CSE 270 (FA25)
Discrete Differential Geometry
Discrete Exterior Calculus

Albert Chern

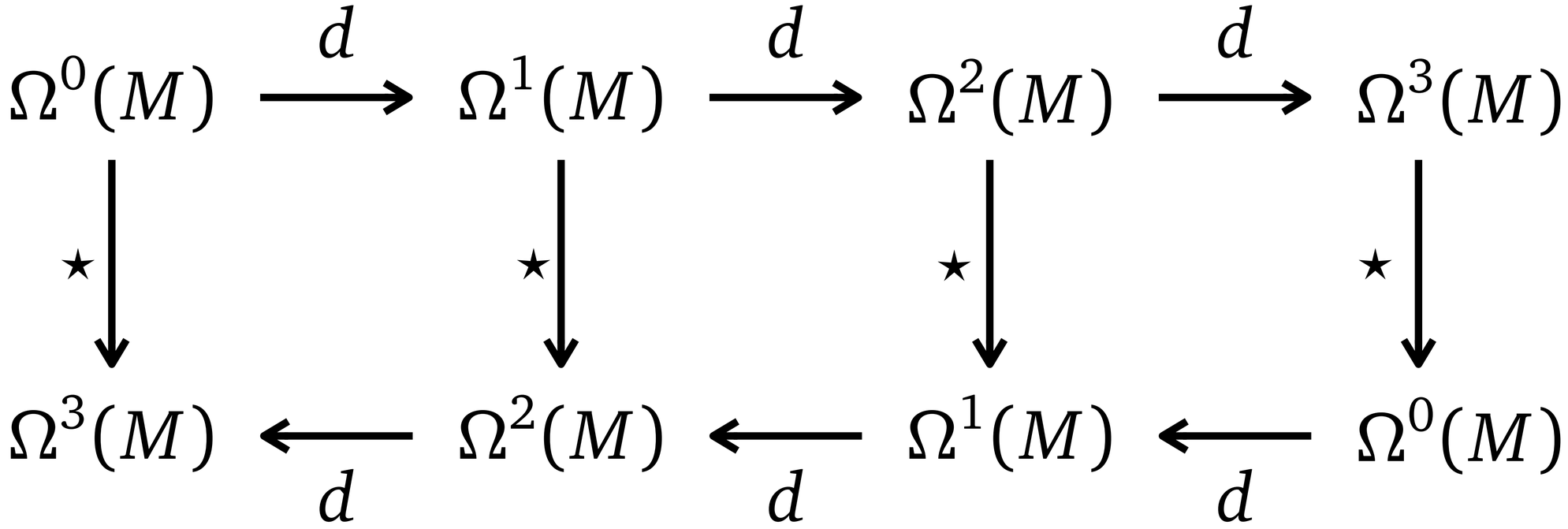
UC San Diego
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Discrete Differential Forms

- Discrete differential forms
- Discrete Hodge stars
- Example: heat equation

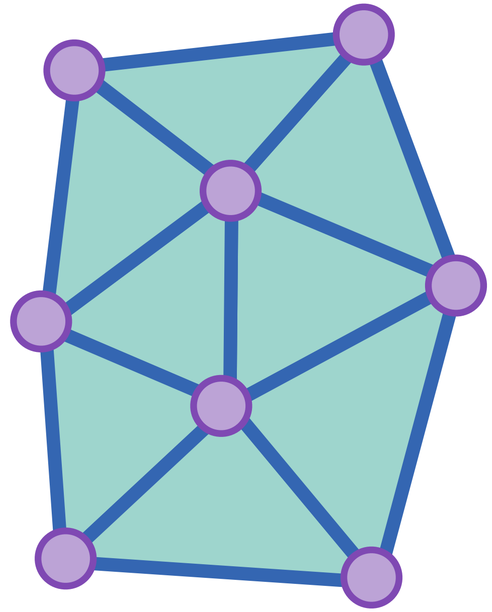
Differential forms

- In the smooth theory, differential *k*-forms are objects to be integrated on a *k*-dimensional geometry.
- Exterior derivatives and Hodge stars form the following diagram:



Discrete differential forms

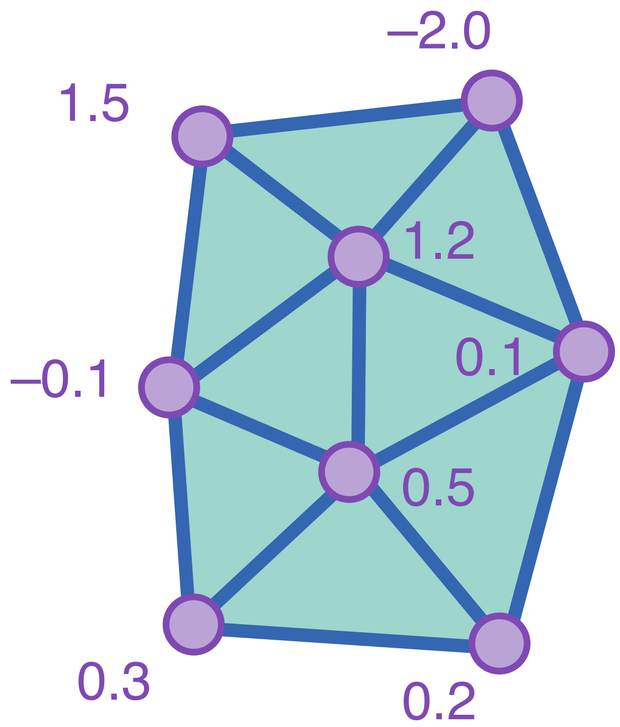
- Consider a discrete triangle mesh $M = (\mathcal{P}, \mathcal{E}, \mathcal{F})$



Discrete differential forms

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- We encode a 0-form $\phi \in \Omega^0(M)$ by its samples on vertices

$$\hat{\phi} : \mathcal{P} \to \mathbb{R} \qquad \hat{\phi}_p = \phi(p) \qquad p \in \mathcal{P}$$



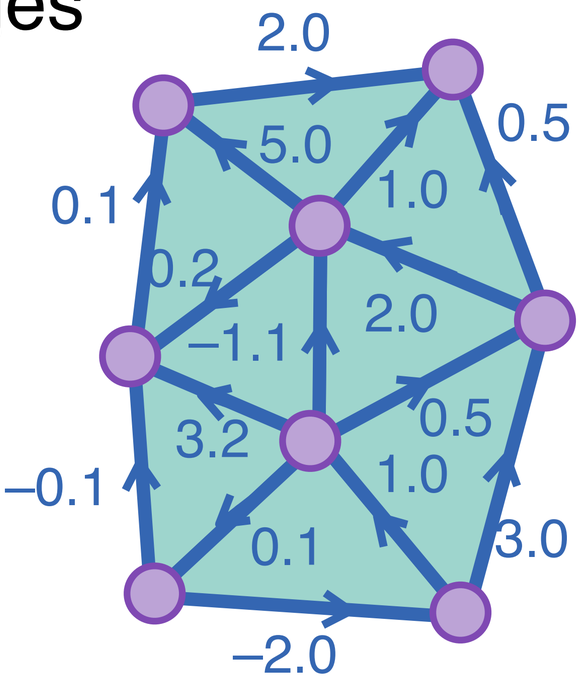
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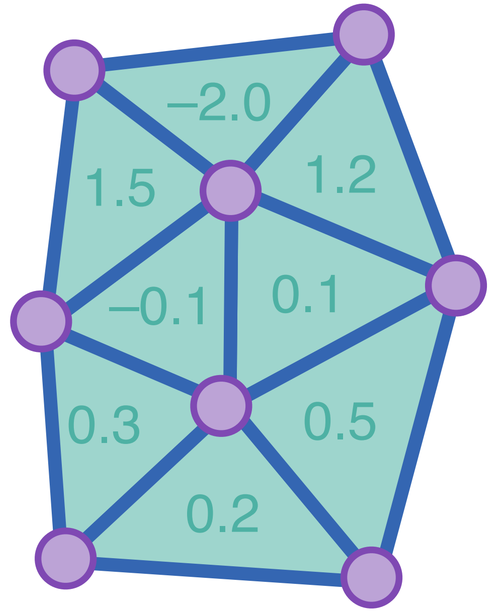
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- Encode 2-form $\omega \in \Omega^2(M)$ by its integrals on faces

$$\hat{\omega} : \mathcal{F} \to \mathbb{R} \qquad \hat{\omega}_f = \int_f \omega \qquad f \in \mathcal{F}$$



Discrete differential forms

- Each discrete *k*-form $\boldsymbol{\omega}$ is an assignment of numbers per simplex on the *k*-skeleton

$$\boldsymbol{\omega} = (\omega_{\sigma_1}, \ldots, \omega_{\sigma_{|M_k|}})^{\mathsf{T}}$$

- Given any *k*-chain $\gamma = \sum_{i=1}^{M_k} c_i \, \sigma_i \in C_k$, represented by the array

$$\mathbf{c} = (c_1, \ldots, c_{|M_k|})^{\mathsf{T}}$$

The evaluation of the *k*-form on the chain is

$$\int_\gamma \omega \; = \sum_{i=1}^{M_k} c_i \omega_{\sigma_i} \; = \mathbf{c}^{\mathsf{T}} \boldsymbol{\omega}$$

- Discrete *k*-forms can be viewed as covectors dual to *k*-chains.

Formal definition

- Let $M = (M_0, M_1, M_2, \ldots, M_n)$ be a simplicial/cell complex
- Recall: space of *k*-chains $C_k = \text{span}_{\mathbb{R} \text{ or } \mathbb{Z}} M_k$

a *k*-chain is a linear combination of *k*-cells

- There are boundary operators making chain spaces a chain complex

$$C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \qquad \partial_k \circ \partial_{k+1} = 0$$

Definition The space of *k*-cochains (discrete *k*-forms)

$$C^k := \left\{ \alpha \,\middle|\, \alpha \colon C_k \xrightarrow{\text{linear}} \mathbb{R} \right\} = C_k^*$$

A *k*-cochain is an assignment of number on each *k*-cell.

Discrete exterior derivative

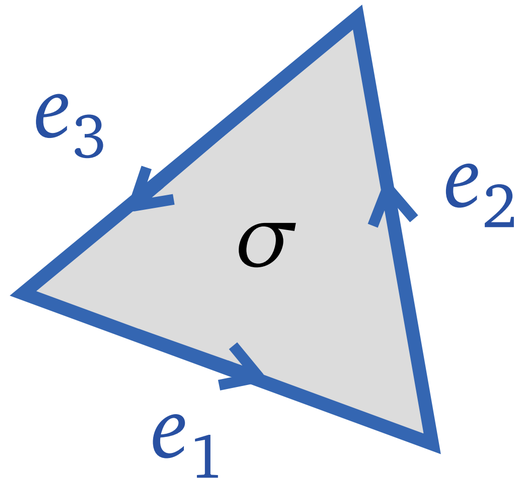
- Representing discrete forms as integrated quantity make defining exterior derivatives easy.
- Take advantage of Stokes' Theorem

$$\int_S d\alpha = \int_{\partial S} \alpha$$

- For example,

$$\widehat{(d\alpha)}_\sigma = \int_\sigma d\alpha \overset{\text{Stokes' Thm}}{=} \int_{\partial\sigma} \alpha$$

$$= \sum_{i=1}^{3} \int_{e_i} \alpha = \sum_{i=1}^{3} \hat{\alpha}_{e_i}$$



Discrete exterior derivative

Definition

The ***discrete exterior derivative*** (or **coboundary operator**) on co-chains is the adjoint of the boundary operator on chains:

$$\mathbf{d}_k : C^k \xrightarrow{\text{linear}} C^{k+1}$$

$$\langle \mathbf{c} \,|\, \mathbf{d}_k \boldsymbol{\alpha} \rangle = \langle \partial_{k+1} \mathbf{c} \,|\, \boldsymbol{\alpha} \rangle \text{ for all } \mathbf{c} \in C_{k+1}$$

- In matrix form: $\mathbf{d}_k = \partial_{k+1}^{\mathsf{T}}$
- Immediate consequence

$$\mathbf{d}_{k+1} \circ \mathbf{d}_k = \mathbf{0}$$

discrete exterior derivatives form a chain complex

$$C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \xrightarrow{d_2} \cdots$$

Discrete Hodge Stars

- Discrete differential forms
- Discrete Hodge stars
- Example: heat equation

Recall Hodge star

- Under orthonormal basis

$$\star(dx_1 \wedge \cdots \wedge dx_k) = dx_{k+1} \wedge \cdots \wedge dx_n$$

orthogonal complementary form

- As an L2 inner product structure for *k*-form

$$\langle\!\langle \alpha, \beta \rangle\!\rangle = \int_M \langle \alpha, \beta \rangle \star 1 = \int_M \alpha \wedge \star\beta \qquad \alpha, \beta \in \Omega^k(M)$$

Dual mesh

Dual Mesh

Triangle Mesh and its Dual

pts oriented edges oriented faces

- Let $M = (M_0, M_1, M_2)$ be a triangle mesh (without boundary for now)

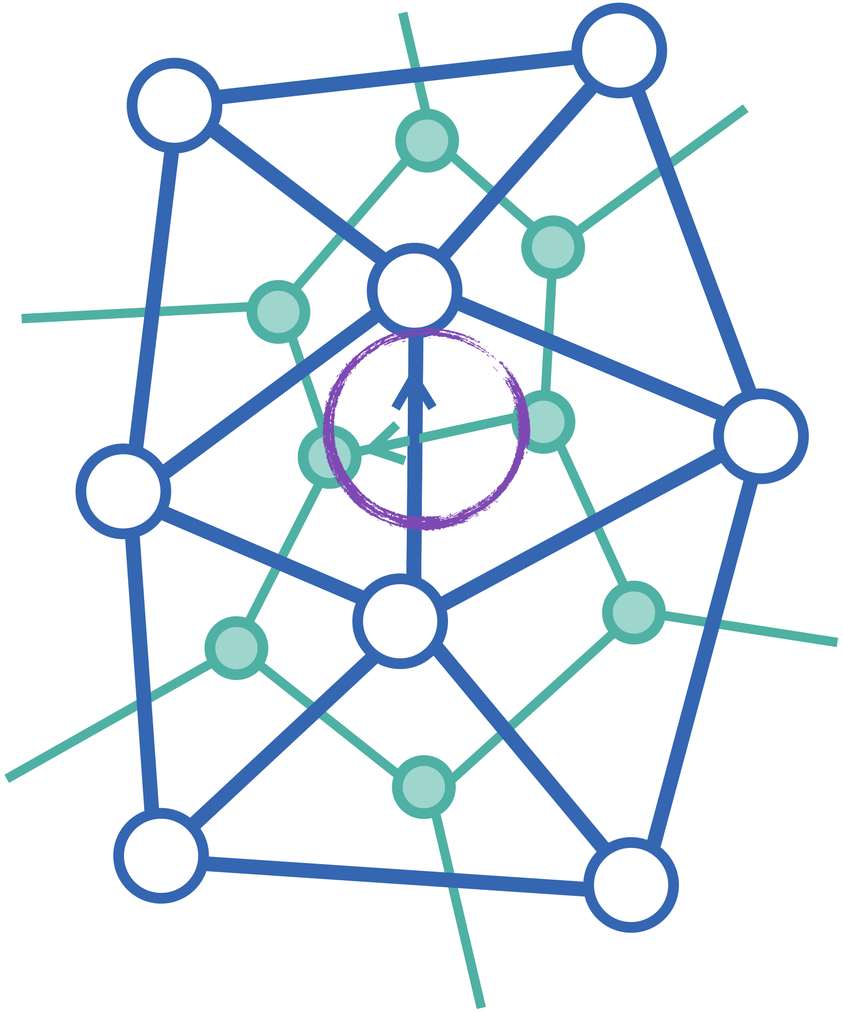
- Consider the *dual mesh*

$$M^* = (M_0^*, M_1^*, M_2^*)$$

dual pts $M_0^* \cong M_2$ primal faces

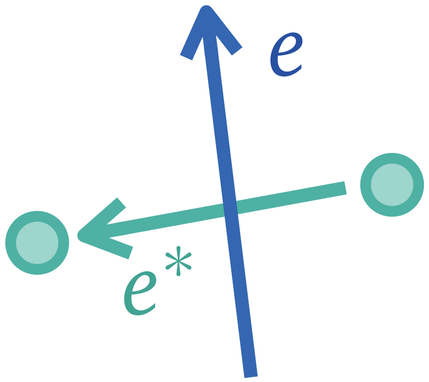
dual edges $M_1^* \cong M_1$ primal edges

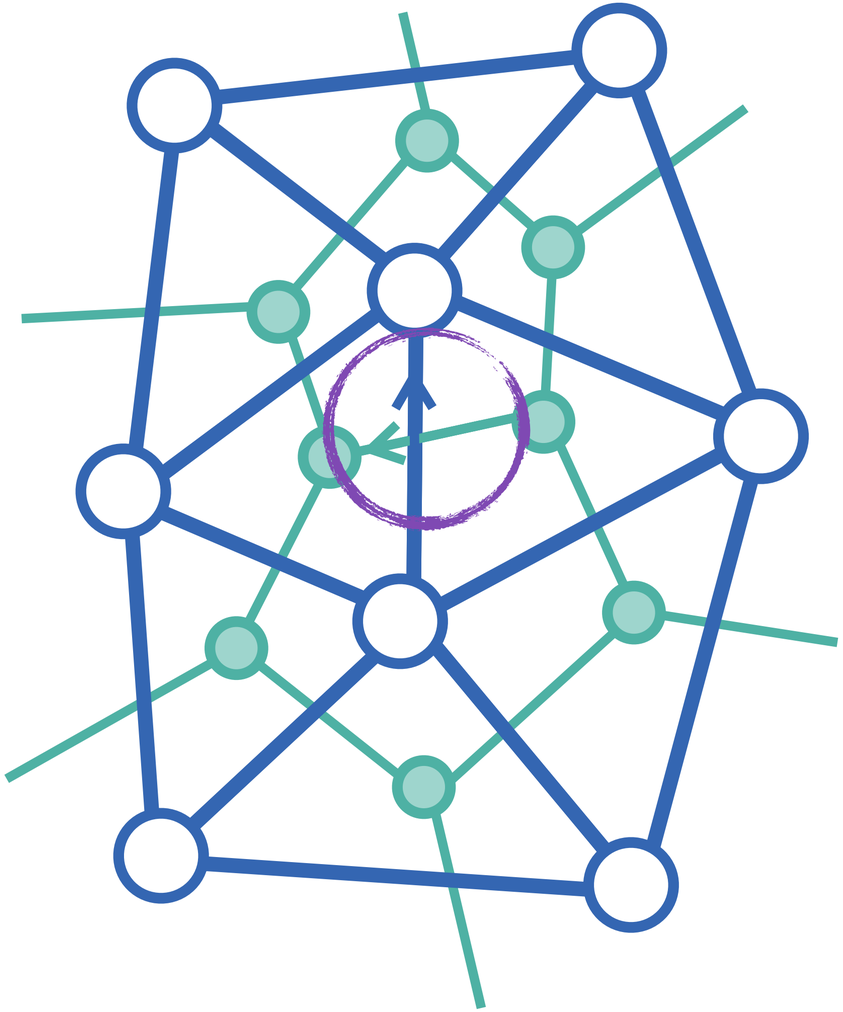
dual faces $M_2^* \cong M_0$ primal pts



Triangle Mesh and its Dual

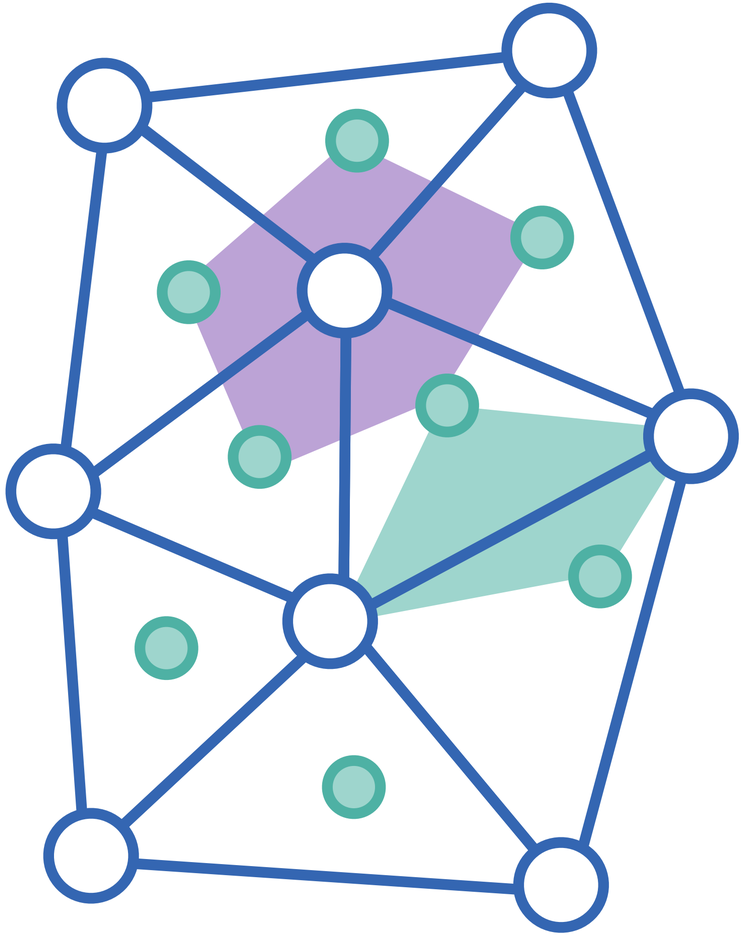
- Orientation of the dual edge e^* is chosen as pointing from the RHS triangle of e to the LHS triangle of e





Dual mesh

- By connecting dual mesh vertices to primal mesh vertices
  - we have notions of “territory” for primal vertex and primal edge.
  - The entire surface can be partitioned into territories of vertices (dual cell) or into territories of edges.



Dual mesh

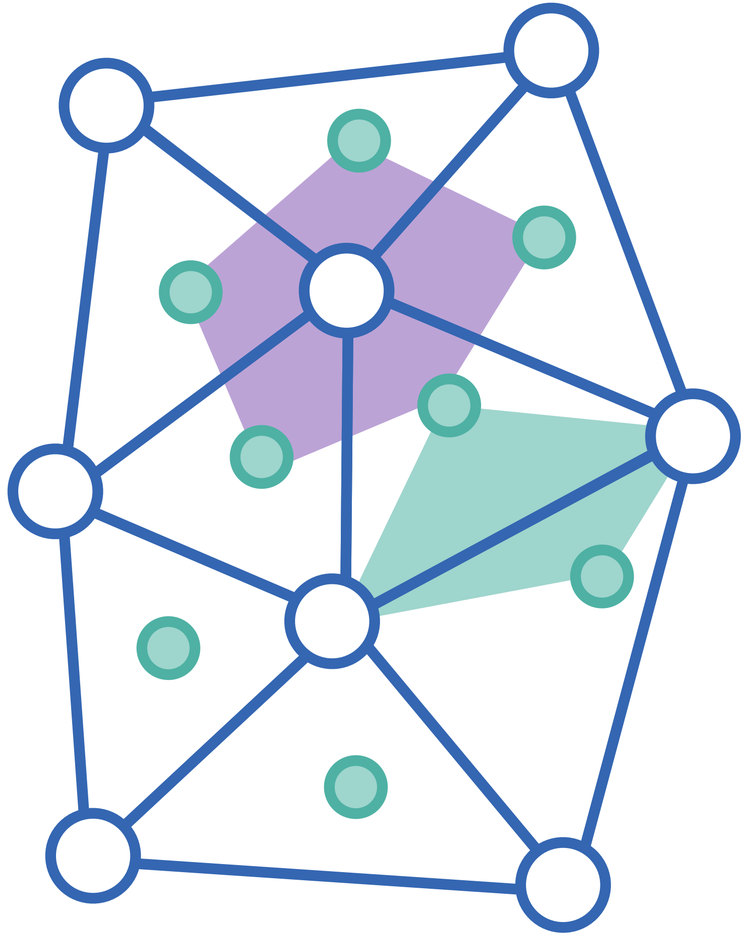
- Edge cell has an area of

projection

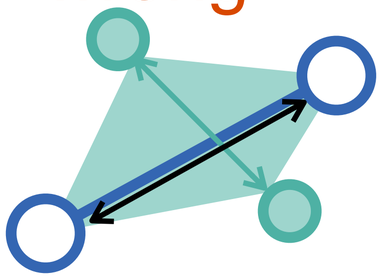
$$\frac{1}{2}|\mathbf{P}_{e^\perp} e^*||e|$$

edge's cross section width

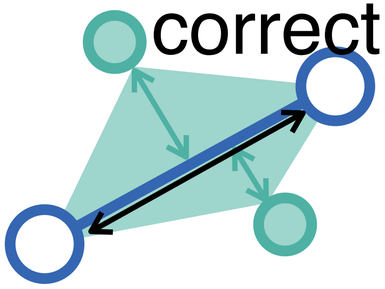
edge length



wrong

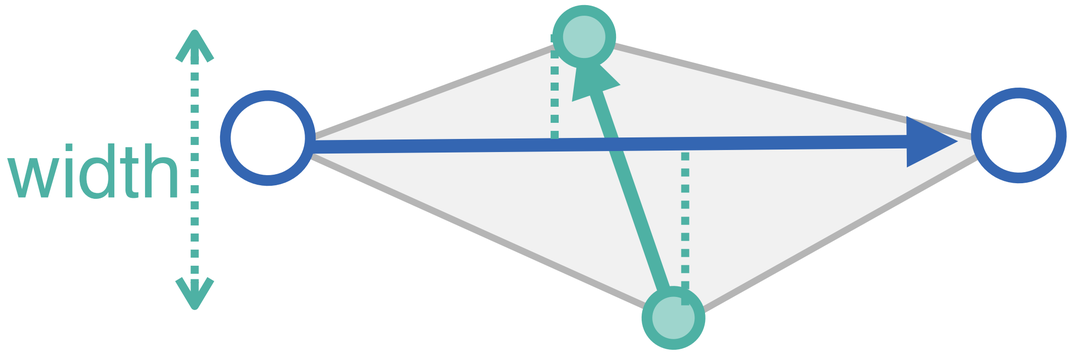


correct



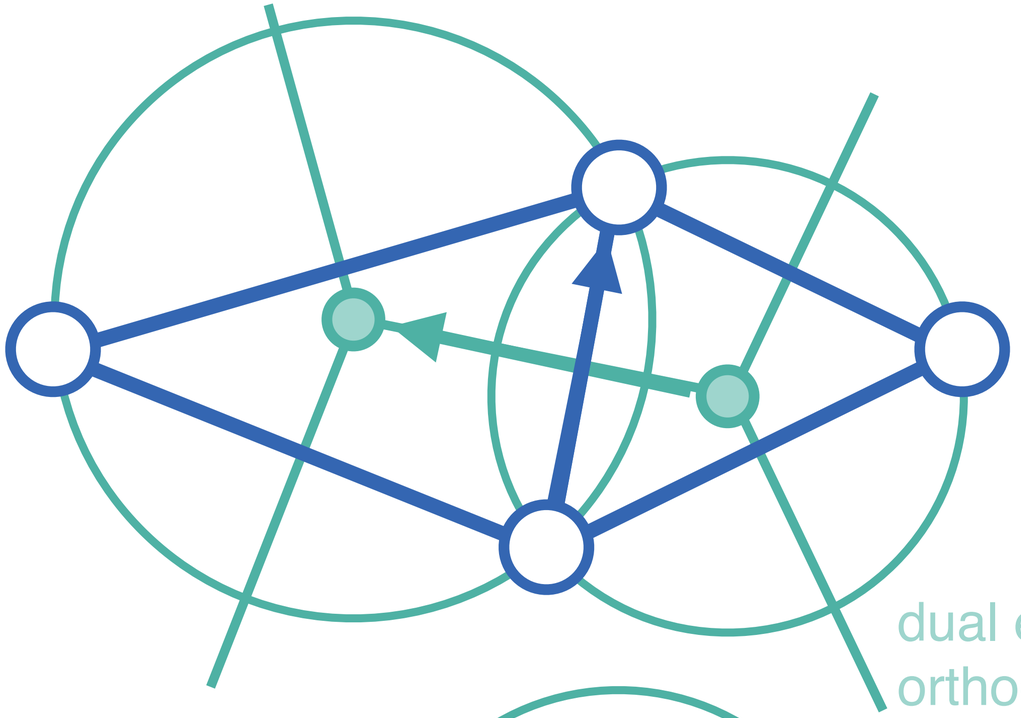
length

width

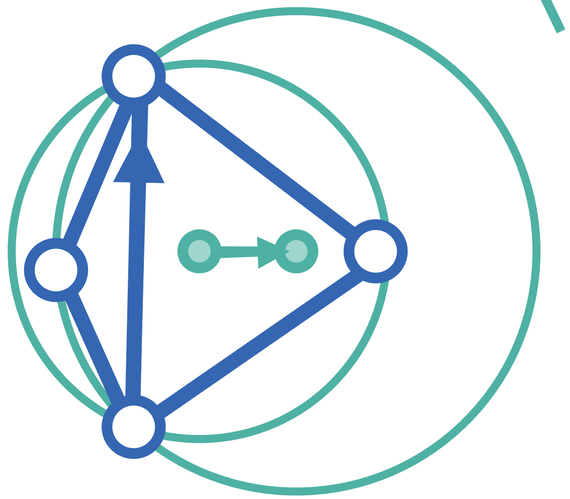


Where to put dual center

- Circumcenter
  - edge width = dual edge length
  - Relies on good triangle mesh



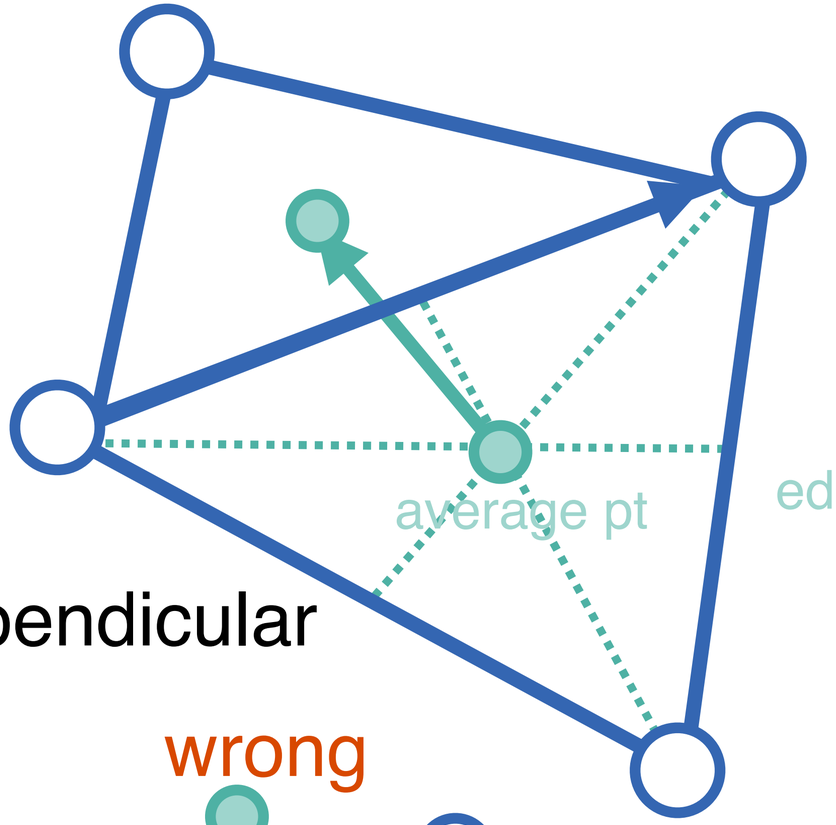
dual edges are the orthogonal bisector of primal edges



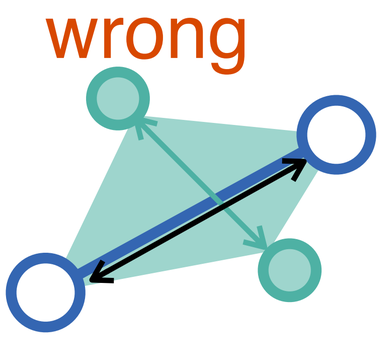
Non-Delaunay mesh: edge weight becomes negative

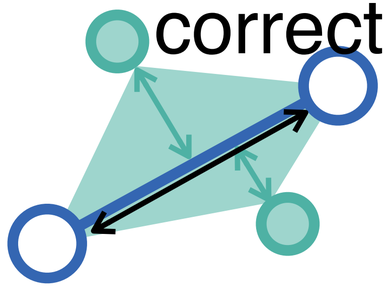
Where to put dual center

- Barycenter
  - Always in the interior of the triangle



  - Dual edge is no longer perpendicular to the primal edge (the “edge width” requires projection)





Discrete Hodge star

Back to Hodge star

Discrete Hodge star

$$\star(dx_1 \wedge \cdots \wedge dx_k) = dx_{k+1} \wedge \cdots \wedge dx_n$$

orthogonal complementary form

- If we put a local orthonormal basis aligned with the simplex
- we observe

$$\frac{1}{|\sigma|}\int_{\sigma} \alpha \approx \frac{1}{|\mathbf{P}_{\sigma^\perp}\sigma^*|}\int_{\mathbf{P}_{\sigma^\perp}\sigma^*} \star\alpha$$

Discrete Hodge star

Definition

Define $\star_k : C^k(M) \xrightarrow{\text{linear}} C^{n-k}(M^*)$ by a diagonal matrix

with

$$(\star_k \alpha)_{\sigma^*} = \frac{|\mathbf{P}_{\sigma^\perp} \sigma^*|}{|\sigma|} \alpha_\sigma$$

$$\star_k = \begin{bmatrix} w_{\sigma_1} & & & \\ & w_{\sigma_2} & & \\ & & \ddots & \\ & & & w_{\sigma_{|M_k|}} \end{bmatrix} \qquad w_\sigma = \frac{|\mathbf{P}_{\sigma^\perp} \sigma^*|}{|\sigma|}$$

On triangle mesh

point masses,
point weights

$$\star_0 = \begin{bmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_{|\mathcal{P}|} \end{bmatrix}$$

edge weights,
edge conductivity,
edge stiffness

$$\star_1 = \begin{bmatrix} w_{e_1} & & & \\ & w_{e_2} & & \\ & & \ddots & \\ & & & w_{e_{|\mathcal{E}|}} \end{bmatrix}$$

1/(face area)

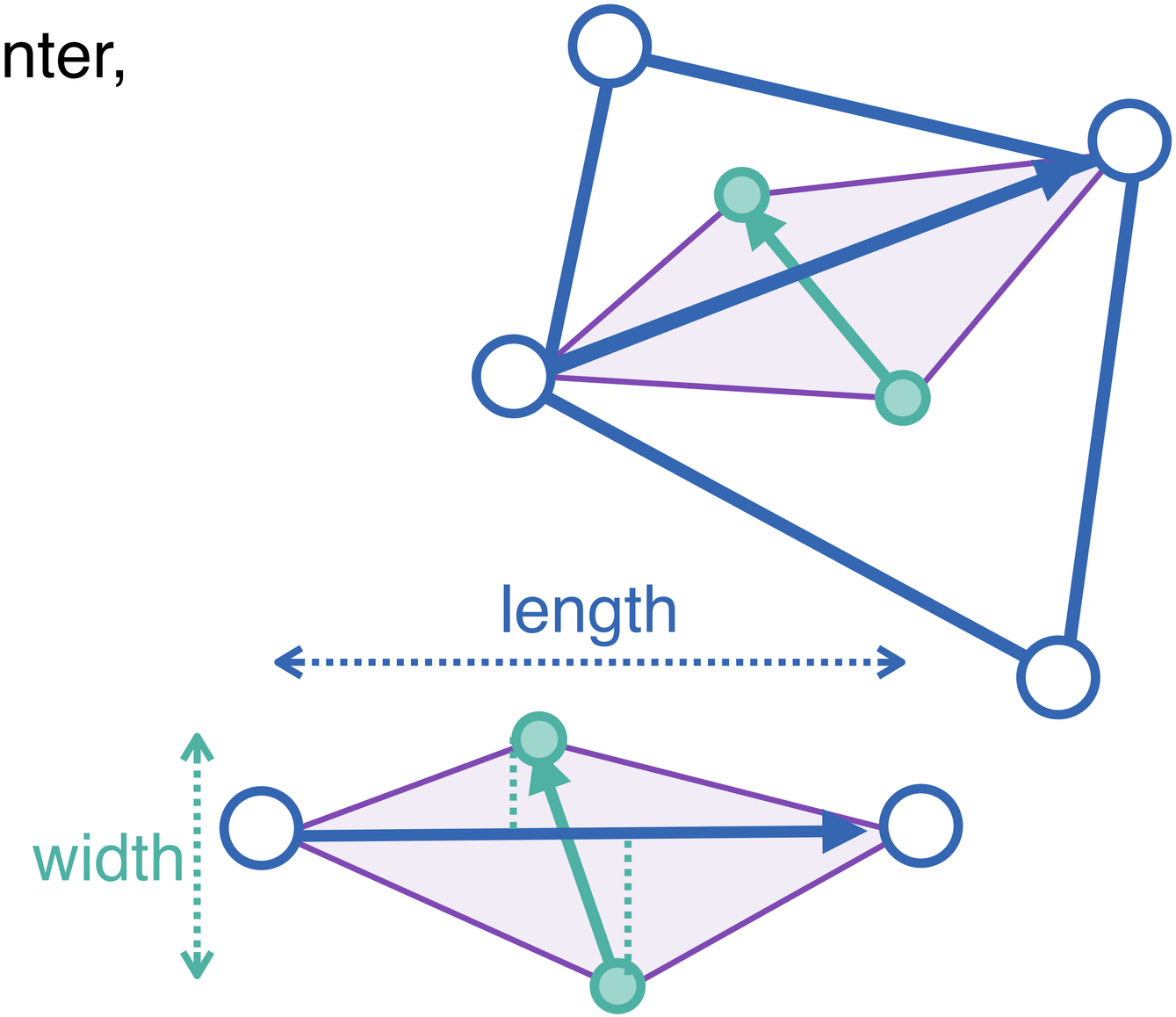
$$\star_2 = \begin{bmatrix} 1/a_{f_1} & & & \\ & 1/a_{f_2} & & \\ & & \ddots & \\ & & & 1/a_{f_{|\mathcal{F}|}} \end{bmatrix}$$

Star on 1-form

- Once we have picked dual center, we build the edge weights

$$w_e = \frac{\text{width}}{\text{length}}$$

$$\star_1 = \begin{bmatrix} w_{e_1} & & & \\ & w_{e_2} & & \\ & & \ddots & \\ & & & w_{e_{|\mathcal{E}|}} \end{bmatrix}$$



Star on 1-form

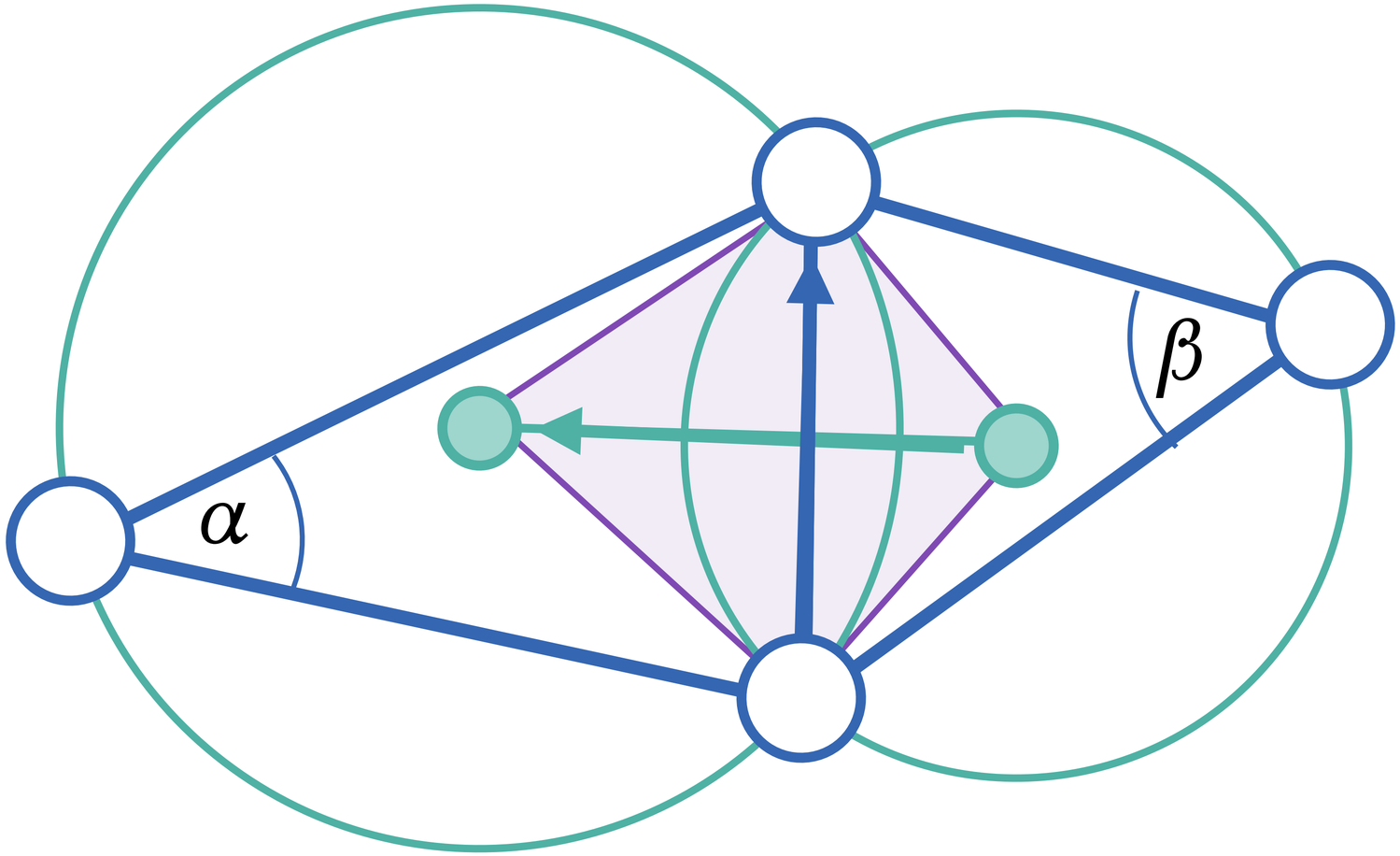
- For circumcentric dual:

$$w_e = \frac{\text{width}}{\text{length}}$$

$$= \frac{1}{2}(\cot\alpha + \cot\beta)$$

(exercise)

known as the
cotangent weight



Star on 0-form

- As for point weight

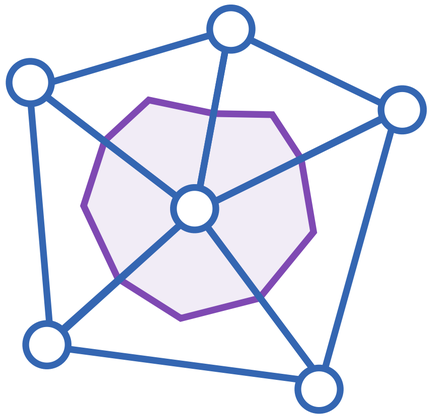
$$m_i = \text{Area}_{\text{pt}_i}$$

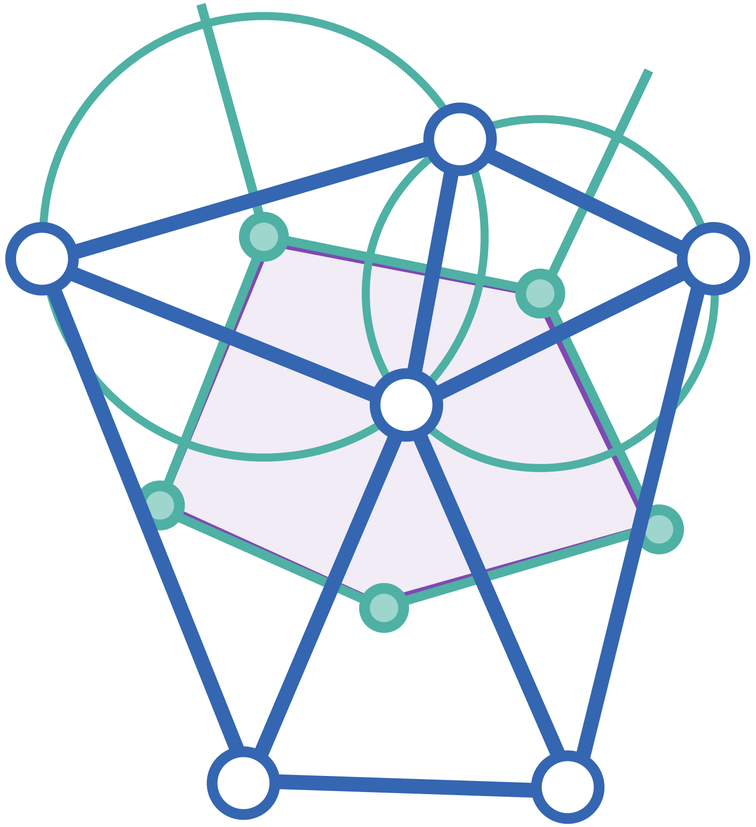
- For circumcentric:

  area of the Voronoi cell

- For barycentric:

$$\frac{1}{3}\sum_{f \succ i} \text{Area}_f$$





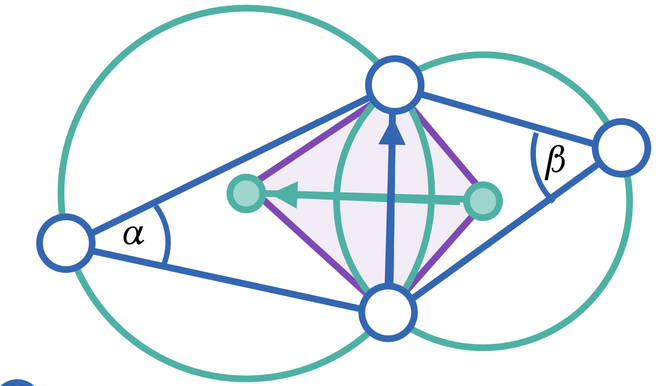
$$\star_0 = \begin{bmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_{|\mathcal{P}|} \end{bmatrix} \in \mathbb{R}^{|\mathcal{P}| \times |\mathcal{P}|}$$

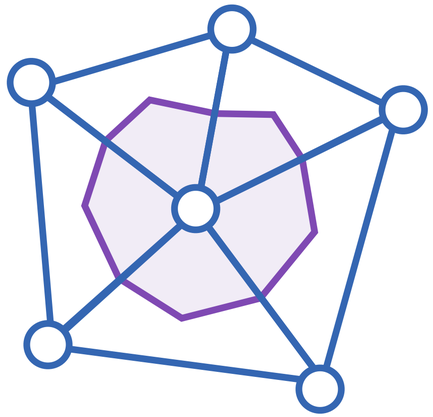
Most common setup

- In practice, we often use circumcentric “cotan” edge weights and the barycentric point area.

$$w_e = \frac{1}{2}(\cot\alpha + \cot\beta)$$

$$m_i = \frac{1}{3}\sum_{f \succ i} \text{Area}_f$$





- It is equivalent to the “piecewise linear Finite Element Method with lumped mass” which has good numerical guarantees.

L2 inner product

Relation to norm-squared
of a differential form

L2 inner product

- Another characterization of Hodge star

$$\langle\!\langle \alpha, \beta \rangle\!\rangle = \int_M \langle \alpha, \beta \rangle \star 1 = \int_M \alpha \wedge \star\beta \qquad \alpha, \beta \in \Omega^k(M)$$

- Discretely

$$\langle\!\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle\!\rangle = \boldsymbol{\alpha}^{\mathsf{T}} [\star_k] \boldsymbol{\beta} = \sum_{i=1}^{M_k} w_{\sigma_k} \alpha_{\sigma_k} \beta_{\sigma_k}$$

L2 inner product

- For 0-forms and 2-forms on triangle mesh

$$\boldsymbol{\phi} \in C^0$$

$$\|\boldsymbol{\phi}\|^2 = \sum_{i=1}^{|\mathcal{P}|} m_i |\phi_i|^2 \approx \int_M |\phi|^2 \, dA$$

$$\boldsymbol{\omega} \in C^2$$

$$\|\boldsymbol{\omega}\|^2 = \sum_{f \in \mathcal{F}} \frac{1}{a_f} |\omega_f|^2 = \sum_{f \in \mathcal{F}} a_f \left| \frac{\omega_f}{a_f} \right|^2 \approx \int_M |\omega|^2 \, dA$$

L2 inner product

- For 1-forms

$$\|\boldsymbol{\alpha}\|^2 = \sum_{e \in \mathcal{E}} \frac{|\text{edge width}|}{|\text{edge length}|} |\alpha_e|^2 \overset{?}{\approx} \int_M |\alpha|^2 \, dA$$

  - Let $\hat{e}$ be normalized edge vector, $\hat{e}^\perp$ its 90° rotation
  - Pointwise $|\alpha|^2 = \alpha[\![\hat{e}]\!]^2 + \alpha[\![\hat{e}^\perp]\!]^2$
  - Now,

$$\int_M |\alpha|^2 \, dA \approx \sum_{e \in \mathcal{E}} \frac{\text{width}_e \text{length}_e}{2} |\alpha|^2$$

$$= \sum_{e \in \mathcal{E}} \frac{\text{width}_e \text{length}_e}{2} \left(\alpha[\![\hat{e}]\!]^2 + \alpha[\![\hat{e}^\perp]\!]^2 \right)$$

(assuming edge orientation are sufficiently random)

$$\approx \sum_{e \in \mathcal{E}} \frac{\text{width}_e \text{length}_e}{2} \left(2 \cdot \alpha[\![\hat{e}]\!]^2 \right) = \sum_{e \in \mathcal{E}} \frac{\text{width}_e}{\text{length}_e} \alpha[\![e]\!]^2$$

More general Hodge stars

- Consider the cochain complex $C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \xrightarrow{d_2} \cdots$
- Its dual complex $C_0 \xleftarrow{d_0^\intercal} C_1 \xleftarrow{d_1^\intercal} C_2 \xleftarrow{d_2^\intercal} \cdots$

$$C^{k*} \cong C_k \cong C^{n-k}(M^*)$$

only when dual mesh can be built
(need manifold-ness)

- Define a "norm" for each *k*-cochain: $\|\alpha\|^2 = \boldsymbol{\alpha}^\intercal [\star_k] \boldsymbol{\alpha}$
- This defines a symmetric positive-definite matrix called Hodge star

$$\star_k : C^k \to C_k$$

doesn't need to be diagonal

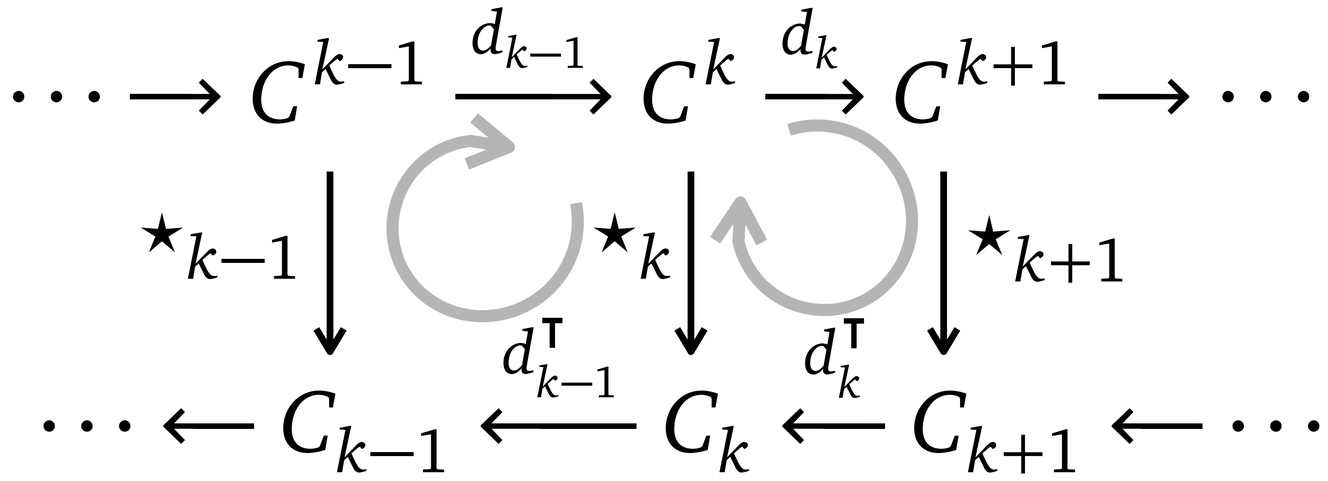
Chain complex with with norms

- We obtain a chain complex with stars (norms)

$$\begin{array}{ccccccc} C^0 & \xrightarrow{d_0} & C^1 & \xrightarrow{d_1} & C^2 & \xrightarrow{d_2} & \cdots \\ \downarrow{\scriptstyle \star_0} & & \downarrow{\scriptstyle \star_1} & & \downarrow{\scriptstyle \star_2} & & \\ C_0 & \xleftarrow{d_0^\intercal} & C_1 & \xleftarrow{d_1^\intercal} & C_2 & \xleftarrow{d_2^\intercal} & \cdots \end{array}$$

- You can build this diagram as long as you have a chain complex with some notion of $\|\cdot\|^2$.

Laplacian



- Laplacian is the 2nd derivative operator given by

$$\pm\Delta = \star_k^{-1} d_k^\intercal \star_{k+1} d_k + d_{k-1} \star_{k-1}^{-1} d_{k-1}^\intercal \star_k$$

- To keep it symmetric and positive (semi)definite, take another $\star_k$

$$\mathbf{L} = d_k^\intercal \star_{k+1} d_k + \star_k d_{k-1} \star_{k-1}^{-1} d_{k-1}^\intercal \star_k$$

Example: Heat Equation

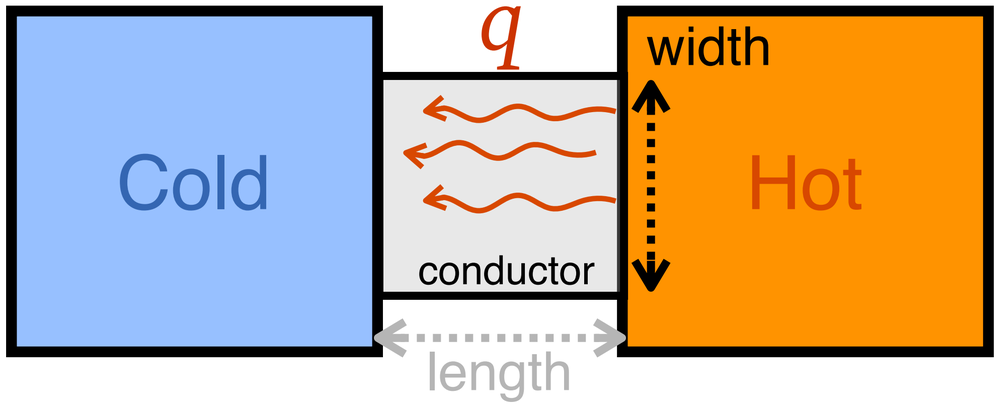
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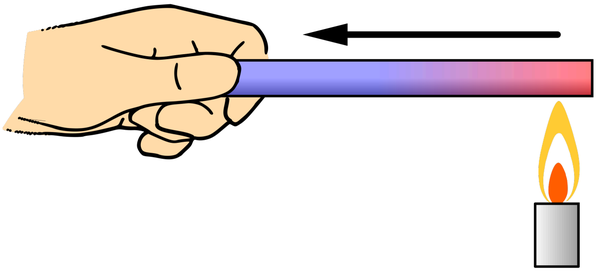
Setup

- Interplay between two physical quantities
  - Temperature (indication of hot/cold) (K or ℃ or ℉)
  - Heat (amount of energy) (J or Calorie)
- Two physical laws:
  - Spatial temperature difference induces heat flow
  - Temperature rises when heat is added to a system

Heat conduction

- Spatial temperature difference induces heat flow





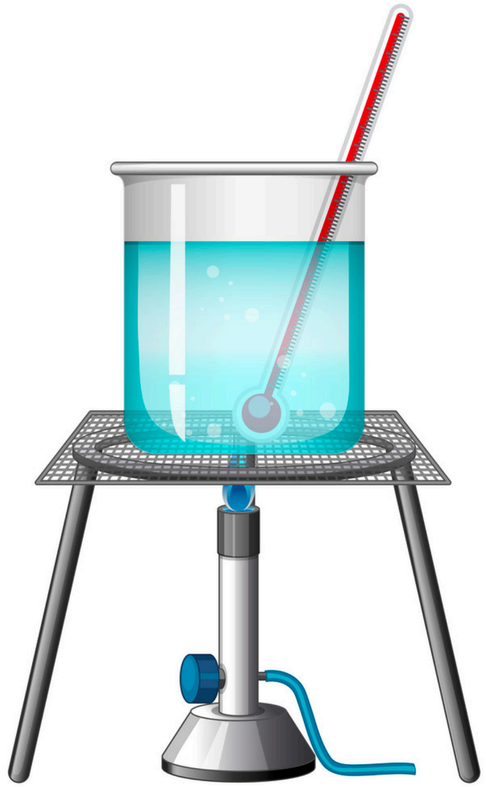
$$\underset{\text{flux } q}{\text{heat}} = -\underbrace{\text{Conductivity}}_{\text{proportional to } \frac{\text{width}}{\text{length}}} \cdot \left(\underset{\text{difference}}{\text{temperature}}\right)$$

(energy per time)

and a coefficient with unit $\frac{\text{J}}{\text{K s}}$

Temperature and heat

- Temperature rises when heat is added to a system



$$\left(\begin{matrix}\text{rise in}\\ \text{temperature}\end{matrix}\right) \cdot \begin{matrix}\text{Heat}\\ \text{capacity}\end{matrix} = \left(\begin{matrix}\text{total inflow}\\ \text{of heat}\end{matrix}\right)$$

(temperature) (energy)

$$\left(\begin{matrix}\text{rate of rise in}\\ \text{temperature}\end{matrix}\right) \cdot \begin{matrix}\text{Heat}\\ \text{capacity}\end{matrix} = \left(\begin{matrix}\text{total inflow}\\ \text{rate of heat}\end{matrix}\right)$$

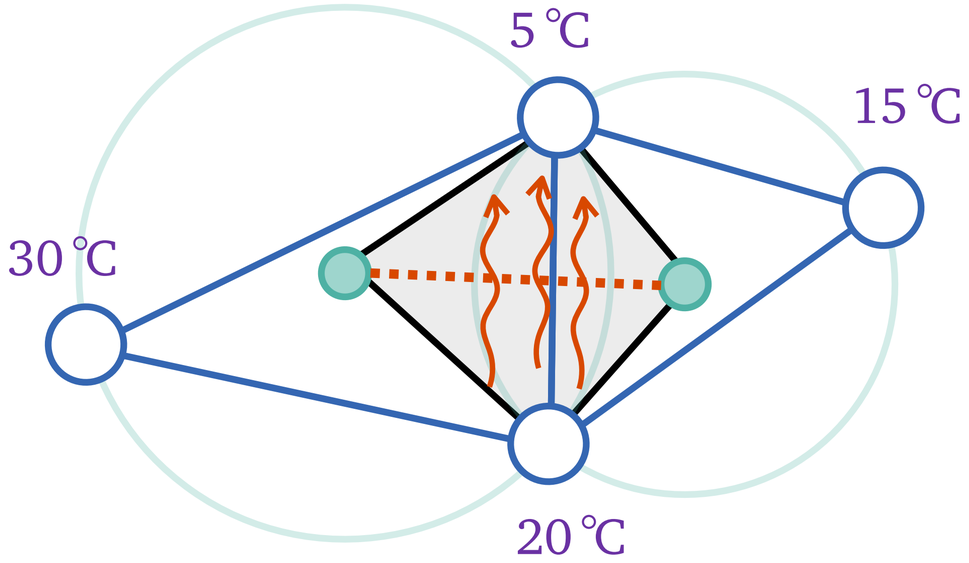
(temperature per time) (energy per time)

proportional to volume and coefficient with unit $\frac{\text{K}}{\text{J}}$

Heat Equation

- Let $M = (M_0, M_1, M_2)$ be a mesh
- Let $u \in C^0(M; \mathbb{R}\,\mathrm{K})$

  temperature-valued 0-form



- Temperature difference

$$\mathbf{d}_0 u \in C^1(M; \mathbb{R}\,\mathrm{K})$$

- Let $w_e \in \mathbb{R}_{>0}\,\frac{\mathrm{J}}{\mathrm{sK}}$ be the heat conductivity of edge e
- Each edge gets a heat flux $q \in C^1(M^*; \mathbb{R}\,\mathrm{J/s})$

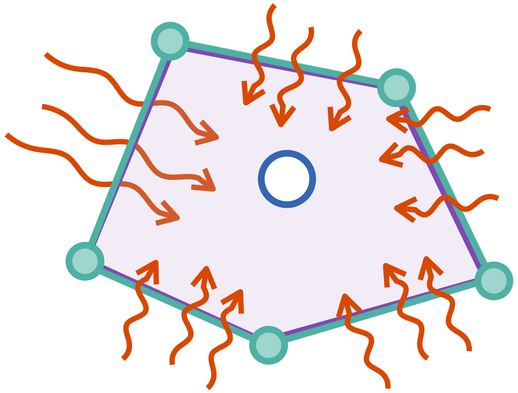
$$q_e = -w_e\,(\mathbf{d}_0 u)_e \qquad \text{or simply} \qquad q = -\star_1 \mathbf{d}_0 u$$

Heat Equation

- Give heat flux $q \in C^1(M^*; \mathbb{R}\,\mathrm{J/s})$

$$(\mathbf{d}_0^\intercal q)_i = \sum_{e \succ i} \begin{cases} -q_e & \text{if } i = \mathrm{src}(e) \\ +q_e & \text{if } i = \mathrm{dst}(e) \end{cases}$$

$$= \text{Total heat flowing into cell } i \text{ per unit time}$$



- Let $m_i \in \mathbb{R}_{>0}\,\frac{\mathrm{K}}{\mathrm{J}}$ be the heat capacity
- Then $m_i \dfrac{\partial u_i}{\partial t} = (\mathbf{d}_0^\intercal q)_i$ or simply $\star_0 \dfrac{\partial u}{\partial t} = \mathbf{d}_0^\intercal q$

Heat Equation

- Putting together

$$q = - \star_1 \mathbf{d}_0 u \qquad \star_0 \frac{\partial u}{\partial t} = \mathbf{d}_0^\intercal q$$

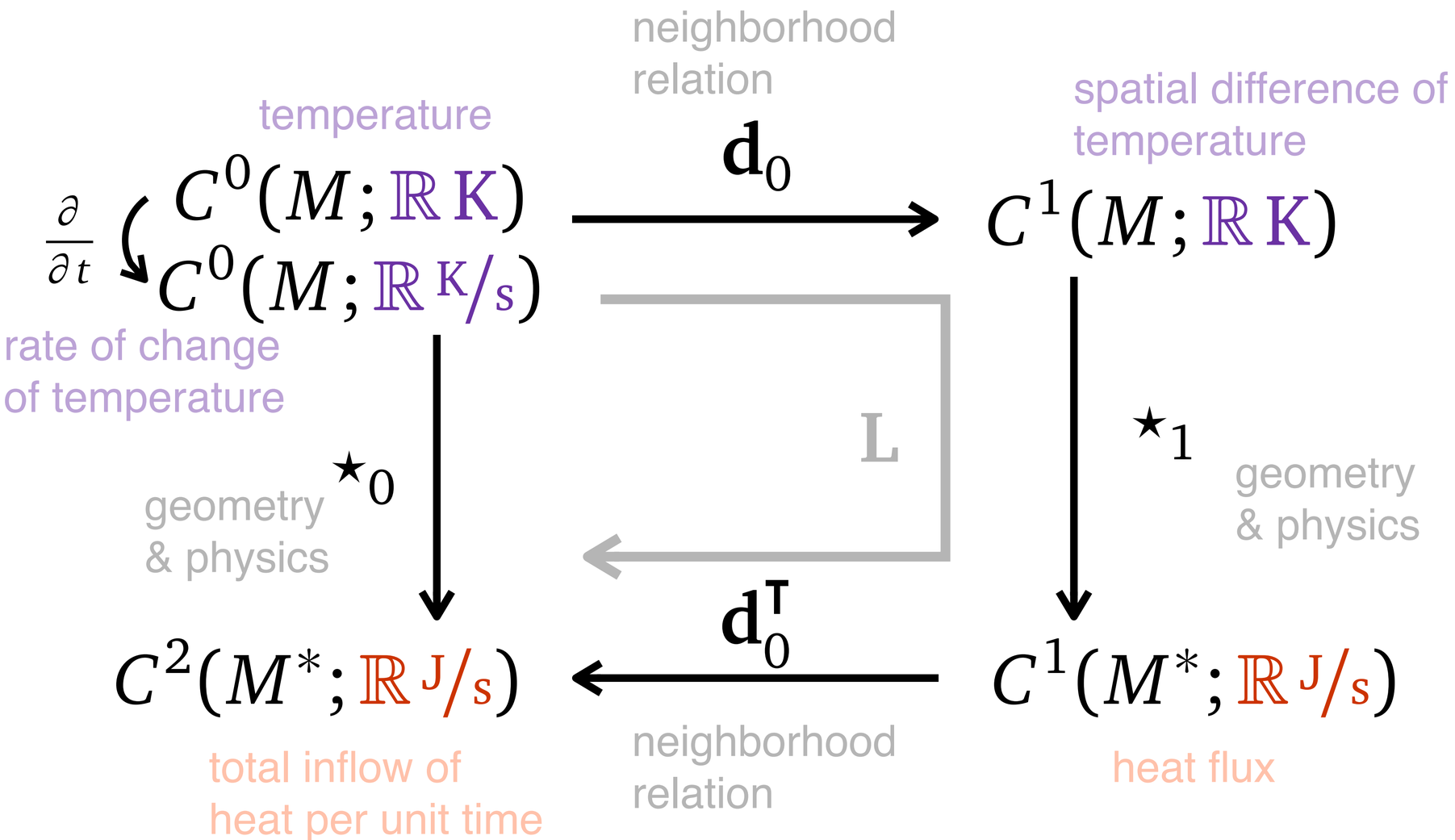
- We get

$$\Longrightarrow \qquad \star_0 \frac{\partial u}{\partial t} = -\mathbf{d}_0^\intercal \star_1 \mathbf{d}_0 u$$

Heat equation

i.e. $\star_0 \frac{\partial u}{\partial t} = -\mathbf{L} u$

Heat Equation



$$\star_0 \frac{\partial u}{\partial t} = -\mathbf{d}_0^\intercal \star_1 \mathbf{d}_0 u$$