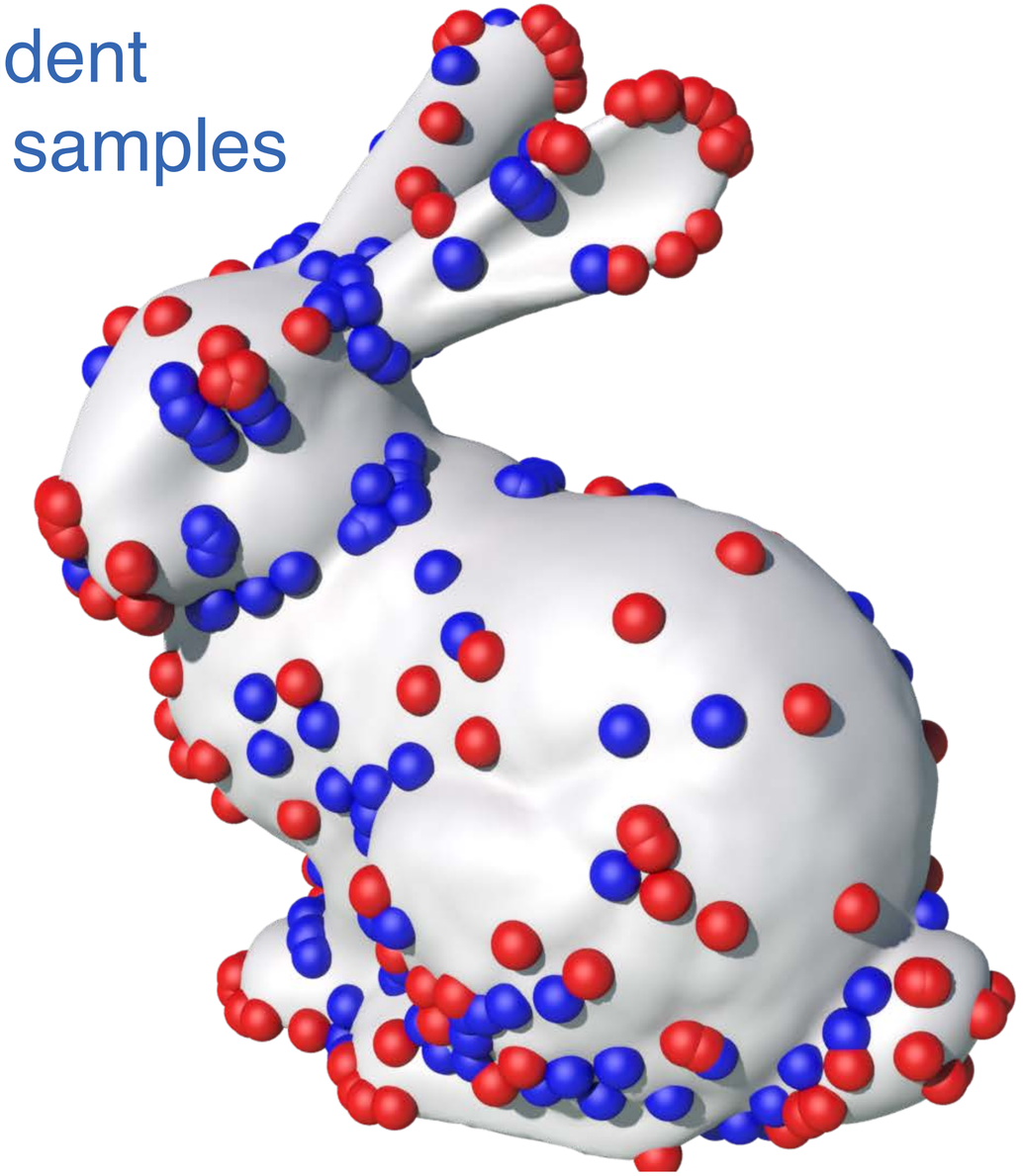
degree = 40

Singularities in optimal n-direction field

singularity
of optimal field



independent
random samples



Gauge theory in physics

- Hairy ball theorem
- Optimal sections of a bundle
- Curvature and singularities
- Gauge theory in physics

Continuous theory

- In the discrete setting

$$(d^{\nabla}\psi)_{ij} = e^{-\mathbb{i}\alpha_{ij}}\psi_j - \psi_i$$

- Towards continuous

$$\psi_j \approx \psi_i + d\psi(\vec{ij}) + \cdots$$

$$e^{-\mathbb{i}\alpha_{ij}} \approx 1 - \mathbb{i}\alpha(\vec{ij}) + \cdots$$

$$(d^{\nabla}\psi)(\vec{ij}) \approx \left(1 - \mathbb{i}\alpha(\vec{ij})\right)\left(\psi_i + d\psi(\vec{ij})\right) - \psi_i$$

$$= d\psi(\vec{ij}) - \mathbb{i}\alpha(\vec{ij})\psi_i + O(|\vec{ij}|^2)$$

$$d^{\nabla}\psi = d\psi - \mathbb{i}\alpha\psi$$

Continuous theory

- Continuous covariant derivative

$$d^{\nabla}\psi = d\psi - \mathbb{i}\alpha\psi$$

$$\boxed{d^{\nabla} = d - \mathbb{i}\alpha\wedge} : \Omega^k(M;\mathbb{C}) \to \Omega^{k+1}(M;\mathbb{C})$$

- Double covariant exterior derivative

$$\begin{aligned} d^{\nabla}d^{\nabla}\psi &= (d - \mathbb{i}\alpha\wedge)(d - \mathbb{i}\alpha)\psi \\ &= \underbrace{\boxed{d\,d\psi}}_{0} \underbrace{\boxed{-\mathbb{i}d(\alpha\psi)}}_{-\mathbb{i}(d\alpha)\psi + \mathbb{i}\alpha\wedge d\psi} - \mathbb{i}\alpha\wedge d\psi - \underbrace{\boxed{\alpha\wedge\alpha\psi}}_{0} \\ &= -\mathbb{i}\beta\psi \end{aligned}$$

curvature of the connection

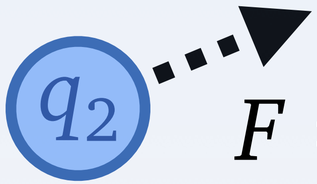
Electrodynamics



charged particle

Electrodynamics





$F = q_2 E$

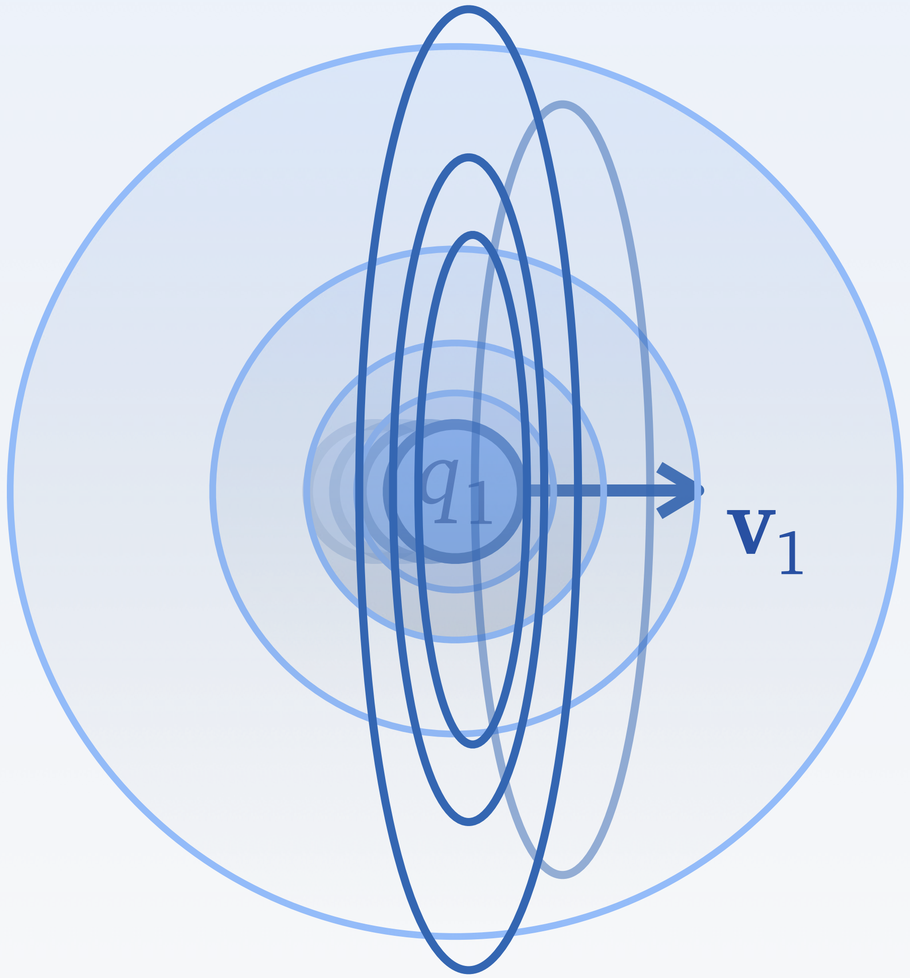
Coulomb electric force

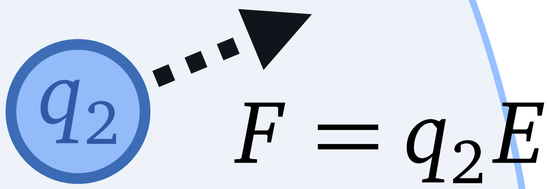
electric field

$E \in \Omega^1(M; \mathbb{R})$

$d \star E = \rho$ charge density n-form

Electrodynamics





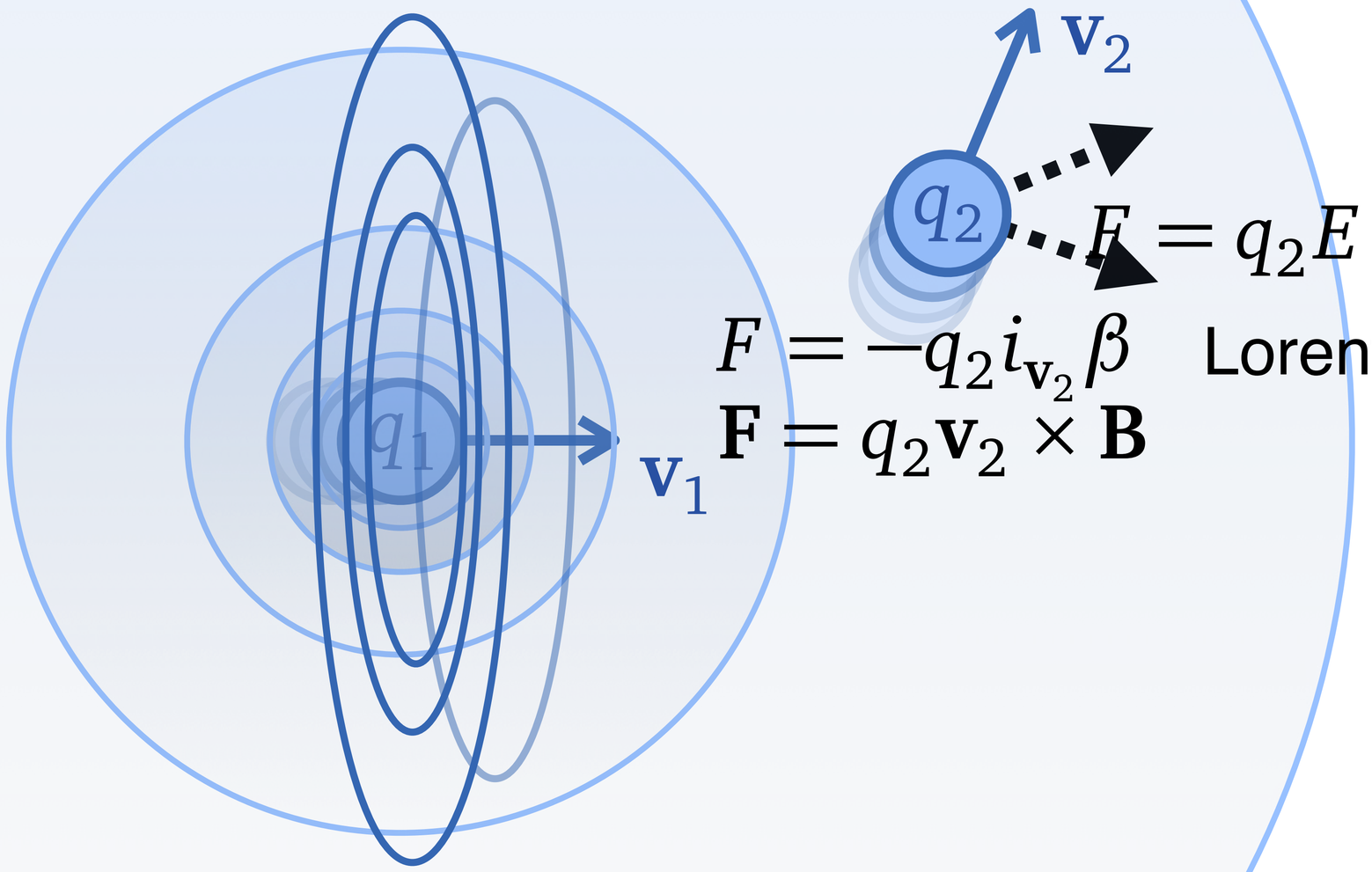
magnetic field

$\beta \in \Omega^2(M; \mathbb{R})$

$d\beta = 0$

$d \star \beta = J$ electric current flux form

Electrodynamics



$F = q_2 E$

$F = -q_2 i_{\mathbf{v}_2} \beta$ Lorentz magnetic force

$\mathbf{F} = q_2 \mathbf{v}_2 \times \mathbf{B}$

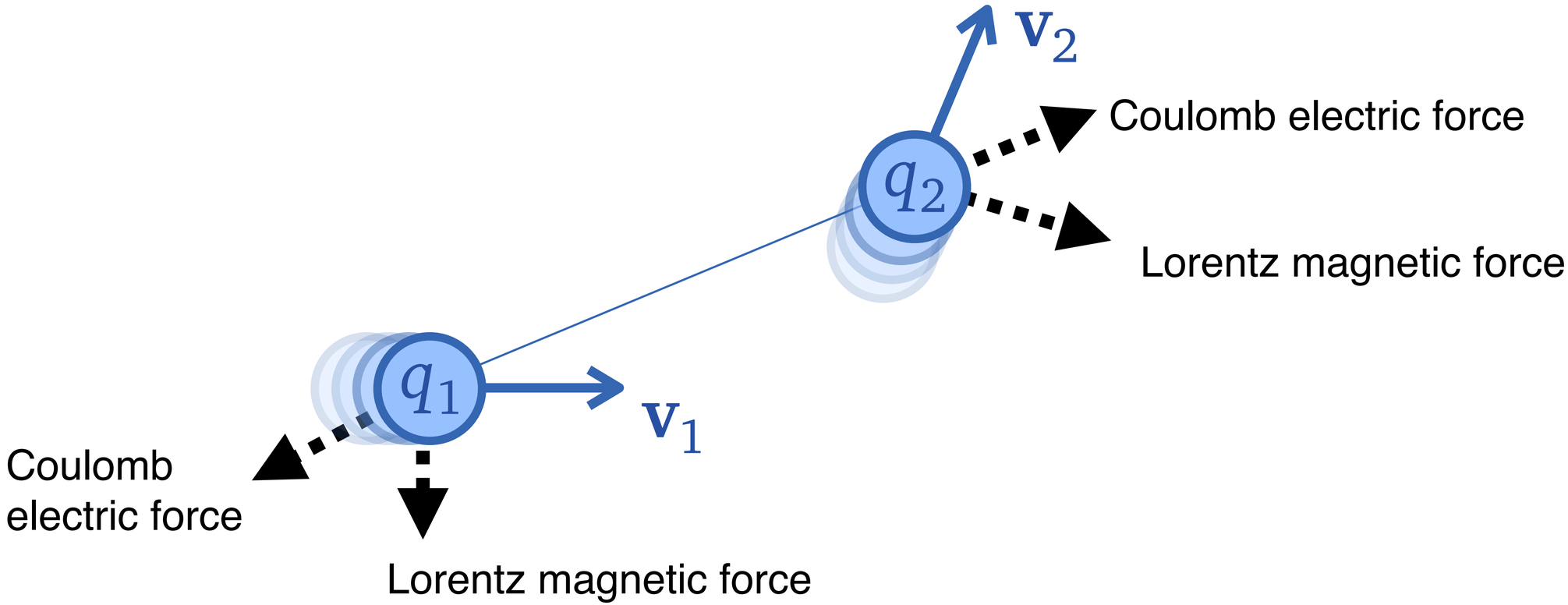
magnetic field

$\beta \in \Omega^2(M; \mathbb{R})$

$d\beta = 0$

$d \star \beta = J$ electric current flux form

Electrodynamics



This figure seems to violate
the conservation of momentum

Vector potential

- The magnetic field $\beta \in \Omega^2(M)$ satisfies $d\beta = 0$
- Kelvin (1851) and Helmholtz (1858): by Hodge decomposition

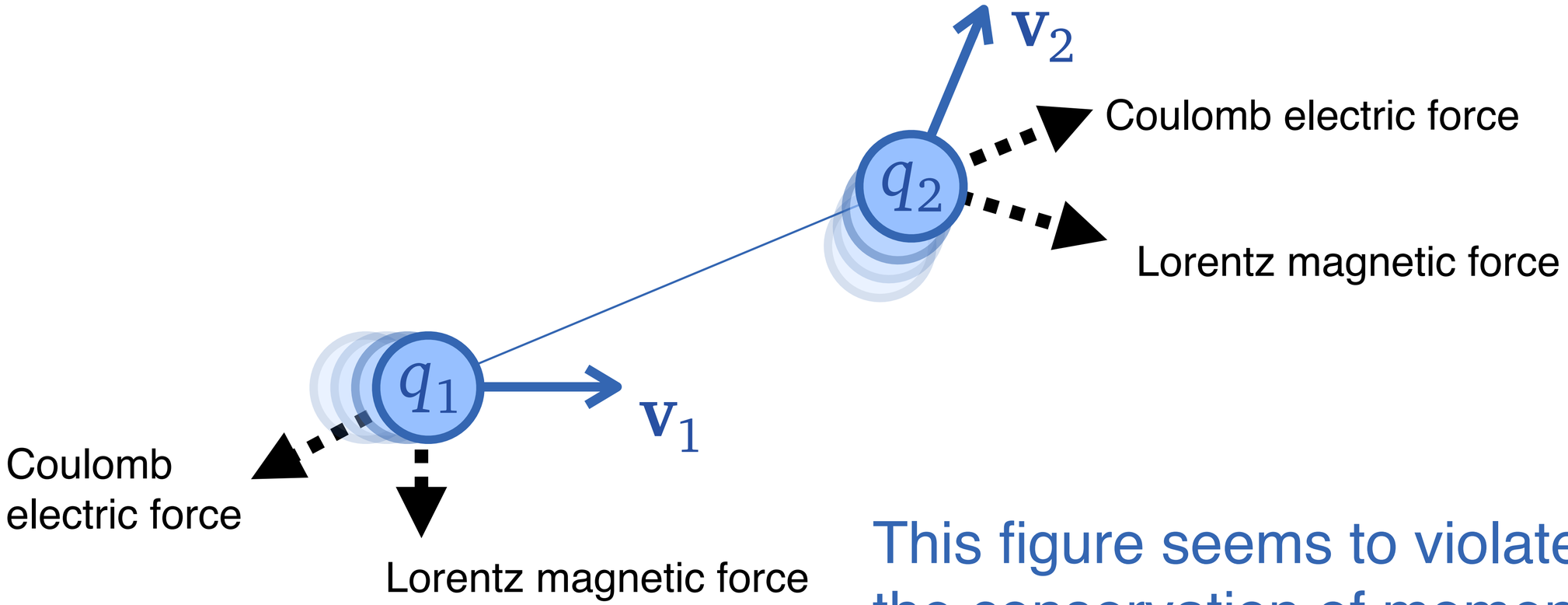
$$\beta = d\alpha$$

- The potential $\alpha \in \Omega^1(M)$ is not unique

$$\alpha, \quad \alpha + d\varphi$$ are equality valid representation.

- A change of choice $\alpha \mapsto \tilde{\alpha} = \alpha + d\varphi$ is called a gauge transformation
- All physically observable quantities should be gauge-invariant.

Electrodynamics

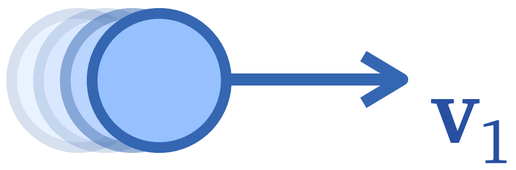


This figure seems to violate
the conservation of momentum

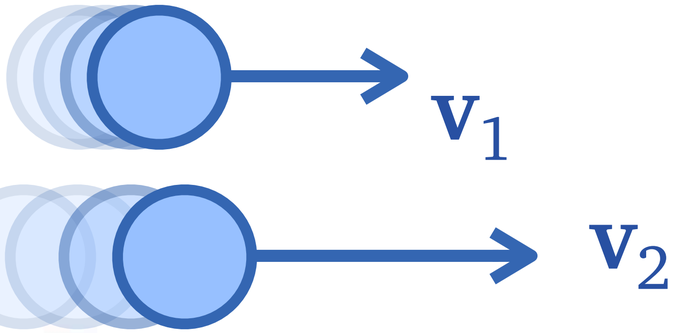
Problem solved if we revise
the momentum as $\mathbf{p}^\flat = m\mathbf{v}^\flat + q\alpha$

(but now the momentum is not gauge invariant)

Quantum mechanics



Quantum mechanics



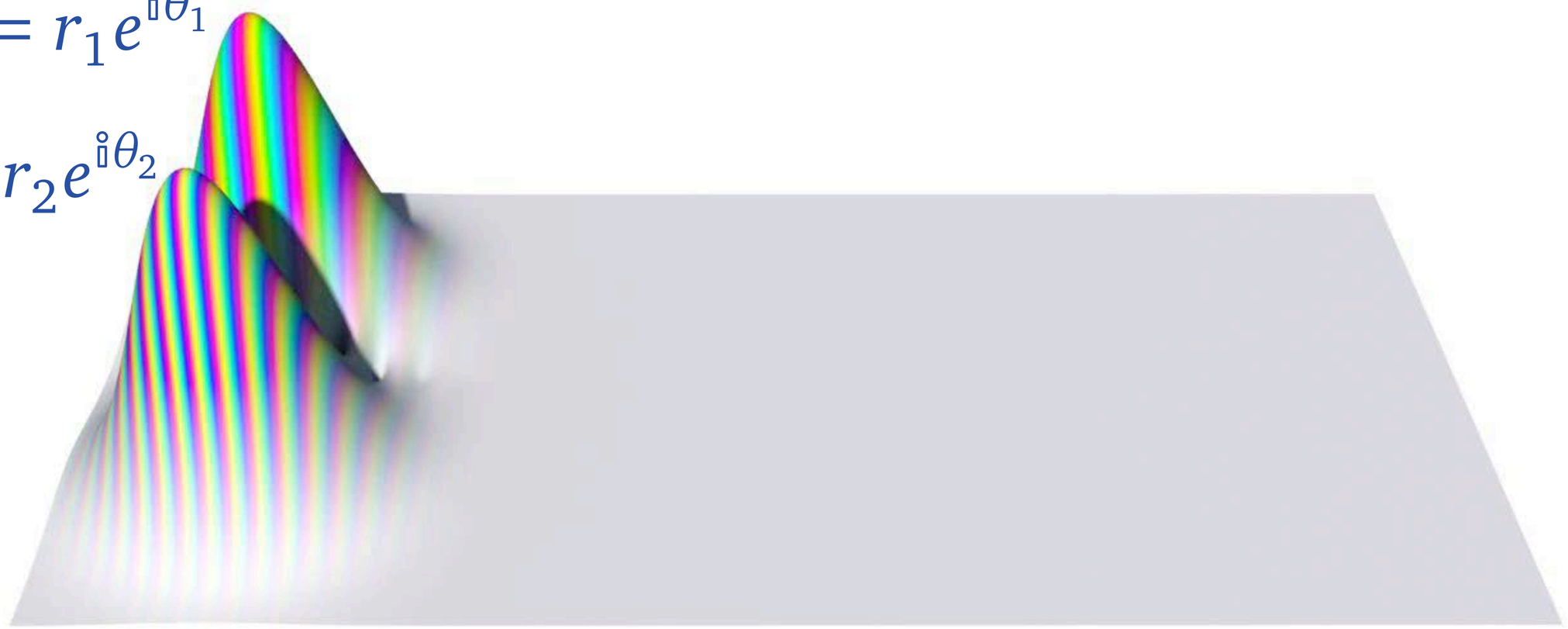
$\mathbf{v}_1$

$\mathbf{v}_2$

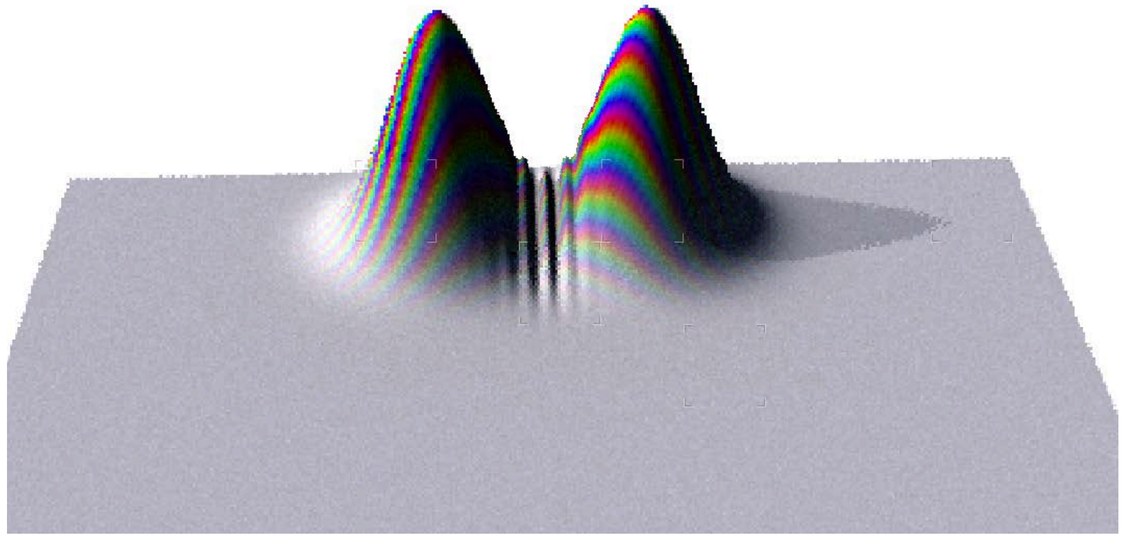
$\mathbf{p}^\flat = r^2 d\theta$

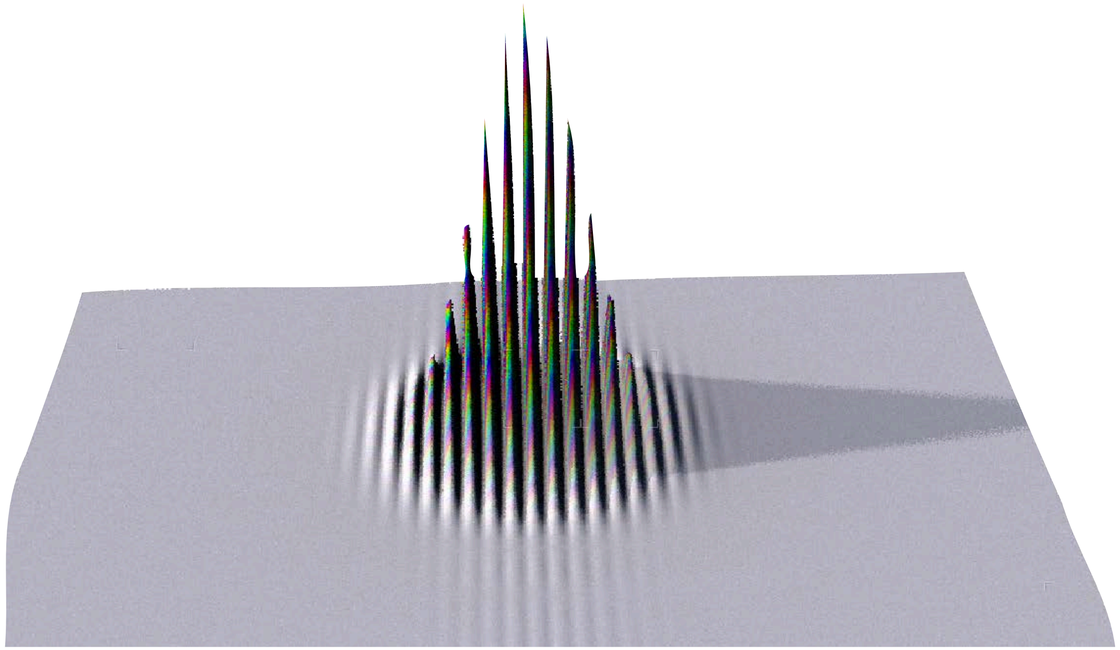
$\psi_1 = r_1 e^{i\theta_1}$

$\psi_2 = r_2 e^{i\theta_2}$



Quantum mechanics





Quantum mechanics

$$\psi = re^{\mathbb{i}\theta}$$

$$\mathbf{p}^\flat = r^2 d\theta$$

$$= \overline{\psi}(-\mathbb{i}d)\psi$$

Total kinetic energy is our Dirichlet energy

$$\int \frac{1}{2m}|\mathbf{p}^\flat|^2 = \frac{1}{2m}\int \overline{\psi} d \star d\psi$$

Quantum electrodynamics

$$\psi = re^{\mathbb{i}\theta}$$

$$\mathbf{p}^\flat = r^2 d\theta + r^2 q\alpha$$

$$= \overline{\psi}(-\mathbb{i}d)\psi + q\alpha\overline{\psi}\psi = \overline{\psi}(-\mathbb{i}(d + \mathbb{i}q\alpha))\psi = \overline{\psi}(-\mathbb{i}d^\nabla)\psi$$

Total kinetic energy is our Dirichlet energy

$$\int \frac{1}{2m}|\mathbf{p}^\flat|^2 = \frac{1}{2m}\int \overline{\psi} d \star d\psi$$

Quantum electrodynamics

$$\psi = re^{\mathbb{i}\theta}$$

$$\mathbf{p}^\flat = r^2 d\theta + r^2 q\alpha$$

$$= \overline{\psi}(-\mathbb{i}d)\psi + q\alpha\overline{\psi}\psi = \overline{\psi}(-\mathbb{i}(d + \mathbb{i}q\alpha))\psi = \overline{\psi}(-\mathbb{i}d^\nabla)\psi$$

Total kinetic energy is our Dirichlet energy

$$\int \frac{1}{2m}|\mathbf{p}^\flat|^2 = \frac{1}{2m}\int \overline{\psi}(\overline{d^\nabla}^\intercal \star d^\nabla)\psi$$

Quantum electrodynamics

Gauge transform in quantum electrodynamics

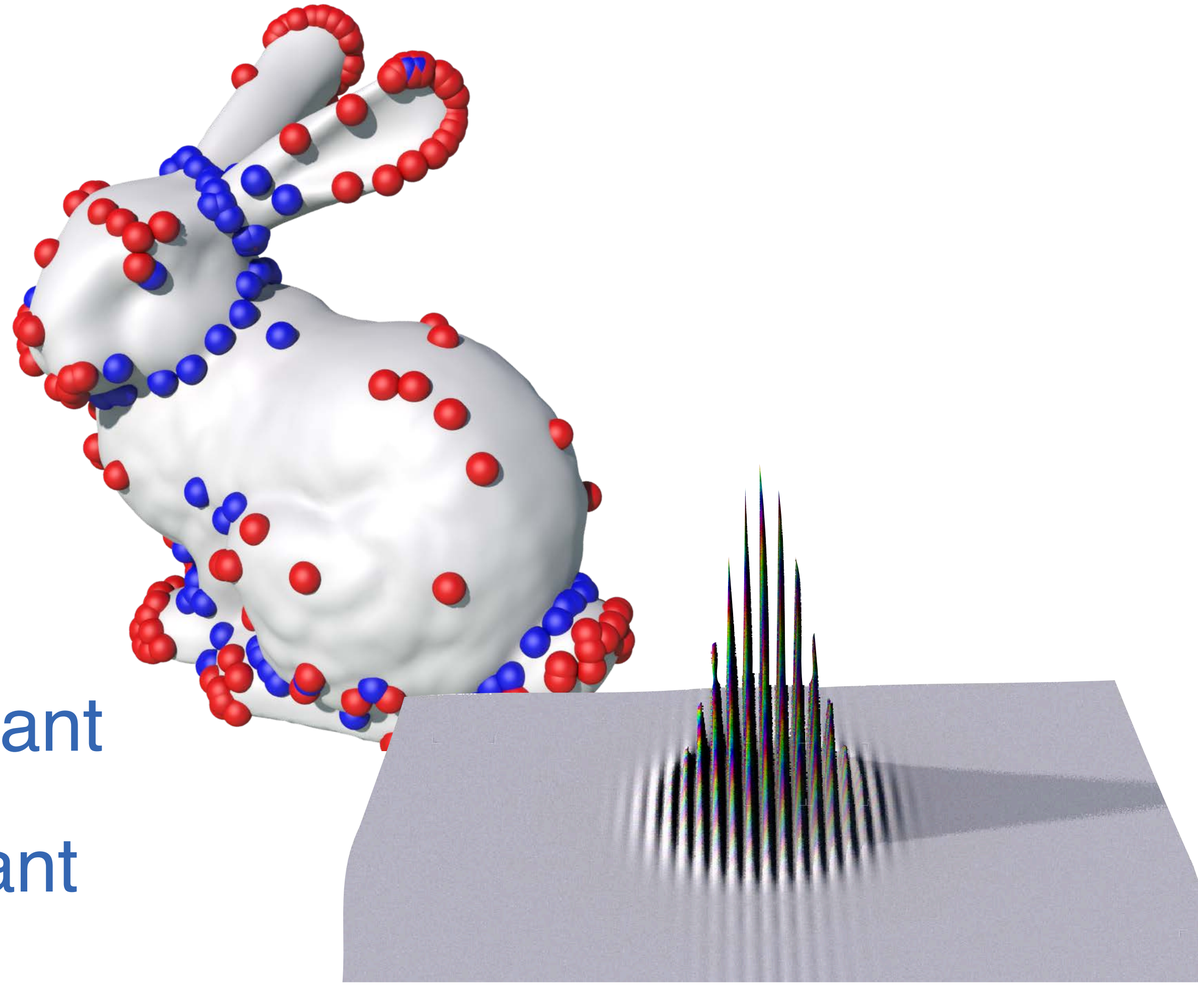
$$\psi \mapsto e^{\mathbb{i}\varphi}\psi$$
$$\alpha \mapsto \alpha + d\varphi$$

Curvature of the bundle

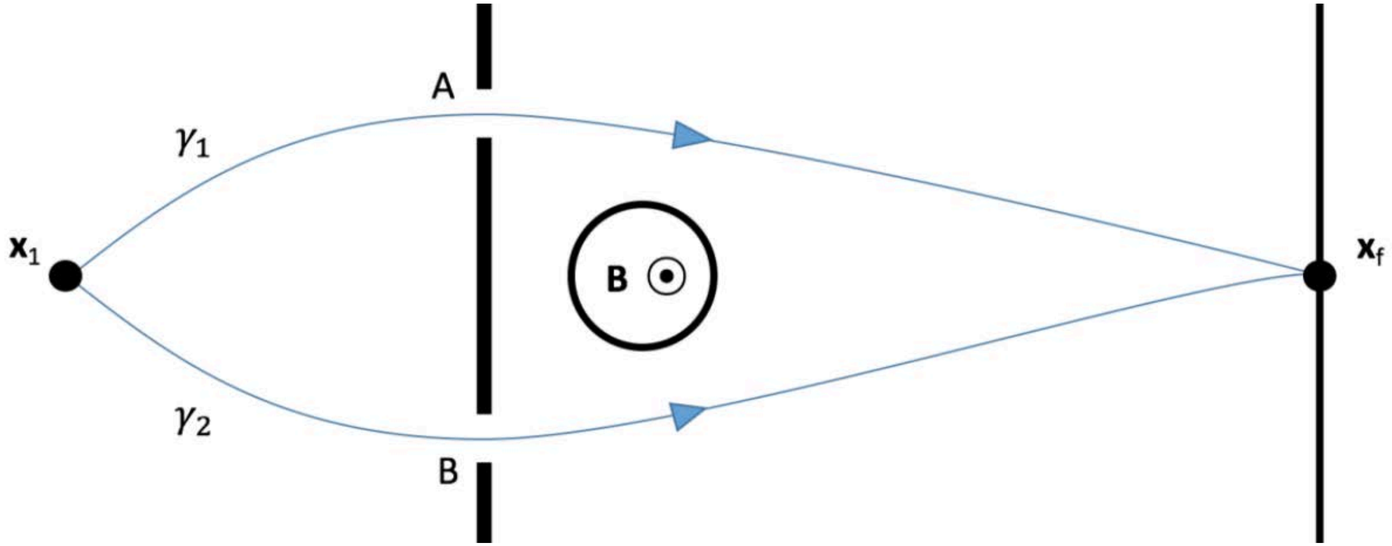
$\beta = d\alpha$ is gauge invariant

Interference pattern is gauge invariant

Singularity location is gauge invariant



Aharonov–Bohm effect



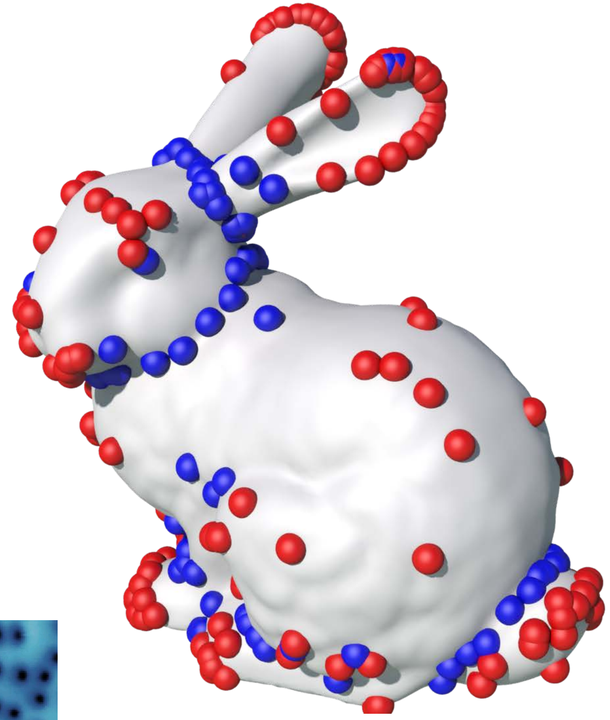
What determines the complex line bundle?

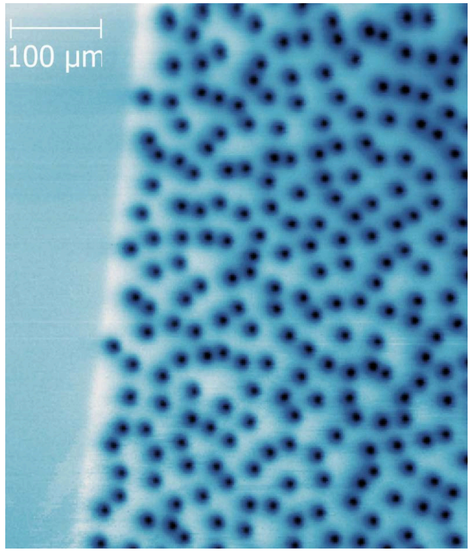
By Hodge decomposition

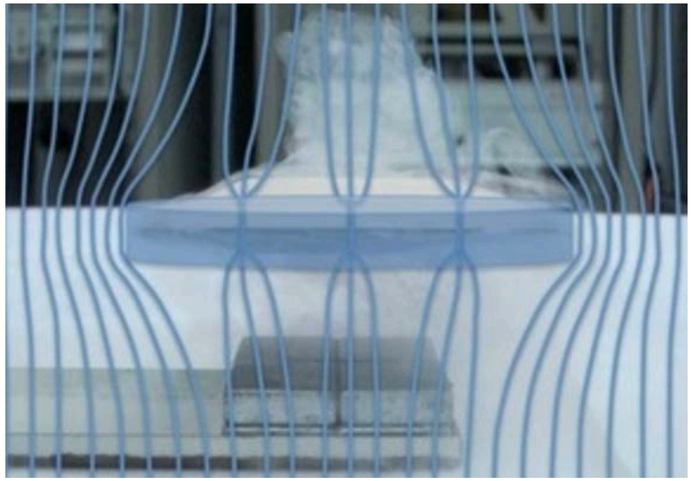
$\beta = d\alpha$ and $\int_{\substack{\text{global} \\ \text{loop}}} \alpha$ allows us to reconstruct α up to a factor of $d\varphi$

Optimal singularity placement

- Prescribe magnetic field β (e.g. as Gaussian curvature) and global period $\int_{\text{loop}}^{\text{global}} \alpha$
- Construct α and thus $L^{\nabla} = \overline{d^{\nabla}}^{\intercal} \star_1 d^{\nabla}$
- Find the singularities of the ground state.







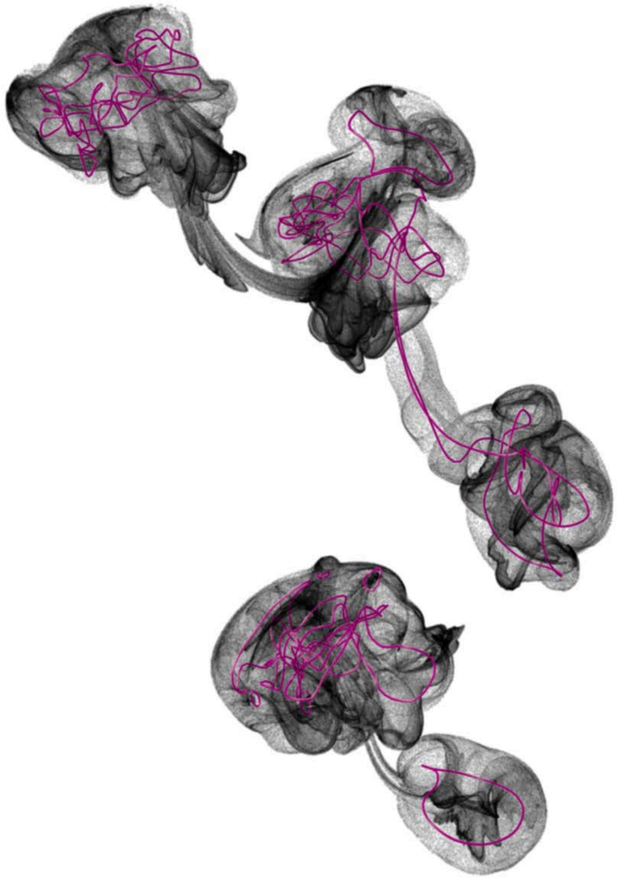
Optimal singularity placement



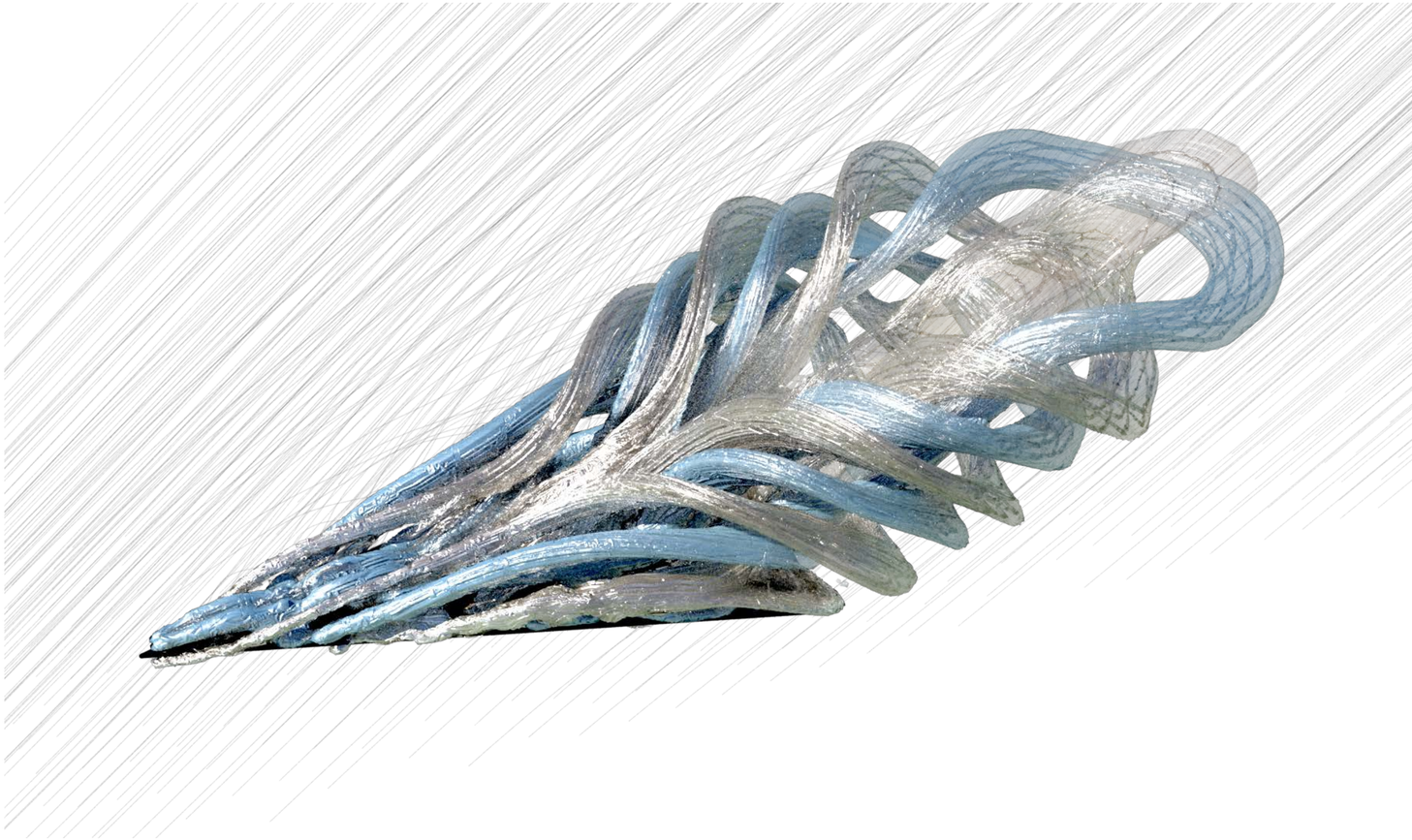
[Knöppel, Crane, Pinkall & Schröder 2015]
Stripe patterns on surfaces

Optimal singularity placement

Complex phase field for vortex detection



[Weißmann, Pinkall & Schröder 2014]
Smoke rings from smoke



[Chern, Knöppel, Pinkall & Schröder 2017]
Inside fluid: Clebsch maps for visualization and processing