CSE 270 (FA25)
Discrete Differential Geometry

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UC San Diego
JACOBS SCHOOL OF ENGINEERING

Course information

Discrete Differential Geometry

- **Instructor:** Albert Chern
- **TA:** Hang (Hesper) Yin
- **Course website:** https://cseweb.ucsd.edu/~alchern/teaching/cse270_fa25/
- We use **Piazza** and **Gradescope**
- Lecture: Tue, Thu 11:00–12:20, Discussion: Mon 15:00–15:50

Course information

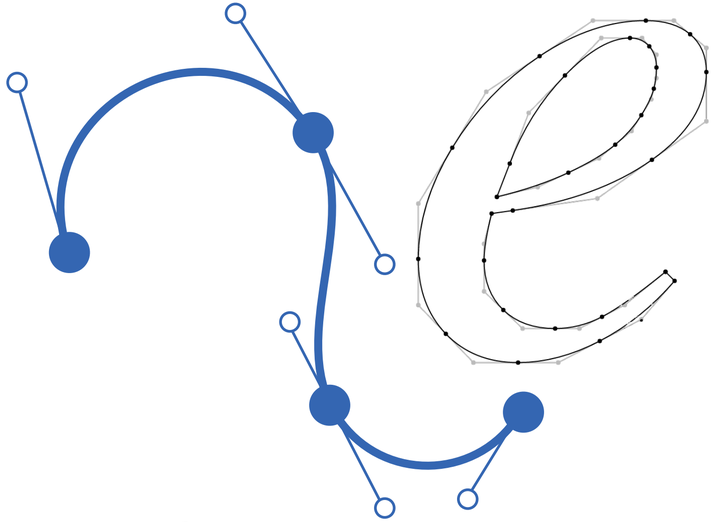
Discrete Differential Geometry

- **Goal:** Differential geometric notations, their discrete theories, and their implementations. Using Houdini software.
- **Applications:** Geometry processing and simulating physical systems.
- **Grade:** HW1–5 (written and coding)
- **Collaboration:** Final submissions should be individual work, but we encourage you to study the math and solve the problems together!

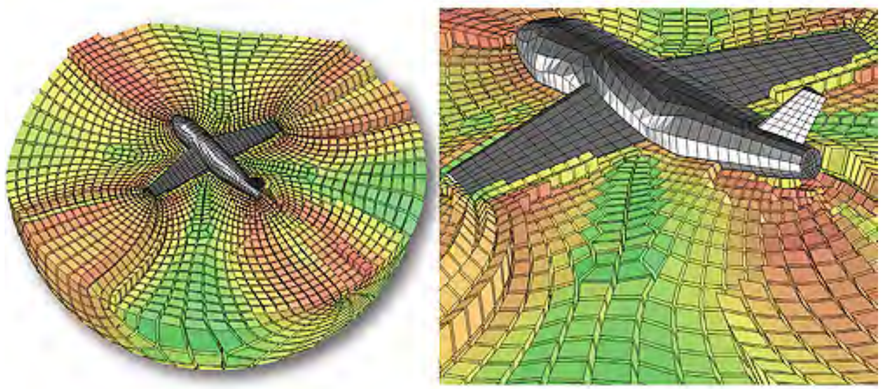
Geometry in Computational World

- Geometry in computations
- What is geometry?
- What is computing?
- Discrete Differential Geometry
- Structure preserving examples
- Roadmap of the course

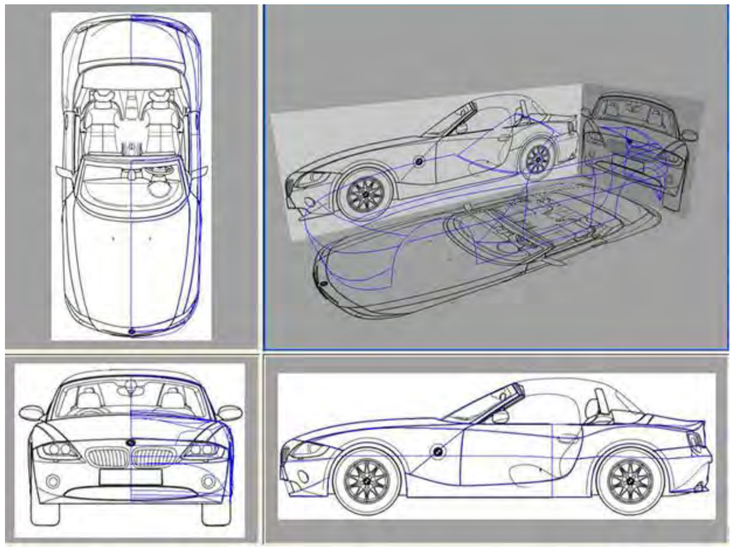
Geometry in graphics and engineering



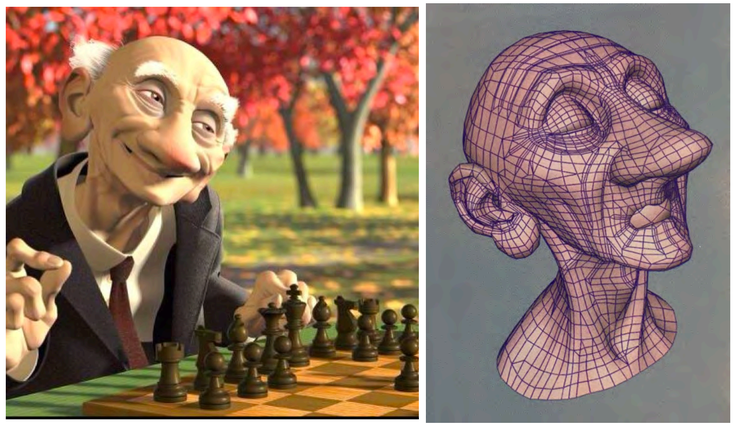
Computer fonts



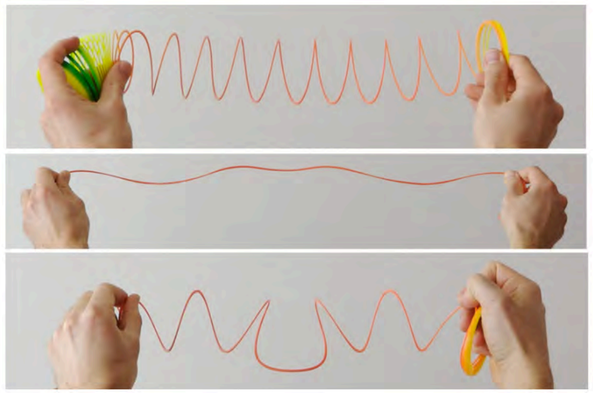
Simulation domain

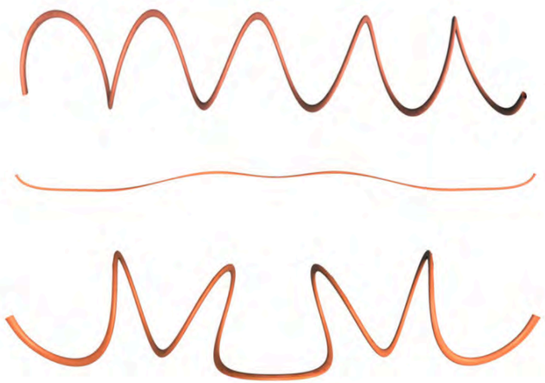


Computer aided design



Modeling and Animation

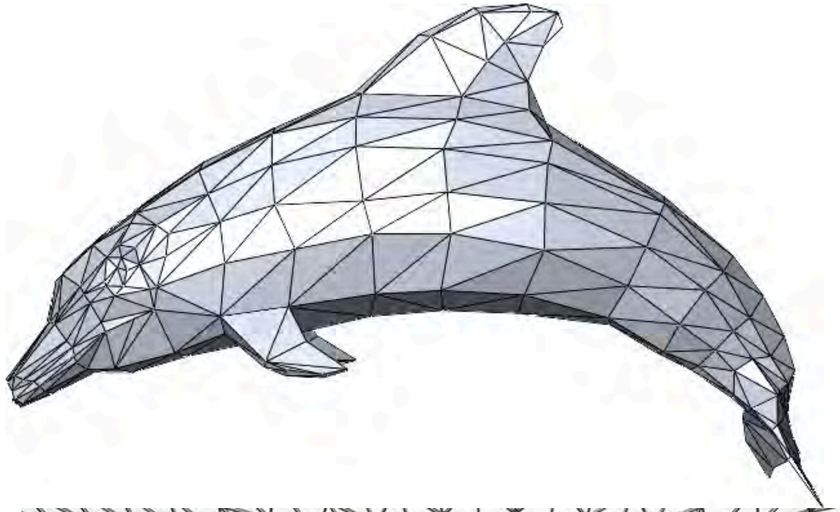


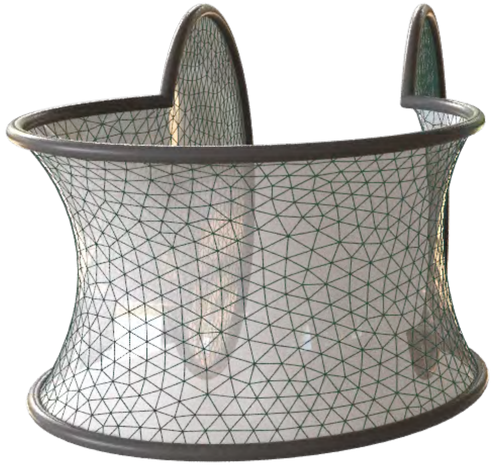


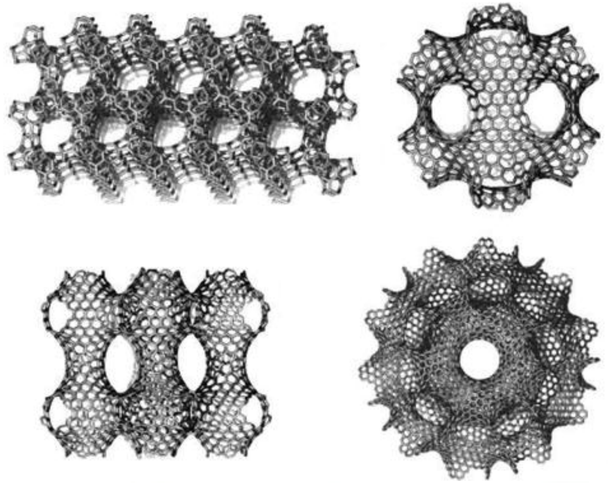
Simulation of physical objects

Digital / discrete geometry

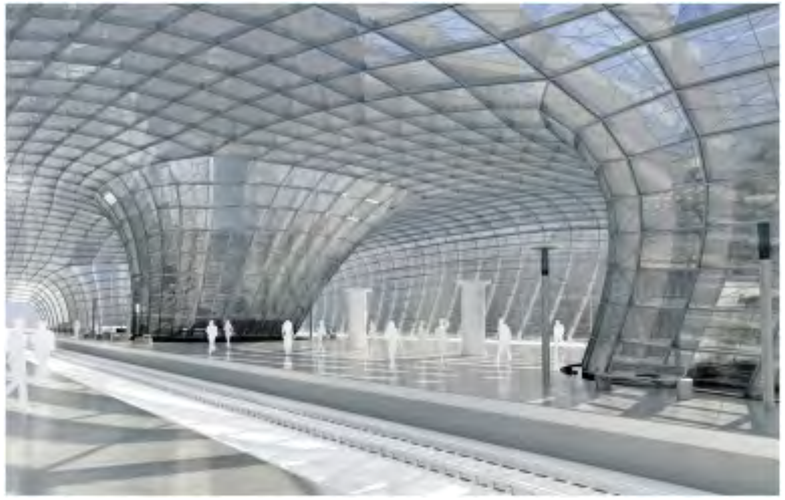
- Digital and discrete geometry mimicking smooth surfaces

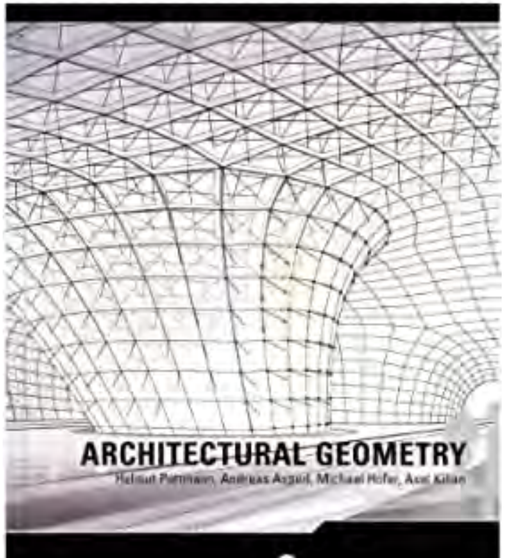












Digital / discrete geometry

Computational,
discrete version

Discrete Differential Geometry

theory for smooth curves & surfaces

Geometry processing

- View discrete surface as a form of “signal data”
  - Traditional signal data: audio & images

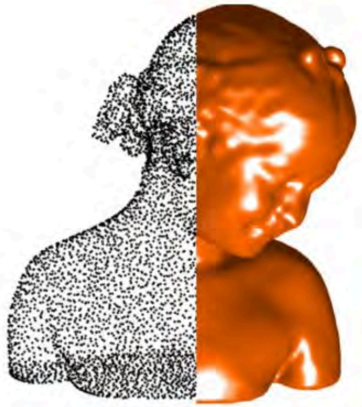




  - Geometric signal
  - Upsampling / downsampling / filtering / aliasing

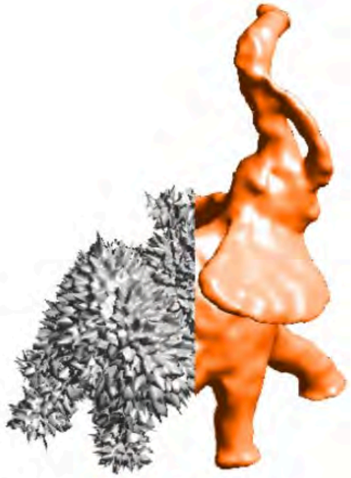


Computing on surfaces

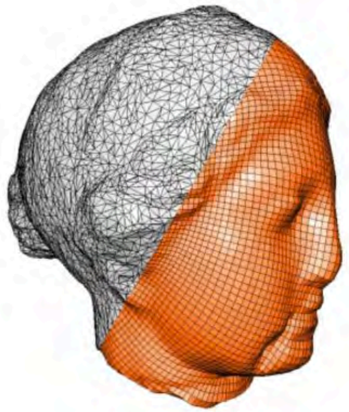


reconstruction





filtering



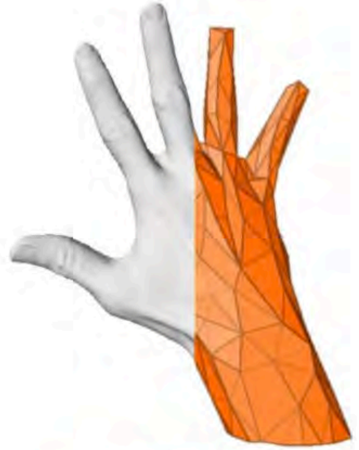
remeshing



shape analysis



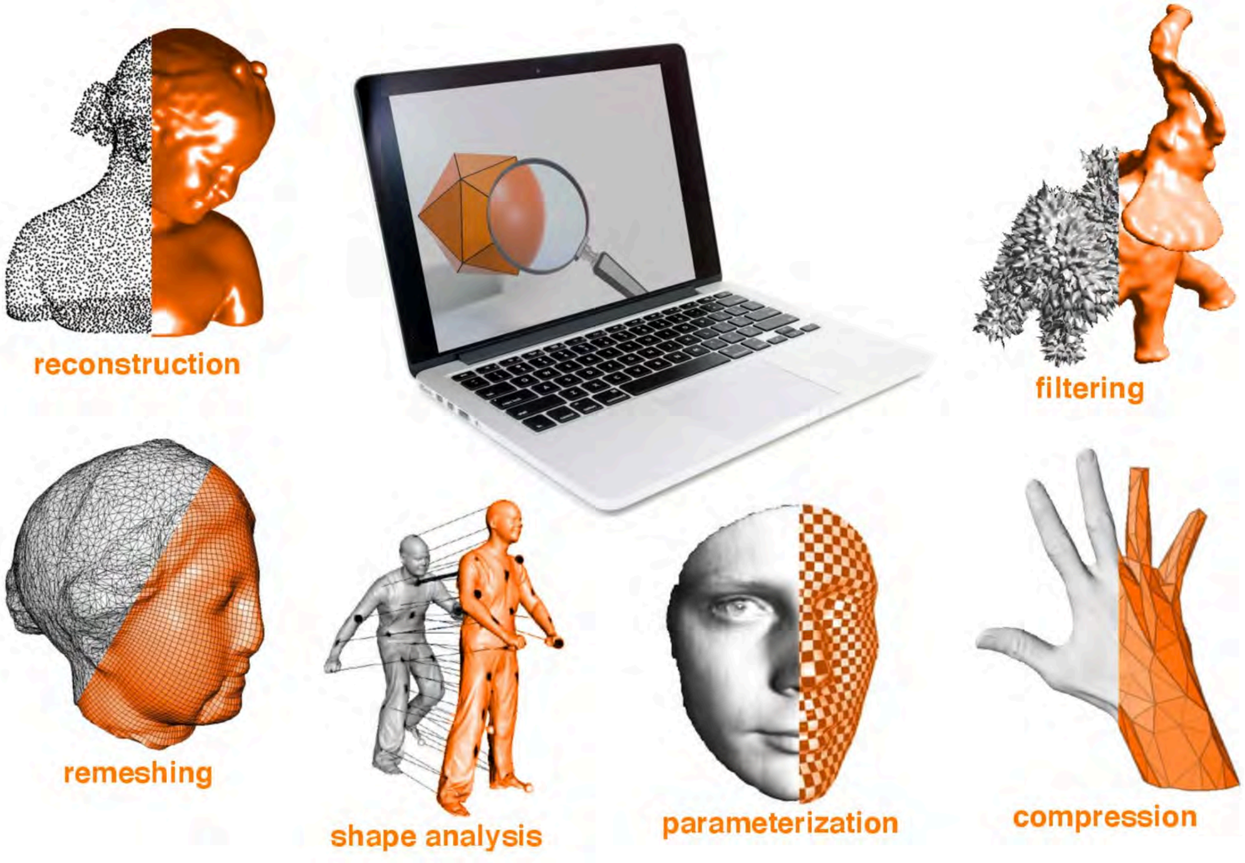
parameterization



compression

Computing on surfaces

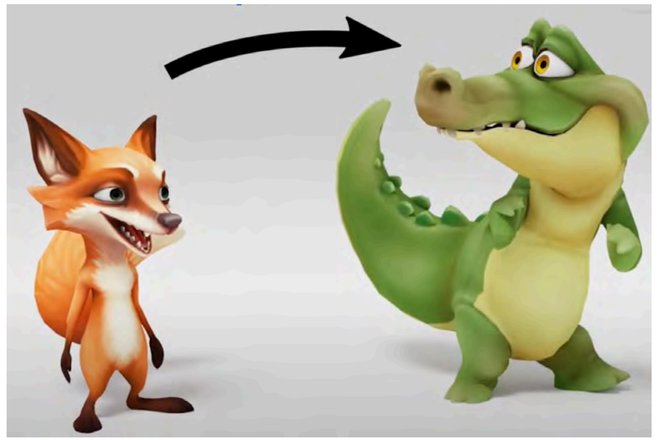
Geometry Processing



Partial Differential Equations (PDEs) on general domains



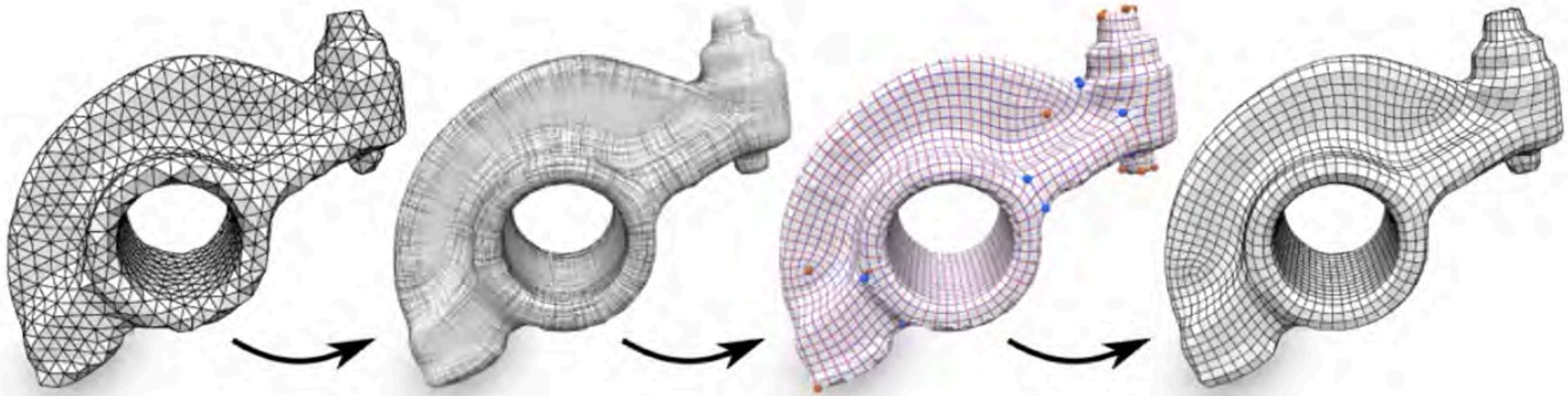
Computing on surfaces

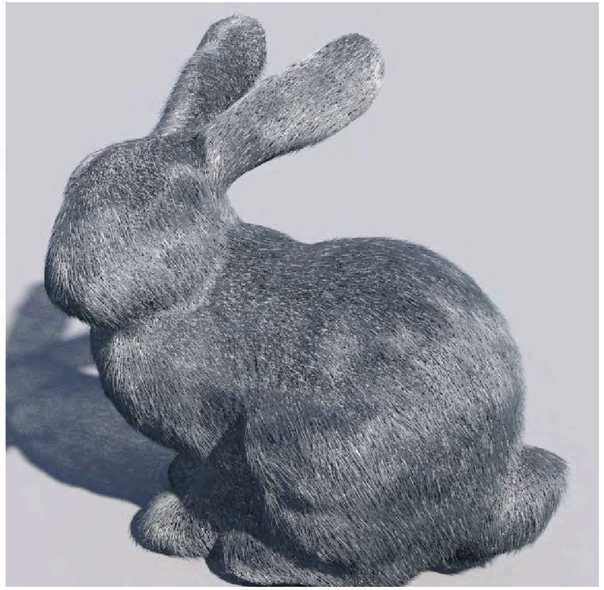




Maps between surfaces

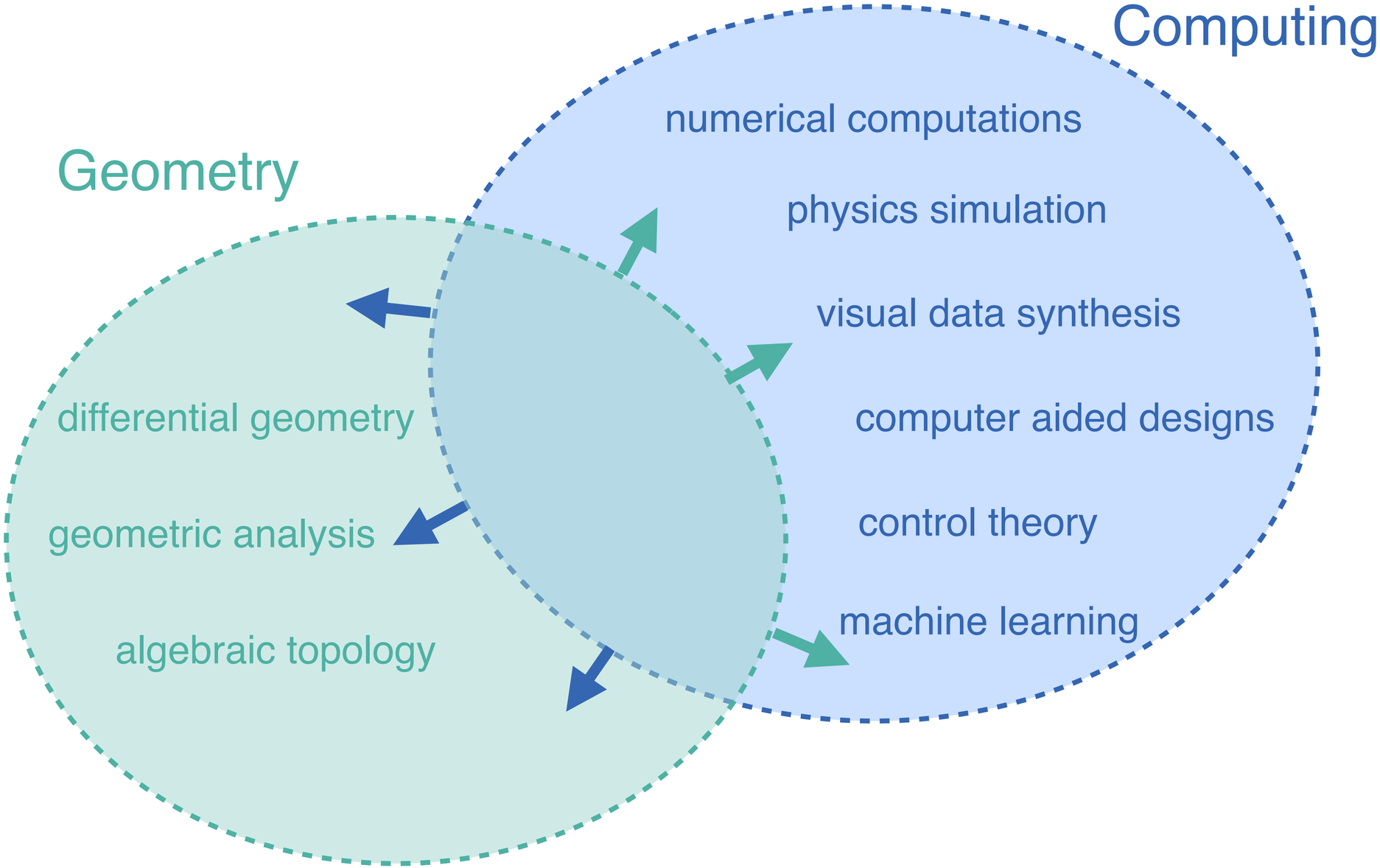






Directional field

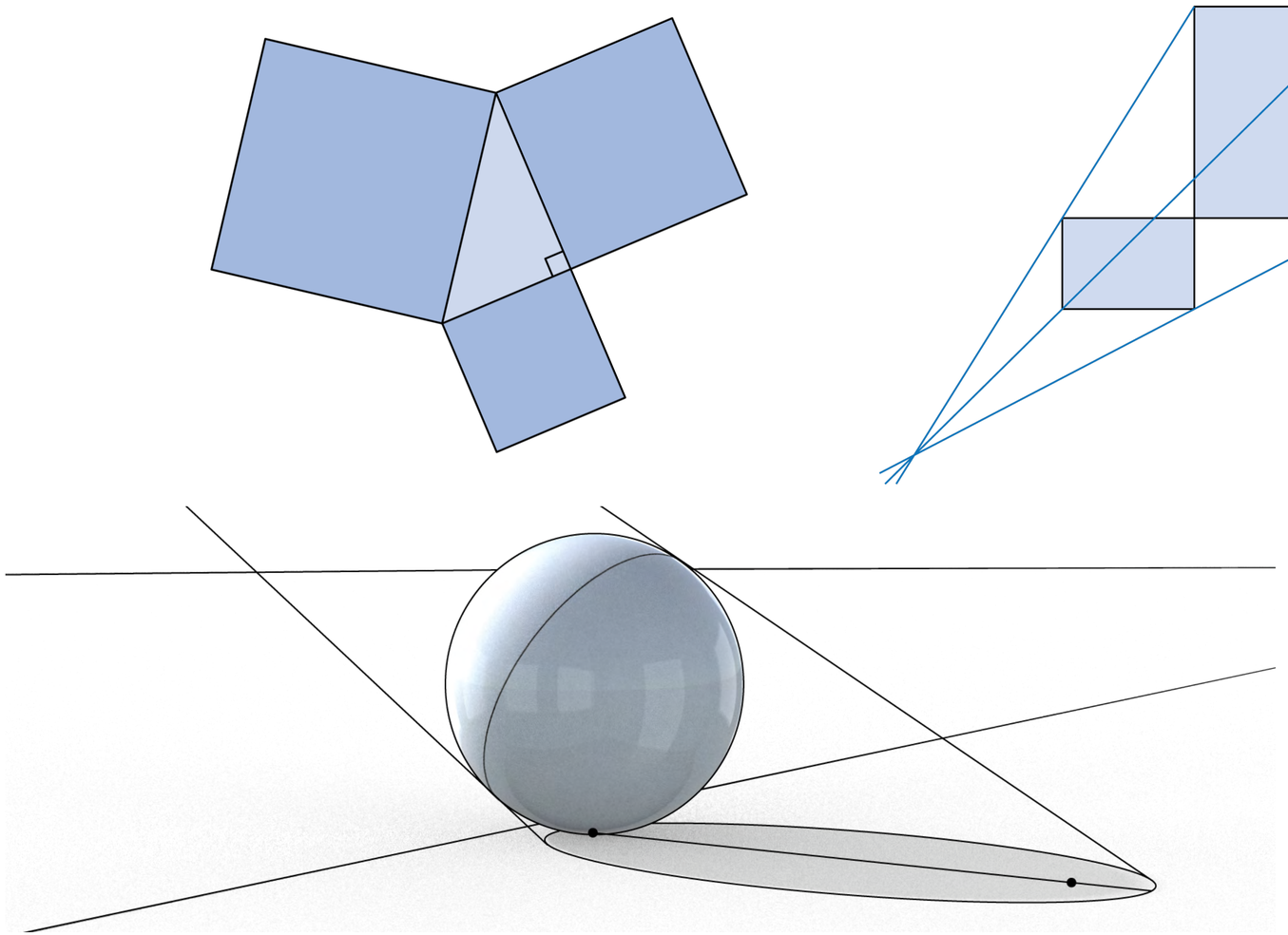
General theme



What is Geometry?

- Geometry in computations
- What is geometry?
- What is computing?
- Discrete Differential Geometry
- Structure preserving examples
- Roadmap of the course

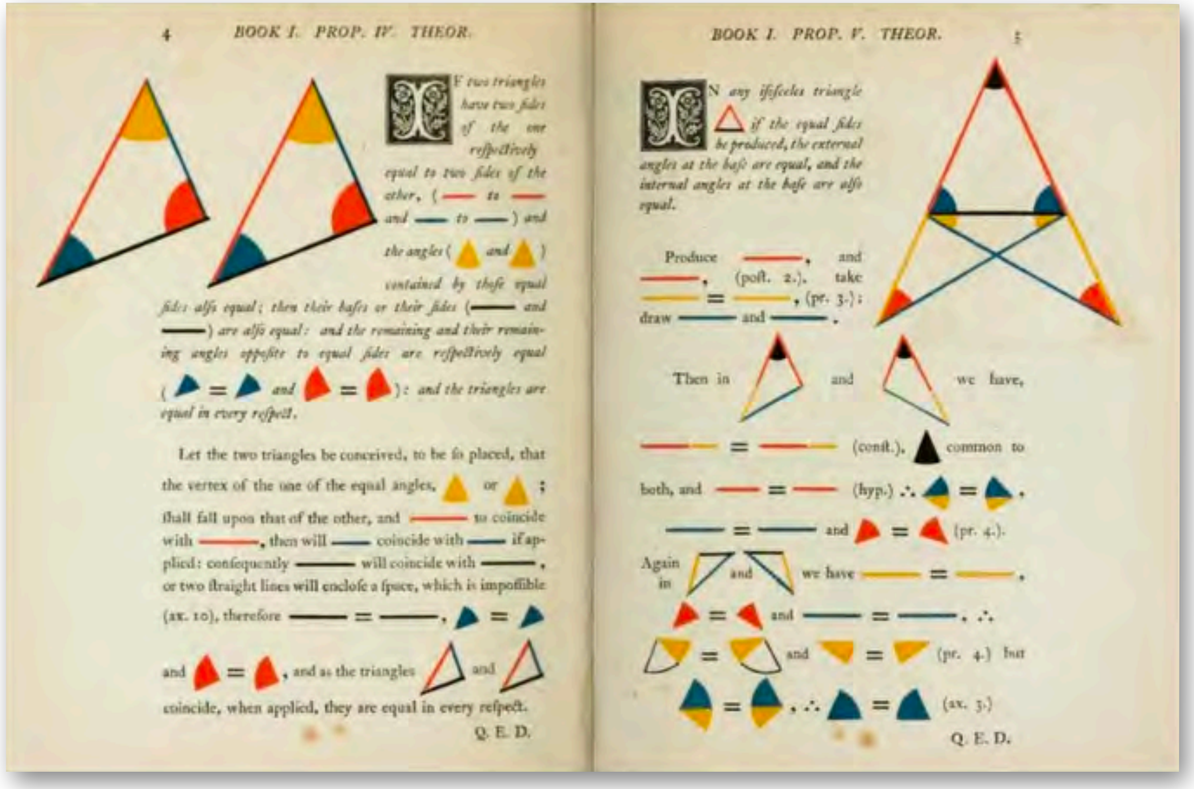
What is geometry?



- Relationship between shapes
- Order of spaces

Euclidean Geometry (300 BC)

- Axiomatic mathematics
  - All geometric propositions are logically deduced from other propositions (self-consistent theory).
  - The theory boils down to definitions objects and their relationships in a space with basic assumptions (postulates/axioms).



4 BOOK I. PROP. IV. THEOR.

IF two triangles have two sides of the one respectively equal to two sides of the other, (— to — and — to —) and the angles (△ and △) contained by those equal sides alſo equal; then their baſes or their sides (— and —) are alſo equal: and the remaining and their remaining angles oppoſite to equal sides are reſpectively equal (△ = △ and △ = △): and the triangles are equal in every reſpect.

Let the two triangles be conceived, to be ſo placed, that the vertex of the one of the equal angles, △ or △ ; ſhall fall upon that of the other, and — to coincide with —, then will — coincide with — if applied: conſequently — will coincide with —, or two ſtraight lines will enclose a ſpace, which is impoſſible (ax. 10), therefore — = —, △ = △ and △ = △, and as the triangles △ and △ coincide, when applied, they are equal in every reſpect.

Q. E. D.

BOOK I. PROP. V. THEOR. 5

IN any iſoſceles triangle △ if the equal ſides be produced, the external angles at the baſe are equal, and the internal angles at the baſe are alſo equal.

Produce —, and —, (poſt. 2.). take — = —, (pr. 3.); draw — and —.

Then in △ and △ we have, — = — (conſt.), △ common to both, and — = — (hyp.) ∴ △ = △, — = — and △ = △ (pr. 4.).

Again in △ and △ we have — = —, △ = △ and — = —. ∴ △ = △ and △ = △ (pr. 4.) but △ = △, ∴ △ = △ (ax. 3.)

Q. E. D.

Euclidean Geometry (300 BC)

- Euclid's 5 postulates of a plane (original version)
  - ✓ We can draw a **line segment** between two points
  - ✓ We can extend a line segment into a **straight line**
  - ✓ We can construct a **circle** by center and radius
  - ✓ All **right-angles** (90°) are equal
  - ✓ If 2 straight-lines intersect another straight line with angle sum less than 180° on one side, then the 2 lines meet at a point on that side.

    (Through a pt not on a given line, there is at most one line through the point that never meets the given line)

Euclidean Geometry (300 BC)

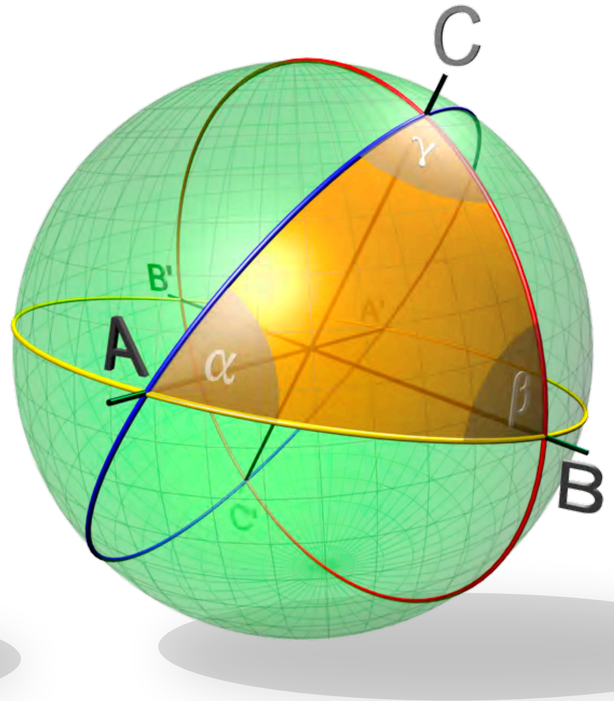
- One can show that the total interior angles of a triangle is 180°.



Spherical Geometry (10th century)

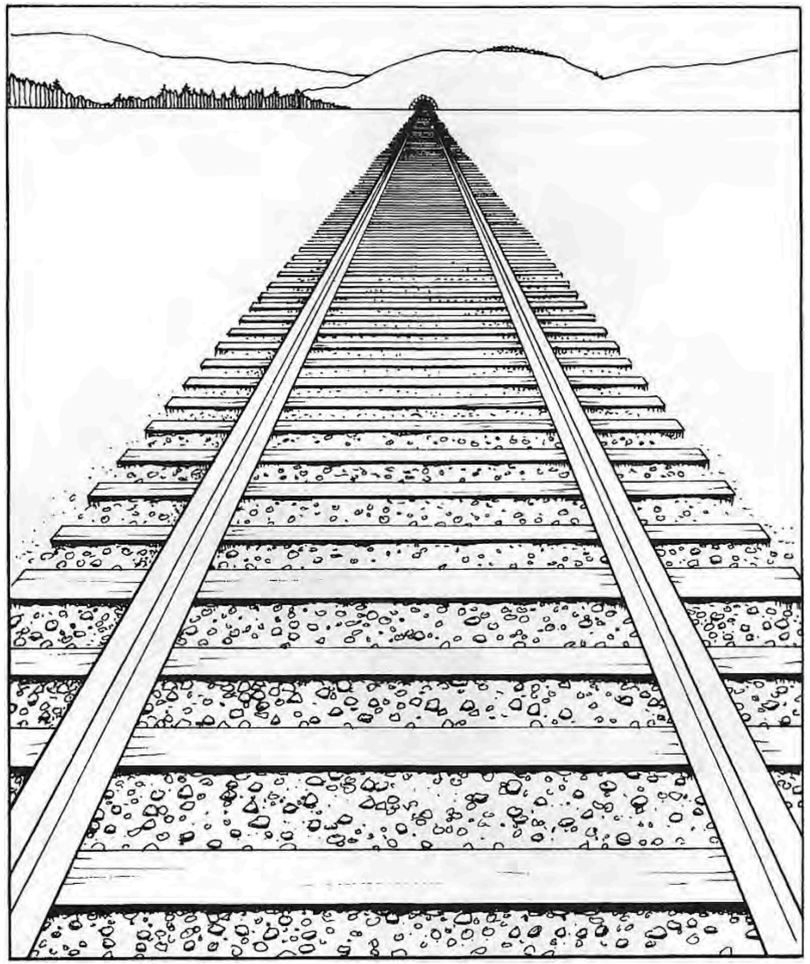
- Motivated by astronomy
- One still defines lines (great circles) and circles (other planar sections)
- Postulates are slightly different.
  - Triangle's interior angle sum is not 180°.





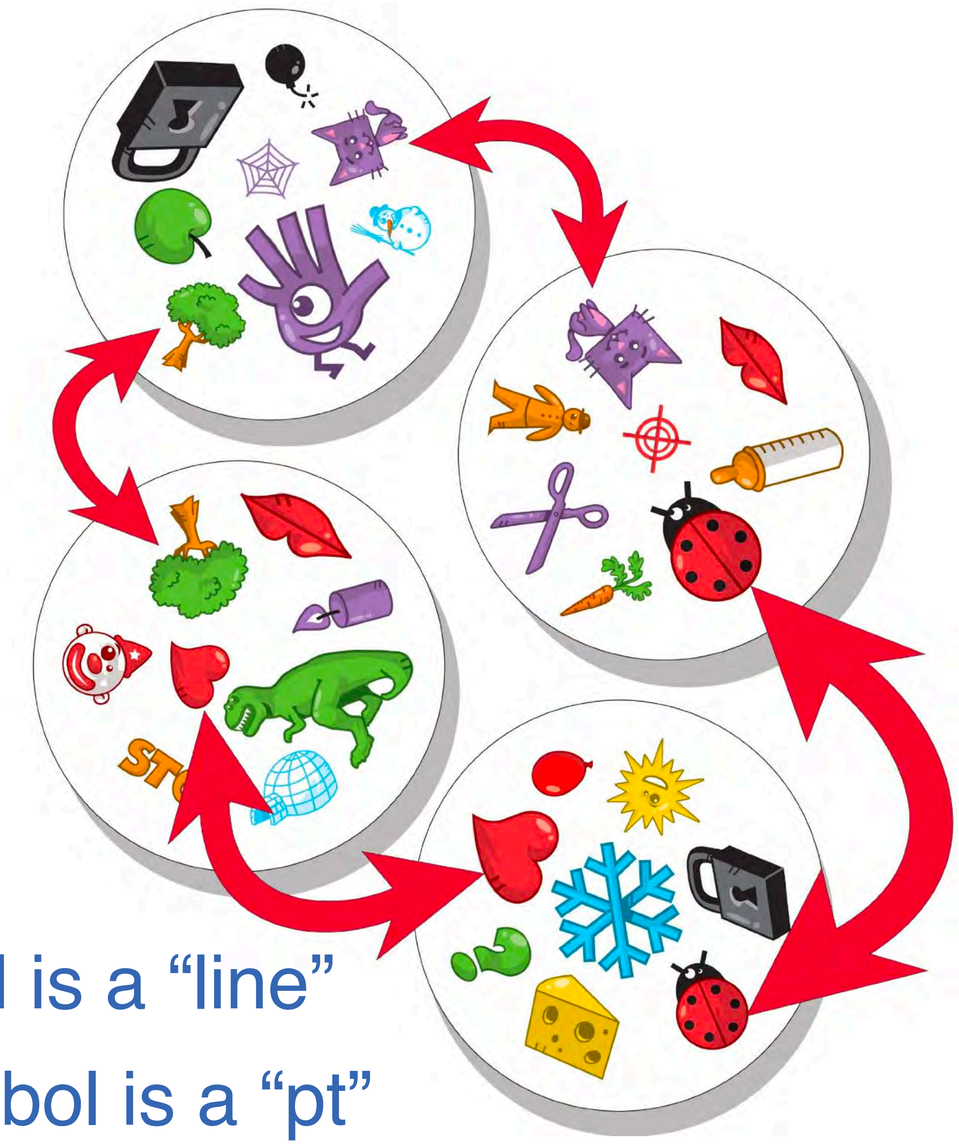
Projective Geometry (15–17th century)

- Renaissance perspective drawing
  - 2D representation of a 3D object.
  - Straight lines must be represented by straight lines.
  - Every pair of lines in a plane will meet exactly once
    - Parallel lines in a plane meet on the horizon of the plane
    - Meeting points of parallel lines are called vanishing points
- Incidence between points and lines
- Axiom for projective plane
  - Two distinct points are contained in a unique line
  - Two distinct lines meet at a unique point



Projective Geometry (15–17th century)

- Axiom for projective plane
  - Two distinct points are contained in a unique line
  - Two distinct lines meet at a unique point
- Duality
  - Swap “lines” and “pts,” swap “meet” and “join”
- Boardgame
  “Spot it” (“Dobble”)

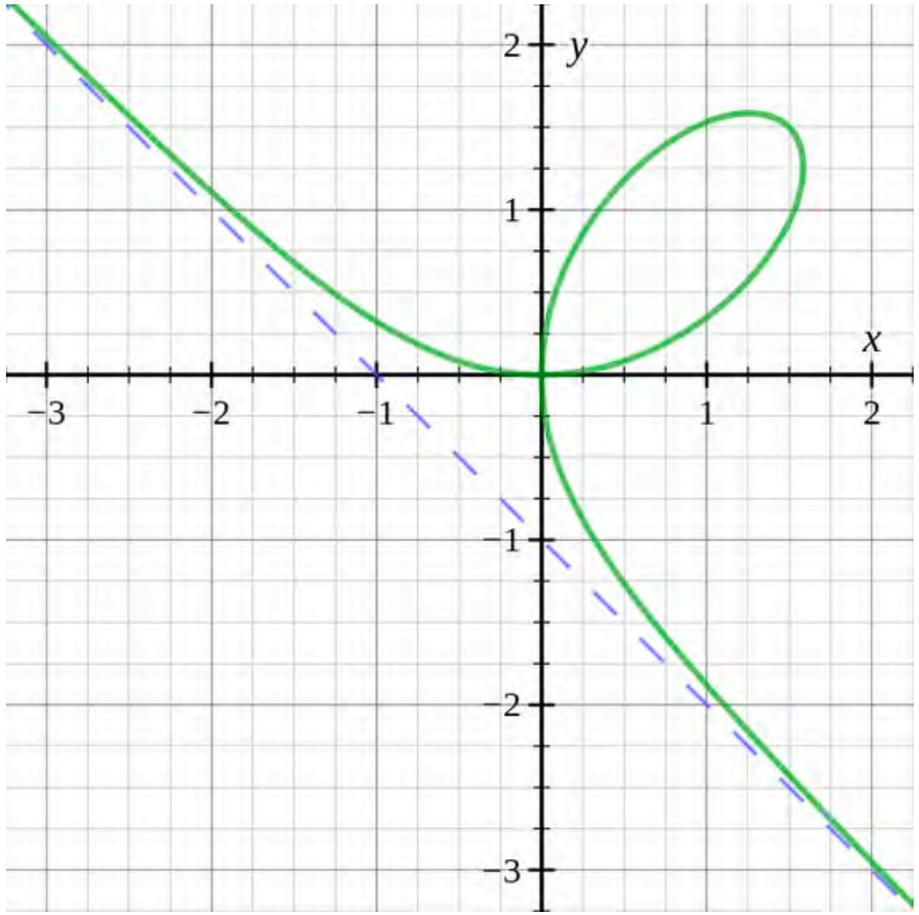


Each card is a “line”

Each symbol is a “pt”

Analytic Geometry (17th Century)

- Represent shapes by equations using Cartesian coordinates.
- Geometric questions become algebraic or calculus problems
- Mainstream approach…
  with caveats:
  - Geometrically intuitive problems may become tedious under coordinates.
  - Calculus problems may be harder to discretize than the original geometric problems.

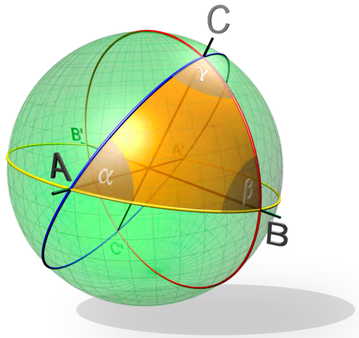


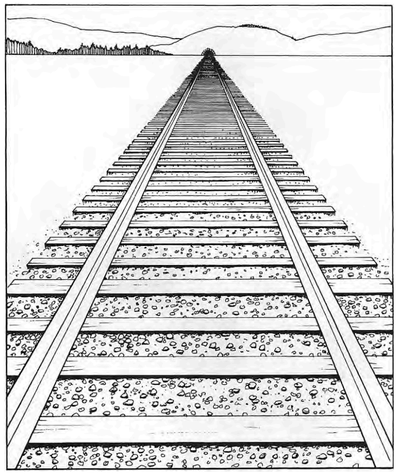
From Euclidean to Non-Euclidean

- For 2000 years, people try to prove Euclid's 5th postulates from the first four (i.e. it should be proven instead of assumed).

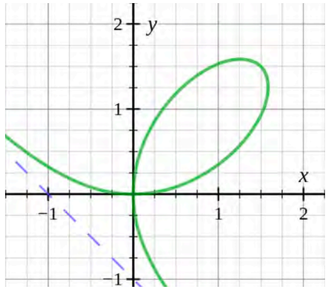
  (Through a pt not on a given line, there is at most one line through the point that never meets the given line)

- Spherical and projective geometries are non-Euclidean but this is ignored because the other four postulates are also slightly different.





- Analytic (coordinate) geometry is almost always Euclidean



Hyperbolic Geometry (1792)

- Bolyai, Lobachevsky, Gauss
- Compared to Euclid's postulates
  - ✓ We can draw a **line segment** between two points
  - ✓ We can extend a line segment into a **straight line**
  - ✓ We can construct a **circle** by center and radius
  - ✓ All **right-angles** (90°) are equal
  - X Through a pt not on a given line, there is at most one line through the point that never meets the given line
  - ✓ Through a pt, there are at least two lines that never meet
- Poincaré disk model
  - Hyperbolic plane can be drawn in a disk, where "lines" are circles perpendicular to the disk boundaries, and "circles" are any other circles

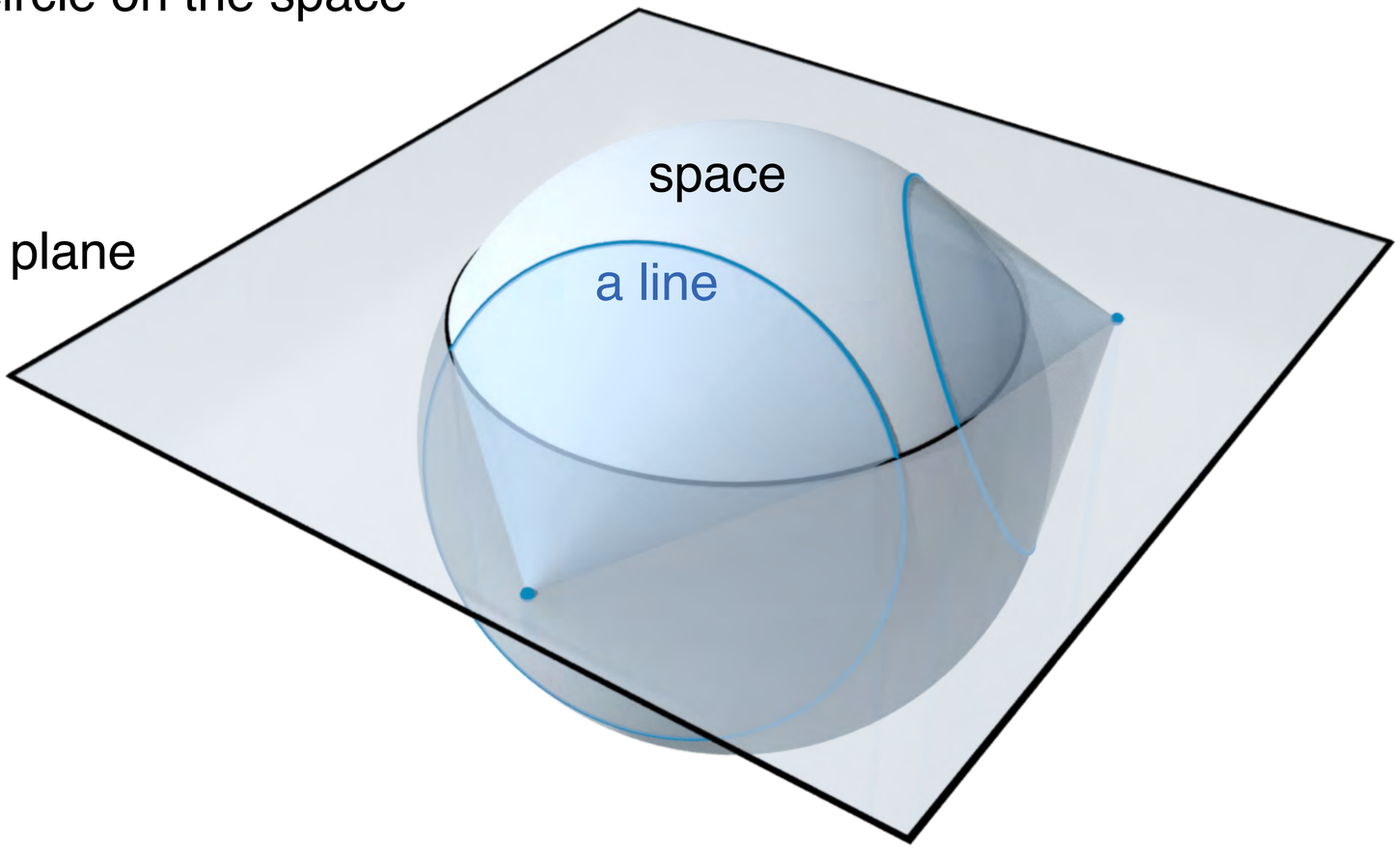


NonEuclidean geometries (19th century)

A plane & a quadratic surface in a projective 3 space yields a spherical, Euclidean, or hyperbolic geometry:

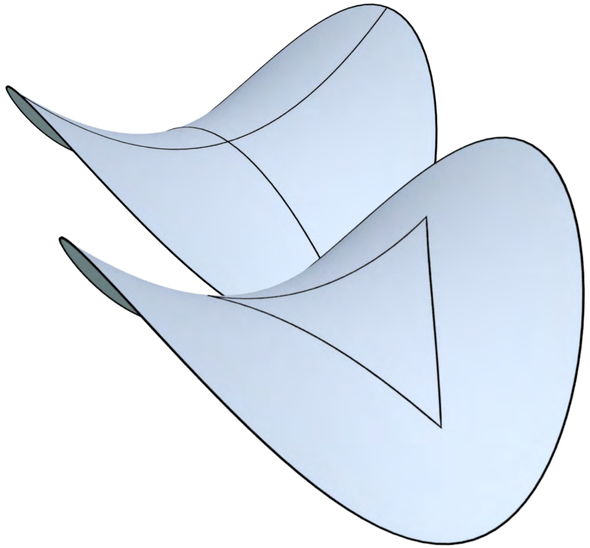
Every point in the ambient
casts a circle on the space

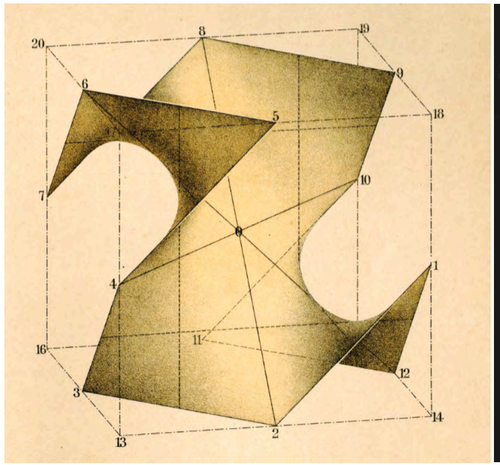
Point on this plane
casts a line



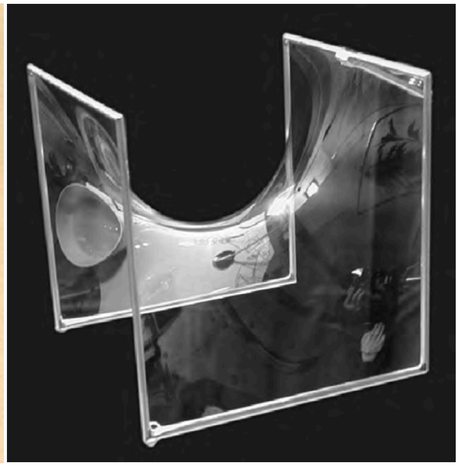
Differential geometry (19th century)

- Study of general curves and surfaces (and manifolds)
  - Applications: Physics, engineering designs, graphics, …

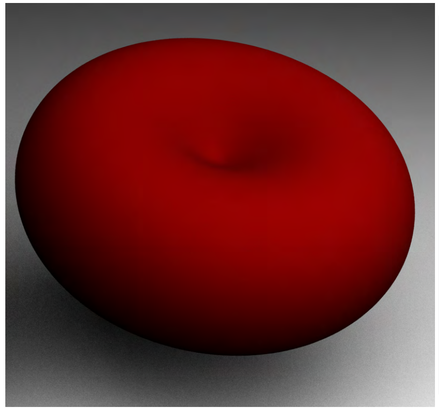


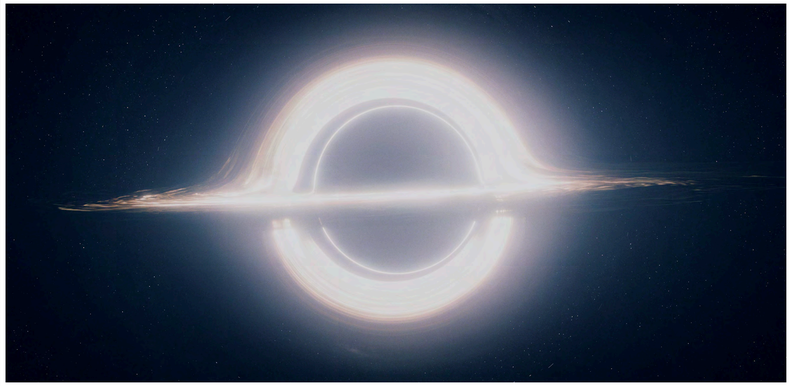


Schwarz 1890



soap films (minimal surface)





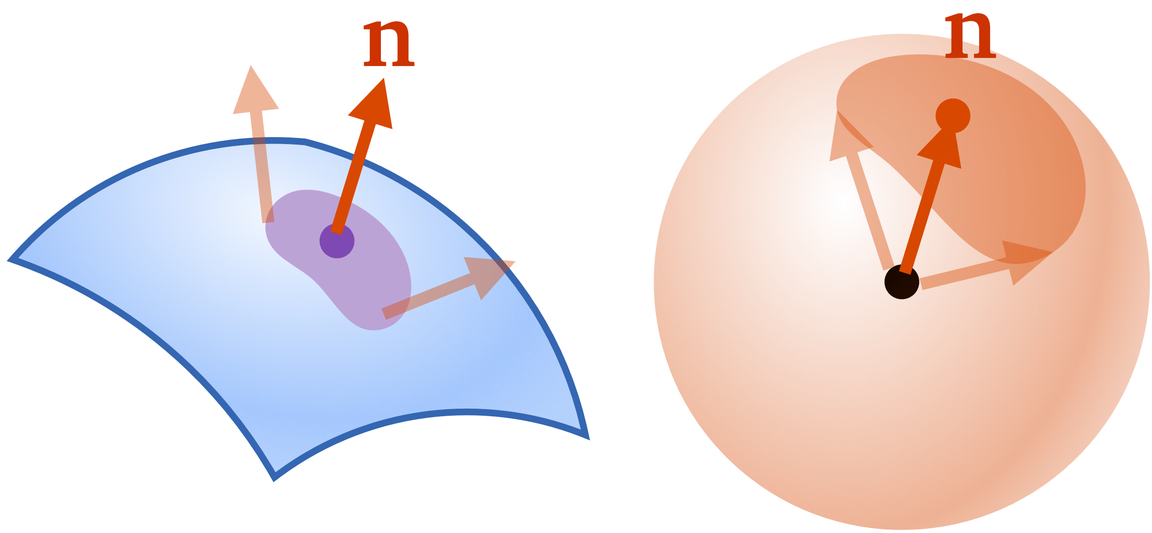




Gauss' "Remarkable Theorem"

- One can describe curvature of a surface by Gaussian curvature:

The solid angle swept by the normal vector (per unit area of the surface)

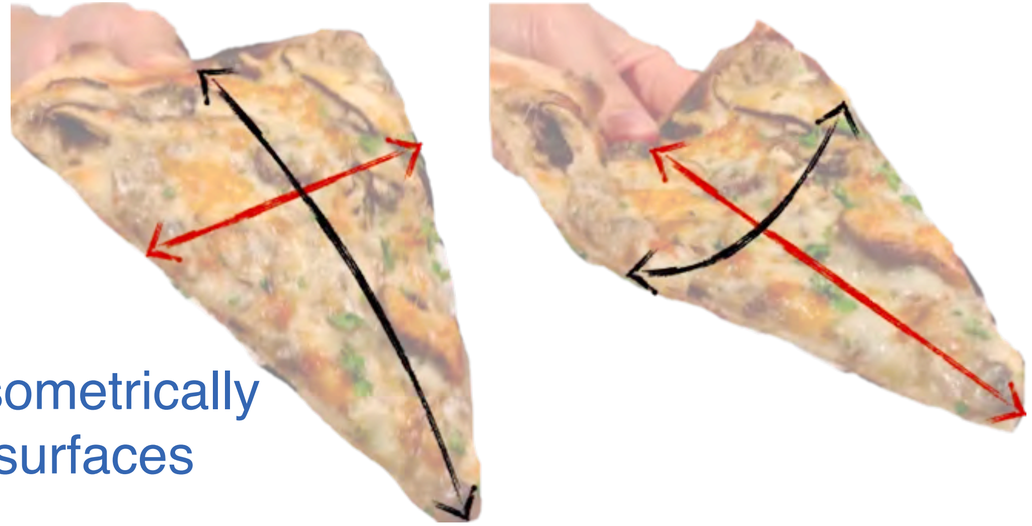


- Remarkable Theorem:
  Gaussian curvature is determined by distances on the surface!

Gauss' "Remarkable Theorem"

- Remarkable Theorem:
  Gaussian curvature is determined by distances on the surface!

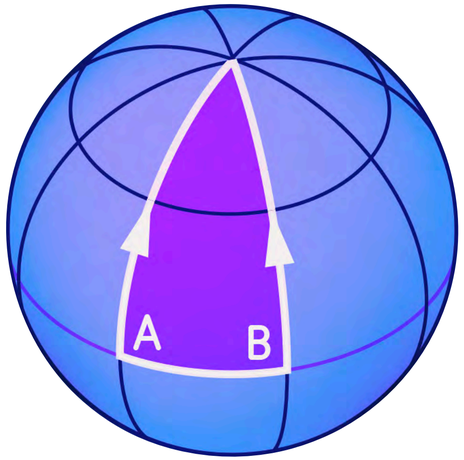


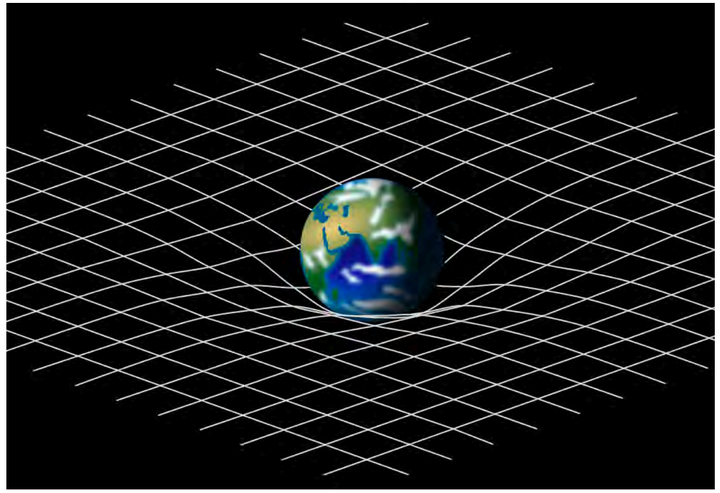


These are isometrically deformable surfaces

Gauss' "Remarkable Theorem"

- Remarkable Theorem:
  Gaussian curvature is determined by distances on the surface!
- **Riemannian geometry (intrinsic geometry):**
  manifolds with some general notion of
  non-Euclidean distances; one may study invariants (like
  Gaussian curvature) of the space.





What is Geometry? (late 19th early 20th)

- There are many geometries throughout history.
  - Euclidean, spherical, hyperbolic, non-Euclidean geometries,…
- What is a unifying concept?

Klein's Erlangen program 1872:

Ingredients of geometry:

- Collection of objects
- Group of transformations
- Invariant notions



Felix Klein 1872

What is Geometry?

Ingredients of geometry:

- Collection of objects
- Group of transformations
- Invariant notions
  - straight lines, planes
  - distance
  - circles, spheres
  - angles
  - area
  - parallelism
- Non-invariant notions
  - additions $\mathbf{p}_1 + \mathbf{p}_2$
  - dot product $\mathbf{p}_1 \cdot \mathbf{p}_2$

$$\left\{ \mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \right\}$$

$$\tilde{\mathbf{p}} = \underbrace{R}_{\text{rotation}}\mathbf{p} + \overbrace{\mathbf{t}}^{\text{translation}}$$

Geometry in Euclidean space

What is Geometry?

Ingredients of geometry:

- Collection of objects
- Group of transformations
- Invariant notions

$$\left\{ \mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \right\}$$

$$\tilde{\mathbf{p}} = R\mathbf{p}$$

rotation

  - straight lines, planes
  - distance
  - circles, spheres
  - angles
  - area
  - parallelism
  - additions $\mathbf{p}_1 + \mathbf{p}_2$
  - dot product $\mathbf{p}_1 \cdot \mathbf{p}_2$

- Non-invariant notions

Geometry in Euclidean vector space

What is Geometry?

Ingredients of geometry:

- Collection of objects
- Group of transformations
- Invariant notions
  - straight lines, planes
  - parallelism
  - additions $\mathbf{p}_1 + \mathbf{p}_2$
- Non-invariant notions
  - distance
  - circles, spheres
  - angles
  - area
  - dot product $\mathbf{p}_1 \cdot \mathbf{p}_2$

$$\left\{ \mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \right\}$$

$$\tilde{\mathbf{p}} = A\mathbf{p}$$

invertible matrix

Geometry in vector space

What is Geometry?

Ingredients of geometry:

- Collection of objects
- Group of transformations
- Invariant notions
  - straight lines, planes
  - parallelism
  - midpoint $\frac{\mathbf{p}_1+\mathbf{p}_2}{2}$
- Non-invariant notions
  - distance
  - circles, spheres
  - angles
  - area
  - additions $\mathbf{p}_1 + \mathbf{p}_2$
  - dot product $\mathbf{p}_1 \cdot \mathbf{p}_2$

$$\left\{ \mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \right\}$$

$$\tilde{\mathbf{p}} = A\mathbf{p} + \mathbf{b}$$

translation

invertible matrix

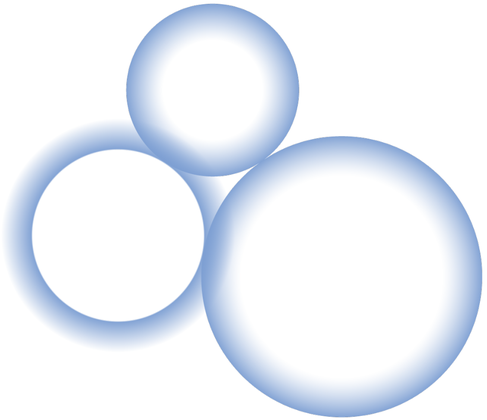
Geometry in affine space

What is Geometry?

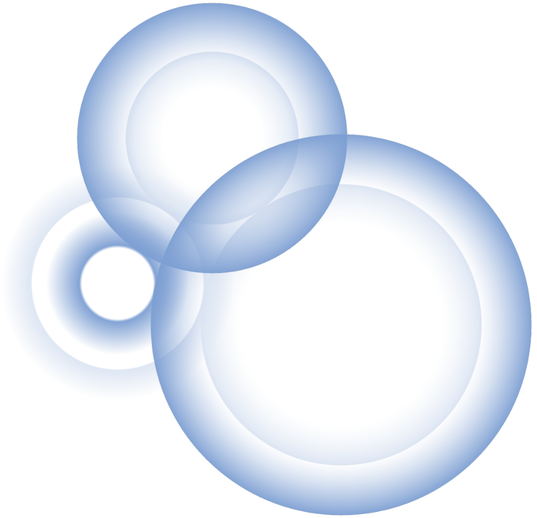
Ingredients of geometry:

- Collection of objects

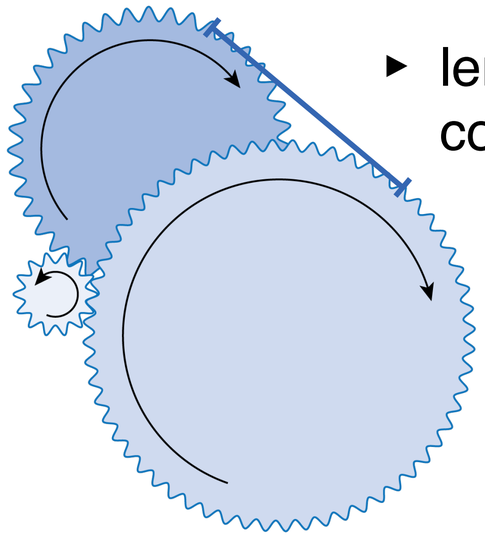
  oriented circles in $\mathbb{R}^2$



- Group of transformations
  - rotate, translate
  - normal offset (propagate wave)



- Invariant notions
  - straight lines
  - oriented contact
  - lengths of common tangents



- Non-invariant notions
  - intersection
  - point

Laguerre geometry of oriented circles

What is Geometry?

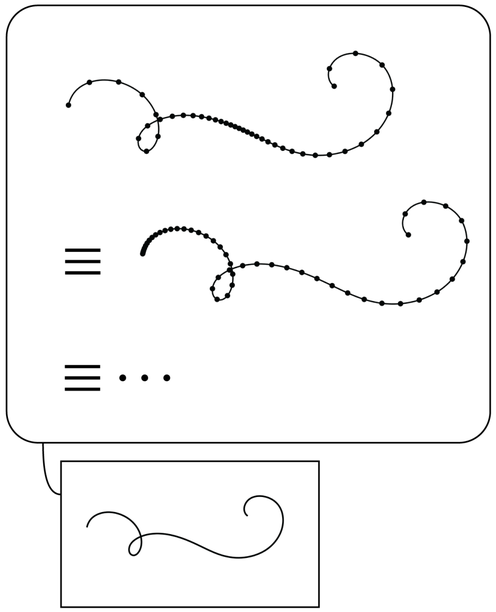
Ingredients of geometry:

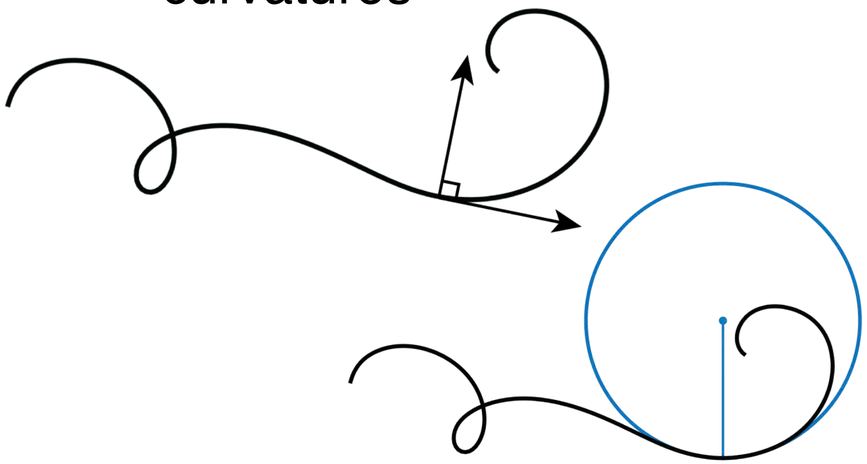
- Collection of objects
- Group of transformations
- Invariant notions
  - tangents, normals
  - curvatures

Particle moving along a trajectory



Reparametrizations





- Non-invariant notions
  - particle speed
  - particle acceleration

Differential geometry of curves

Modern differential geometry

- Provides a version of calculus that is invariant under re-parametrization

$$\hat{\mathbf{x}} = \frac{x\left(\sqrt{x^2+y^2}\hat{\mathbf{r}} + z\hat{\boldsymbol{\theta}}\right) - y\sqrt{x^2+y^2+z^2}\hat{\boldsymbol{\varphi}}}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}$$
$$\hat{\mathbf{y}} = \frac{y\left(\sqrt{x^2+y^2}\hat{\mathbf{r}} + z\hat{\boldsymbol{\theta}}\right) + x\sqrt{x^2+y^2+z^2}\hat{\boldsymbol{\varphi}}}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}$$
$$\hat{\mathbf{z}} = \frac{z\hat{\mathbf{r}} - \sqrt{x^2+y^2}\hat{\boldsymbol{\theta}}}{\sqrt{x^2+y^2+z^2}}$$

$$\operatorname{grad} f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$$
$$= \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}}$$

multivariable calculus

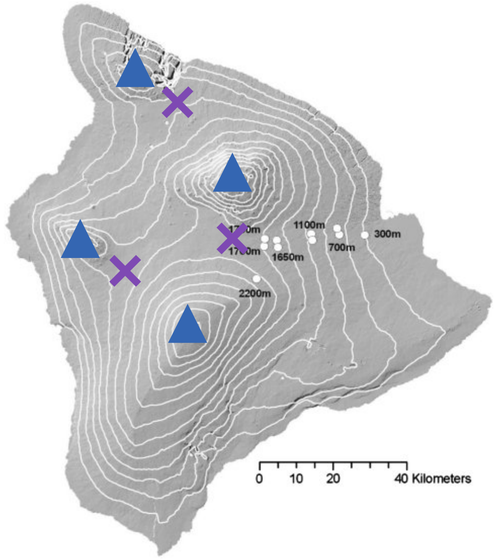
coordinate geometry

$$\operatorname{grad} f = (df)^\sharp$$

modern differential geometry

Rigidity in differential geometry

- “Physical laws” in geometries
  - Do you know that:



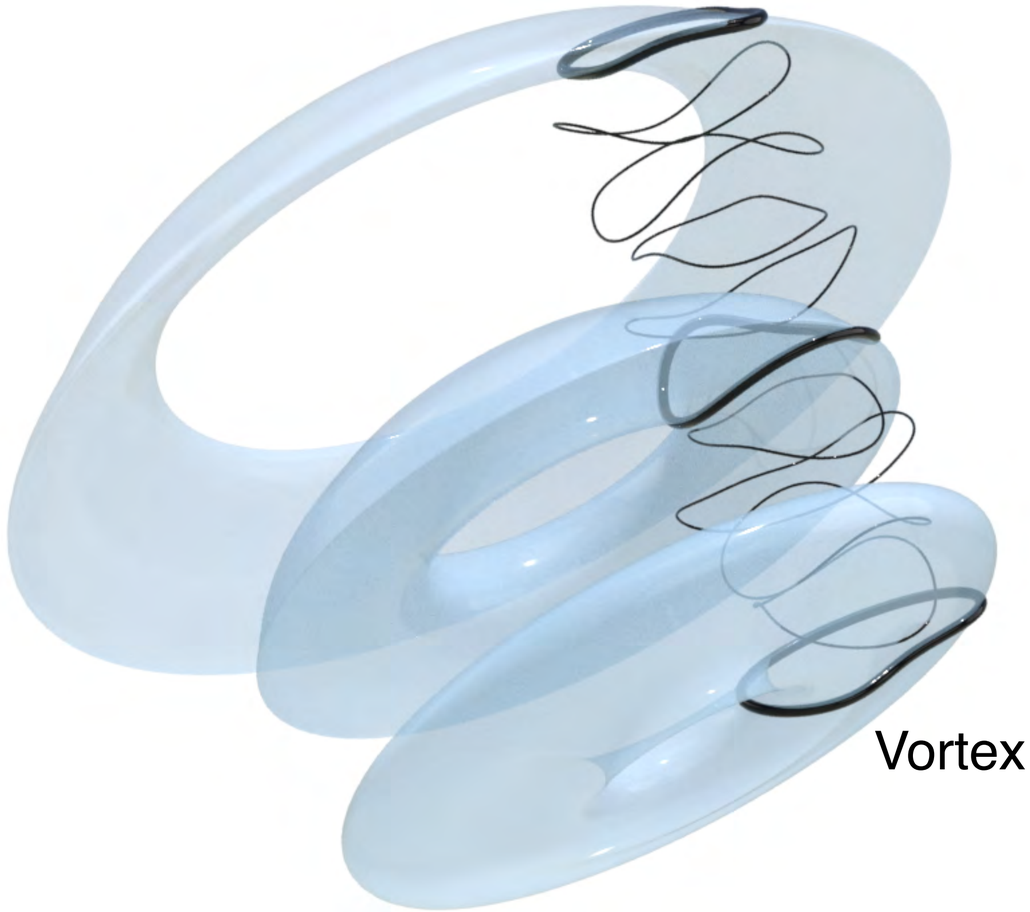
number of summits
+ number of bottoms
– number of saddles = 1

  - Total Gaussian curvature is shape independent

$$\int K(\quad) = \int K(\quad) = \int K(\quad) = 4\pi$$

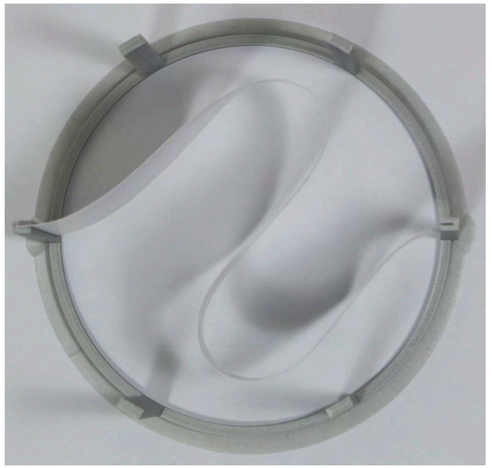
Rigidity in differential geometry

- Invariants in physical system

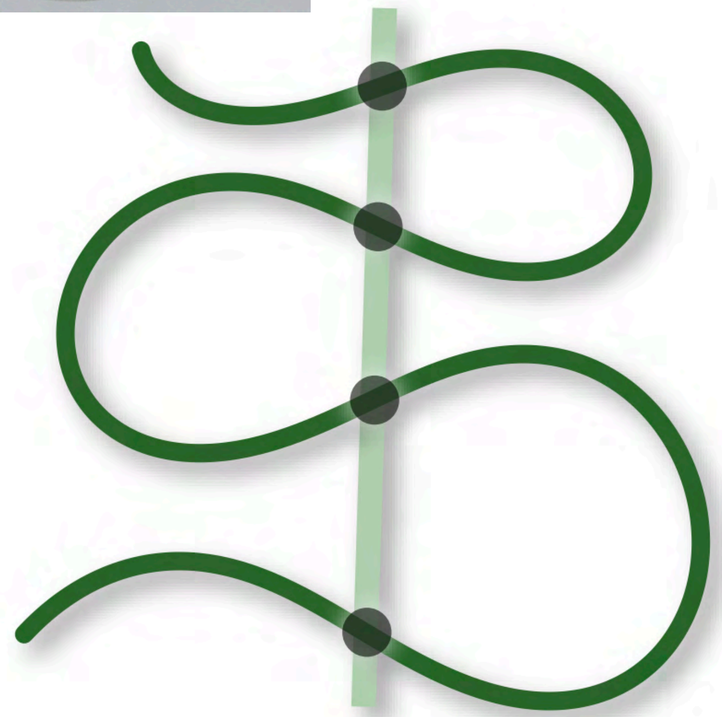


Vortex ring motion

Volume of revolution about any axis is conserved



Elastic wire
at equilibrium
has collinear
inflection points



Why do we study geometry?

- Geometry is the study of shapes and spaces, as well as symmetries and invariants
- Emphasize on structures independent of artificial coordinates and representations
  - Operate calculus at an intuitive level
  - Many physical phenomena are consequences of rigidity of geometric laws
  - Reduction of complexity

What is Computing?

- Geometry in computations
- What is geometry?
- What is computing?
- Discrete Differential Geometry
- Structure preserving examples
- Roadmap of the course

What is computing?

Three stages:

- **Modeling**
  - Turning (real)-world problems into mathematical equations
- **Mathematical Analysis**
  - Making sense from the equations
- **Numerical computation**
  - Approximate solutions to the equations efficiently and robustly

Classical approaches

Three stages:

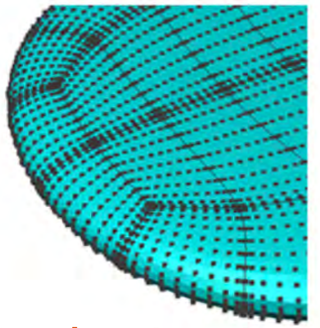
- **Modeling**
  - Turning (real)-world problems into mathematical equations
- **Mathematical Analysis**
  - Making sense from the equations
- **Numerical computation**
  - Approximate solutions to the equations efficiently and robustly

- Usually first establish coordinates
- Then equation is based on the coordinates

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\varphi(x,y,z)=f(x,y,z).$$

- Hard to generalize to general spaces without a natural coordinate

$$\Delta f=\frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j f\right)$$



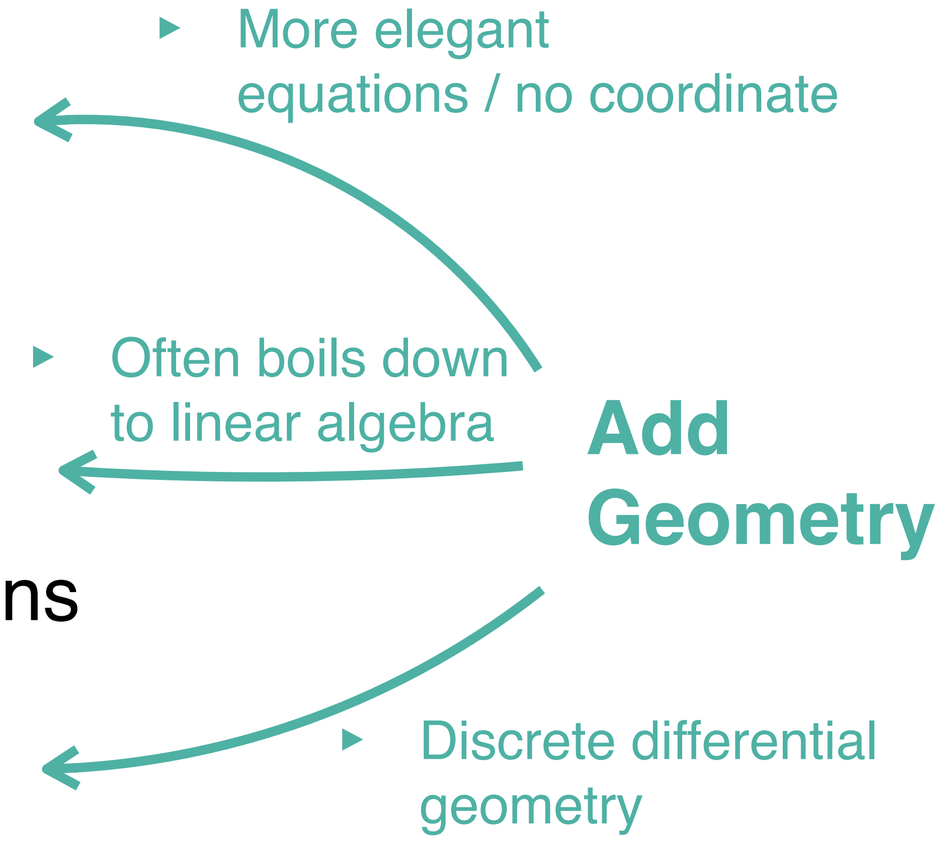
- Hard to see the algebraic structure behind the equations

- The only available tool to turn the equation into numerics is approximation theory

Geometric approach to computing

Three stages:

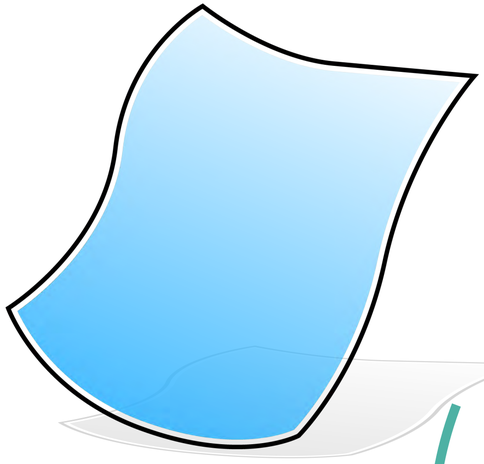
- **Modeling**
  - Turning (real)-world problems into mathematical equations
- **Mathematical Analysis**
  - Making sense from the equations
- **Numerical computation**
  - Approximate solutions to the equations efficiently and robustly



Discrete Differential Geometry (DDG)

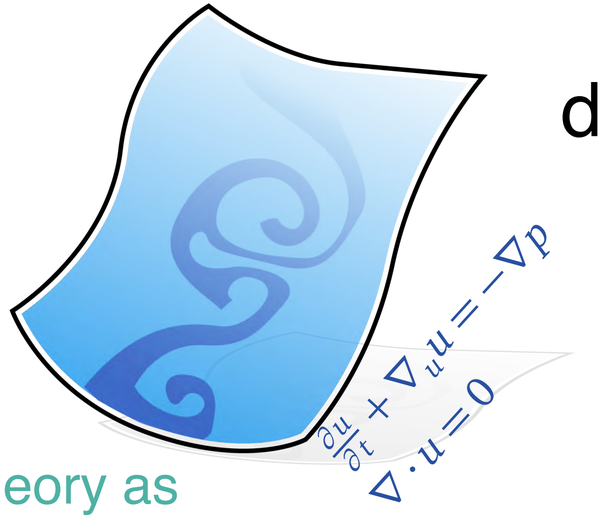
- Geometry in computations
- What is geometry?
- What is computing?
- Discrete Differential Geometry
- Structure preserving examples
- Roadmap of the course

Discretized v.s. Discrete



modeling
& analysis

Invariants,
Symmetries

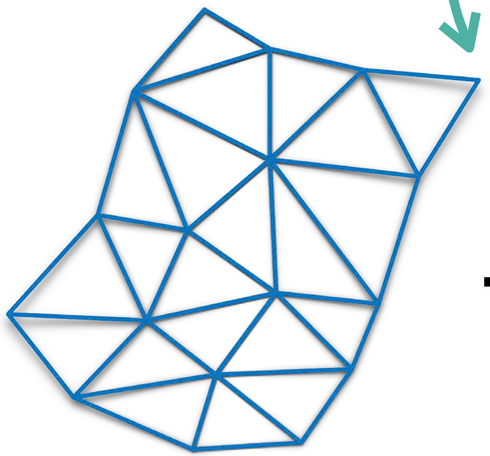


discretization

Approximation
theory



Use the smooth theory as
blueprint and develop a discrete theory

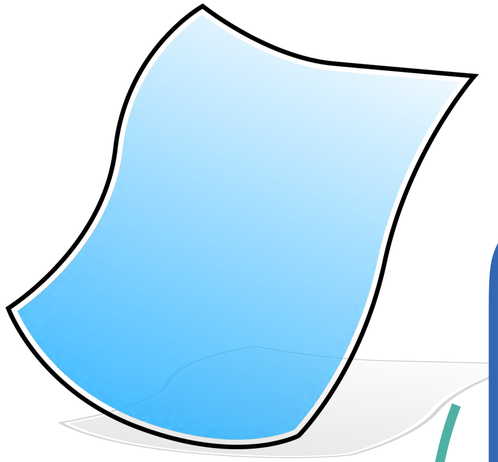


discrete
differential
geometry



Structure preserving:
There are properties
which are exact rather
than approximation

Discretized v.s. Discrete



modeling
& analysis

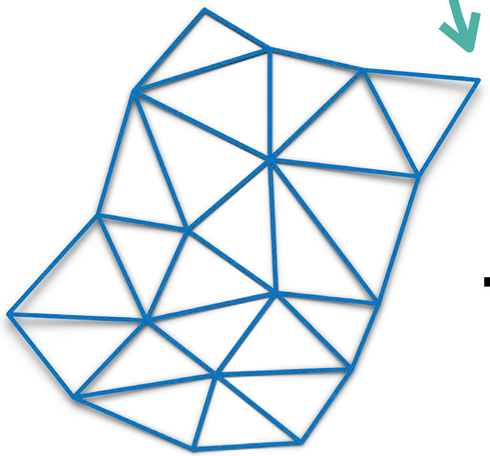


discretization



Discretize the theory
instead of
discretizing the equation

U
b



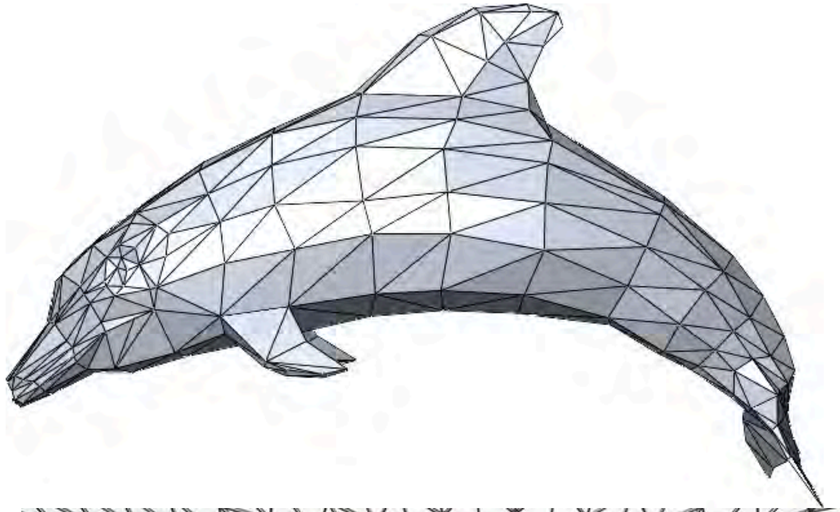
differential
geometry

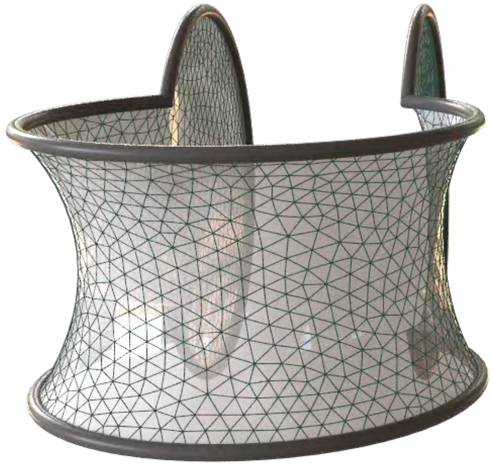


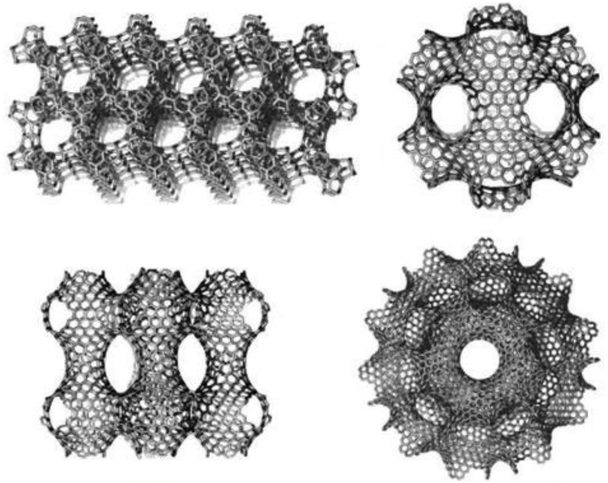
Structure preserving:
There are properties
which are exact rather
than approximation

Digital / discrete geometry

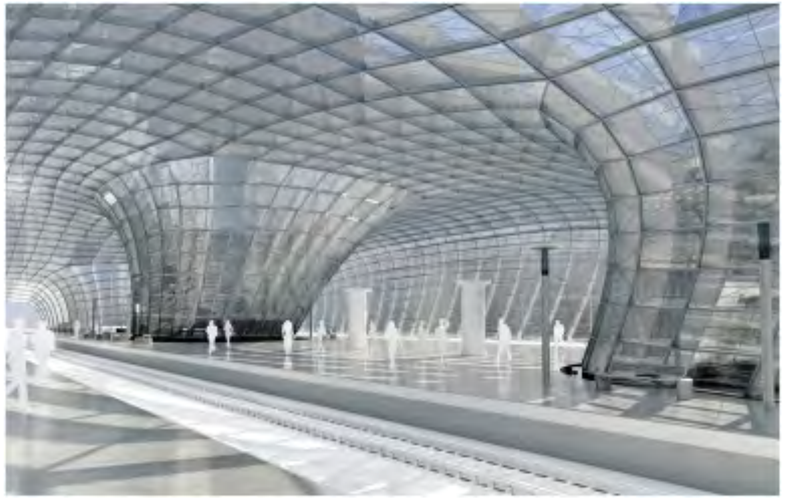
- Treat them as discrete geometries in their own right

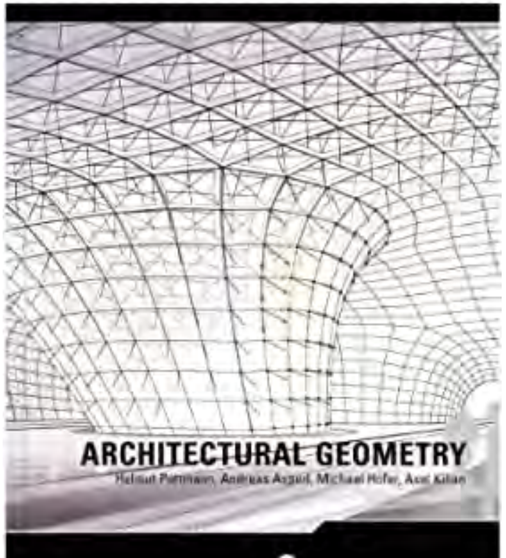












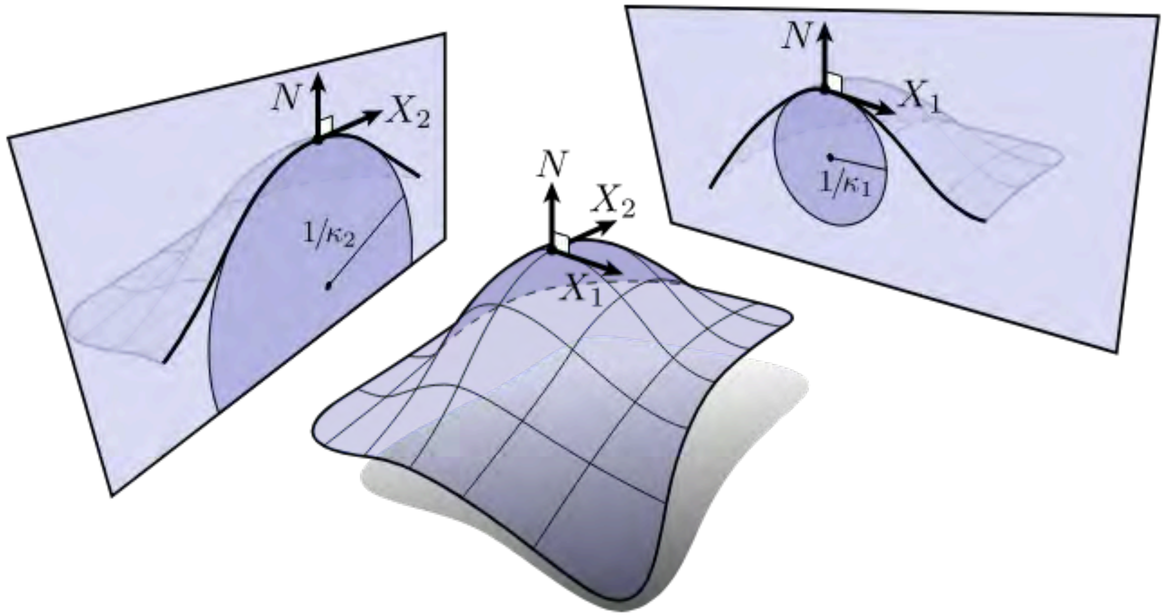
Examples of Structure-Preserving Discrete Theory

- Geometry in computations
- What is geometry?
- What is computing?
- Discrete Differential Geometry
- Structure preserving examples
- Roadmap of the course

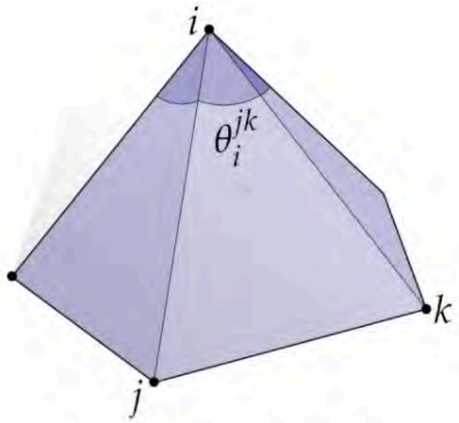
Discrete Gauss–Bonnet Theorem

$$\int_{\substack{\text{entire}\\ \text{surface}}} \kappa_1 \kappa_2 \, dA = 2\pi\chi$$

$\chi =$ 2 for sphere
0 for torus
-2 for two-holed torus

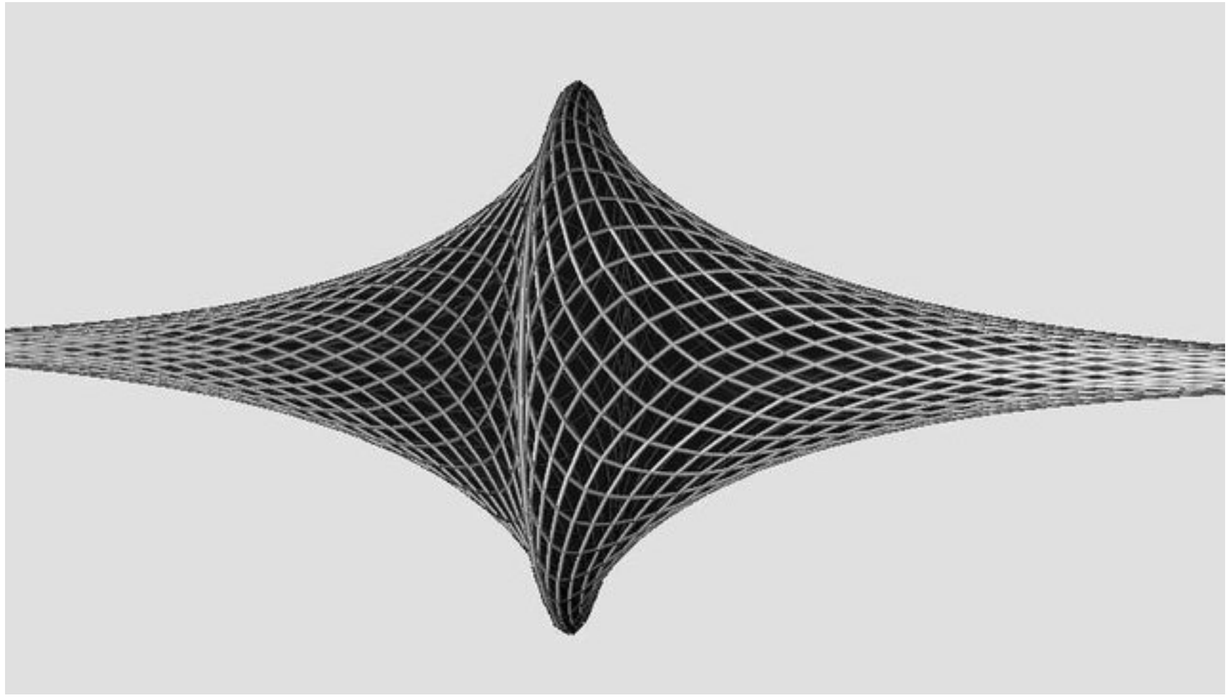


$$\sum_{i:\text{vertices}} \Omega_i = 2\pi\chi$$



$$\Omega_i := 2\pi - \sum_{ijk} \theta_i^{jk}$$

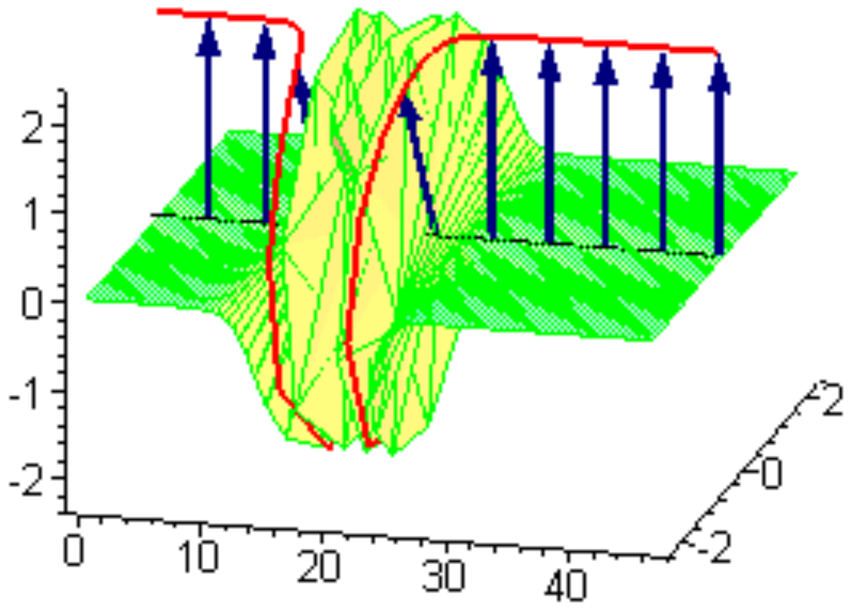
Discrete surface and integrable system



Constant (negative) Gauss curvature surface

Sine–Gordon equation

$$\frac{\partial^2}{\partial t^2}\varphi + \frac{\partial^2}{\partial x^2}\varphi + \sin\varphi = 0$$



Discrete surface and integrable system

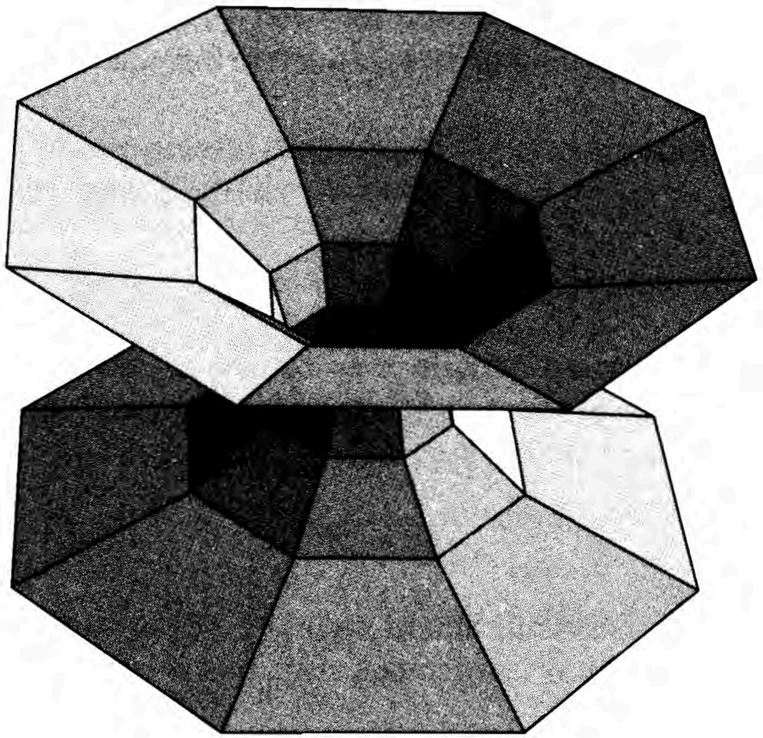
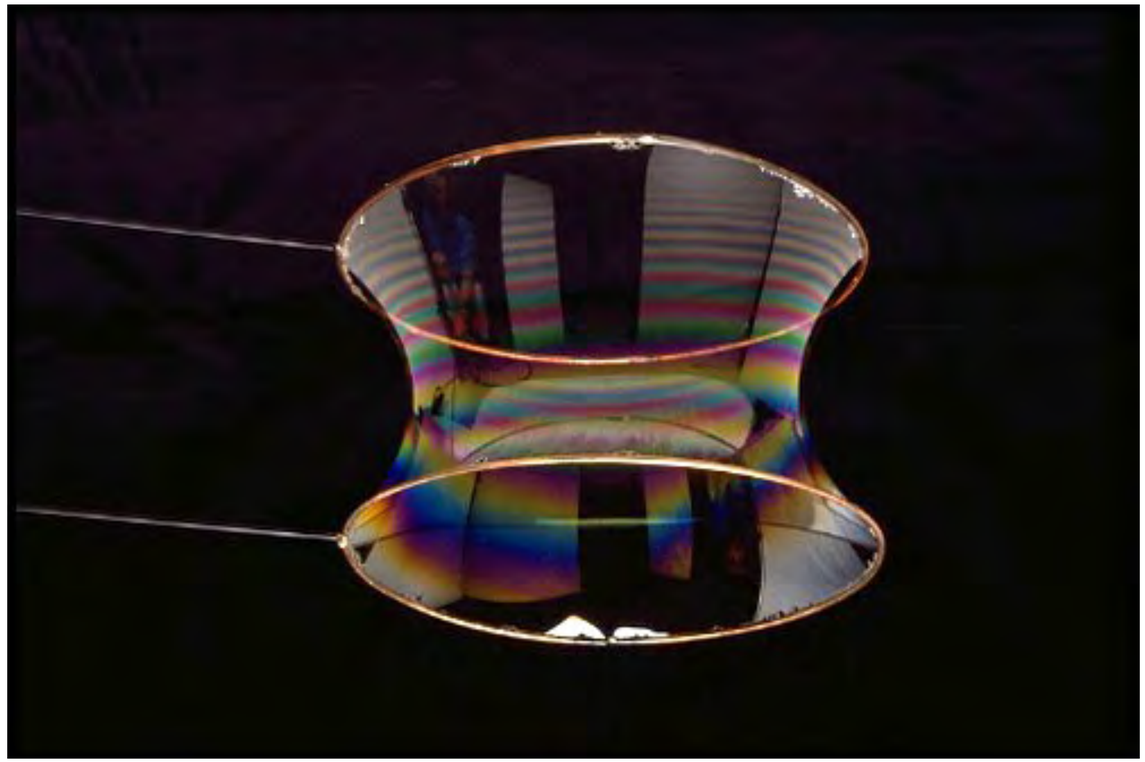


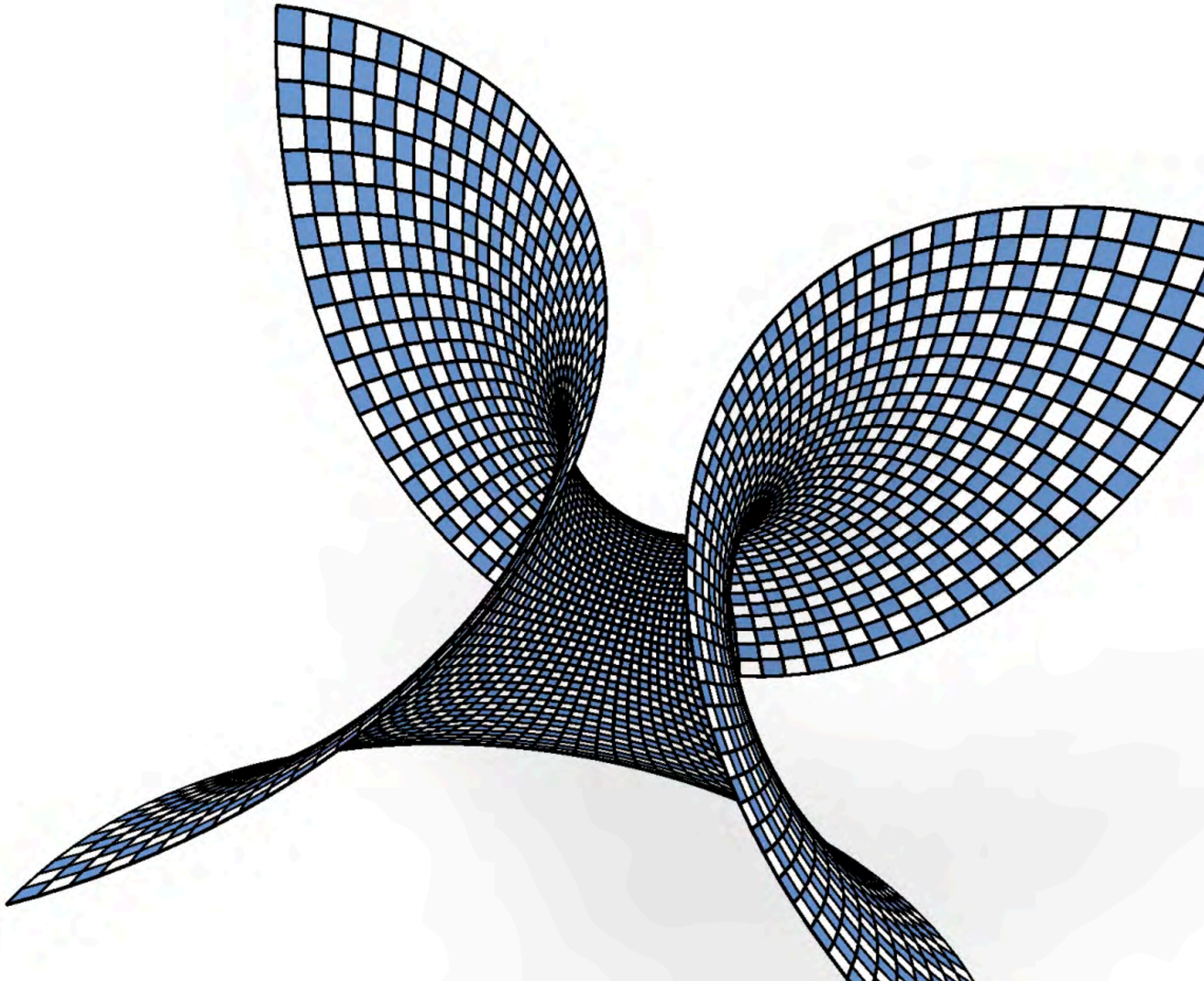
Figure 10. Discrete catenoid



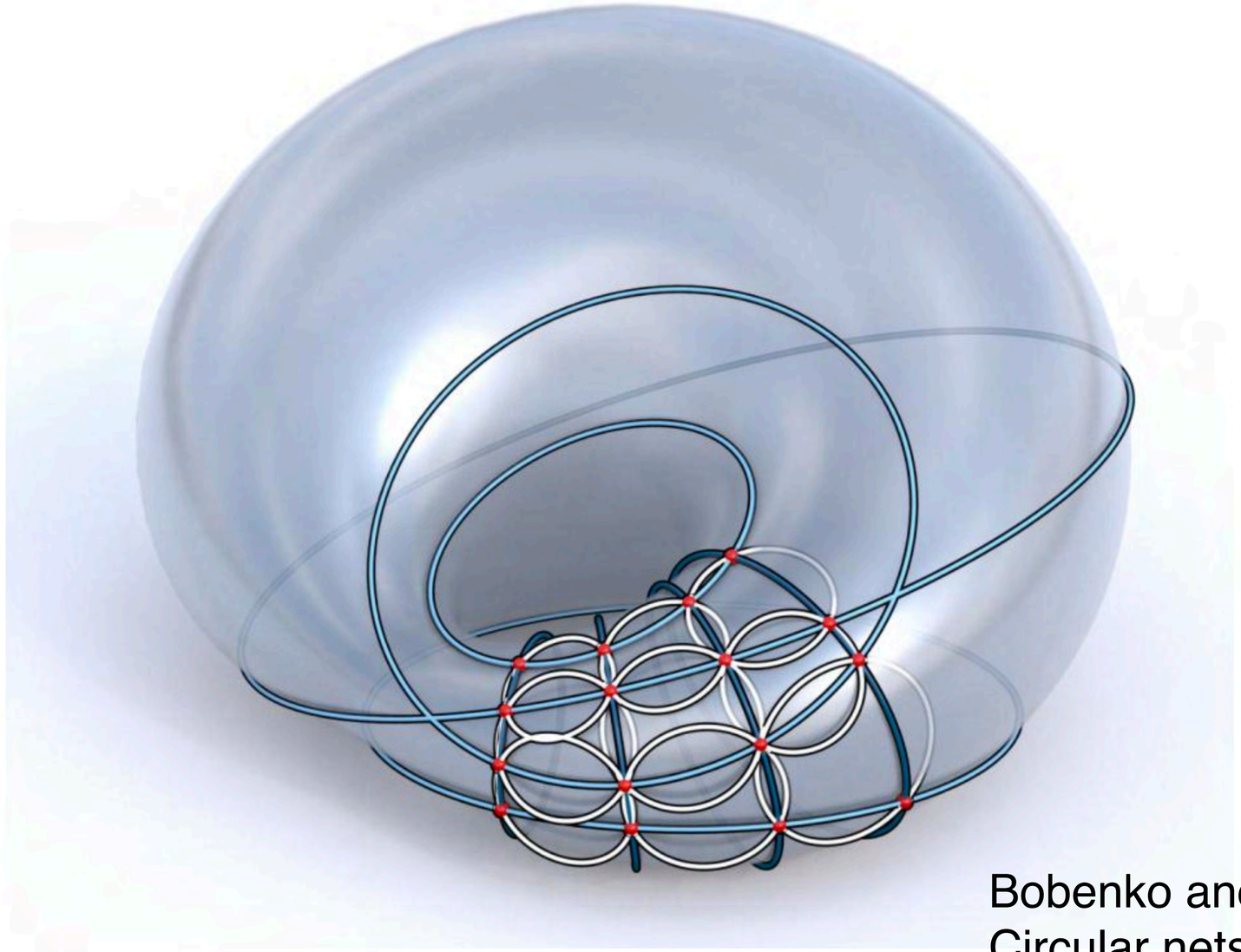
Surface conformally parameterized along curvature lines.

Special surfaces and coordinates



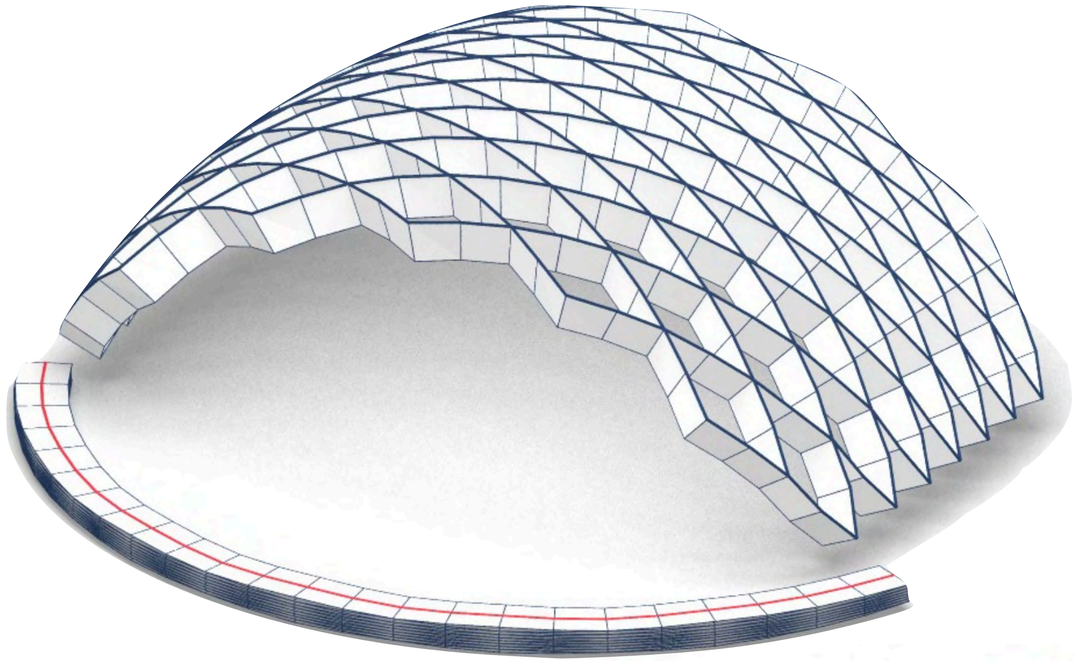


Special surfaces and coordinates

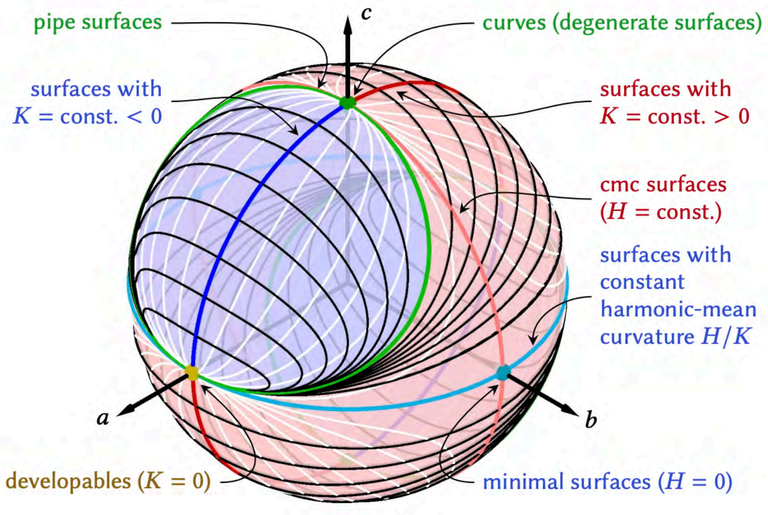


Bobenko and Fairley 2023
Circular nets with spherical parameter lines

Special surfaces and coordinates

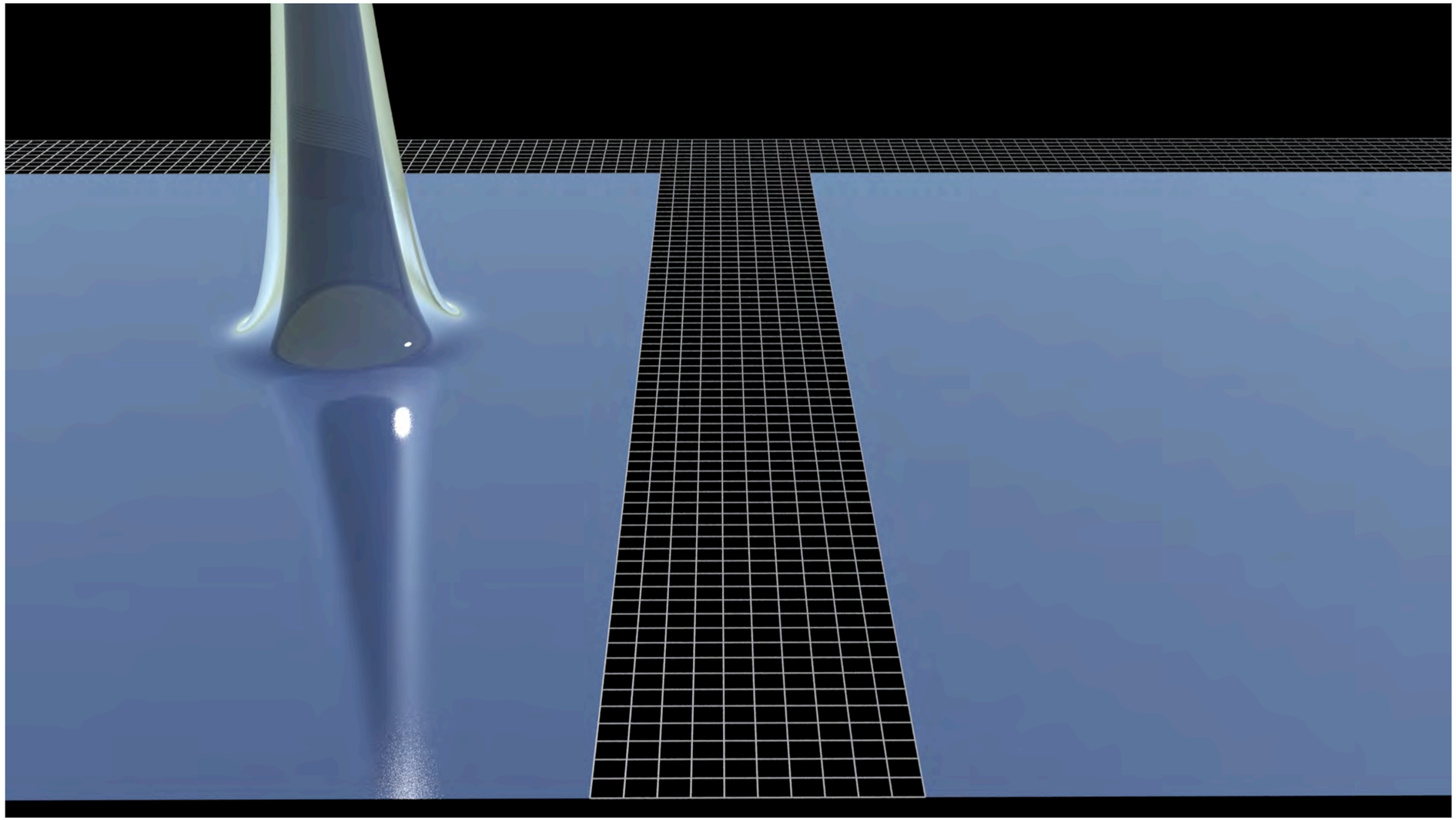




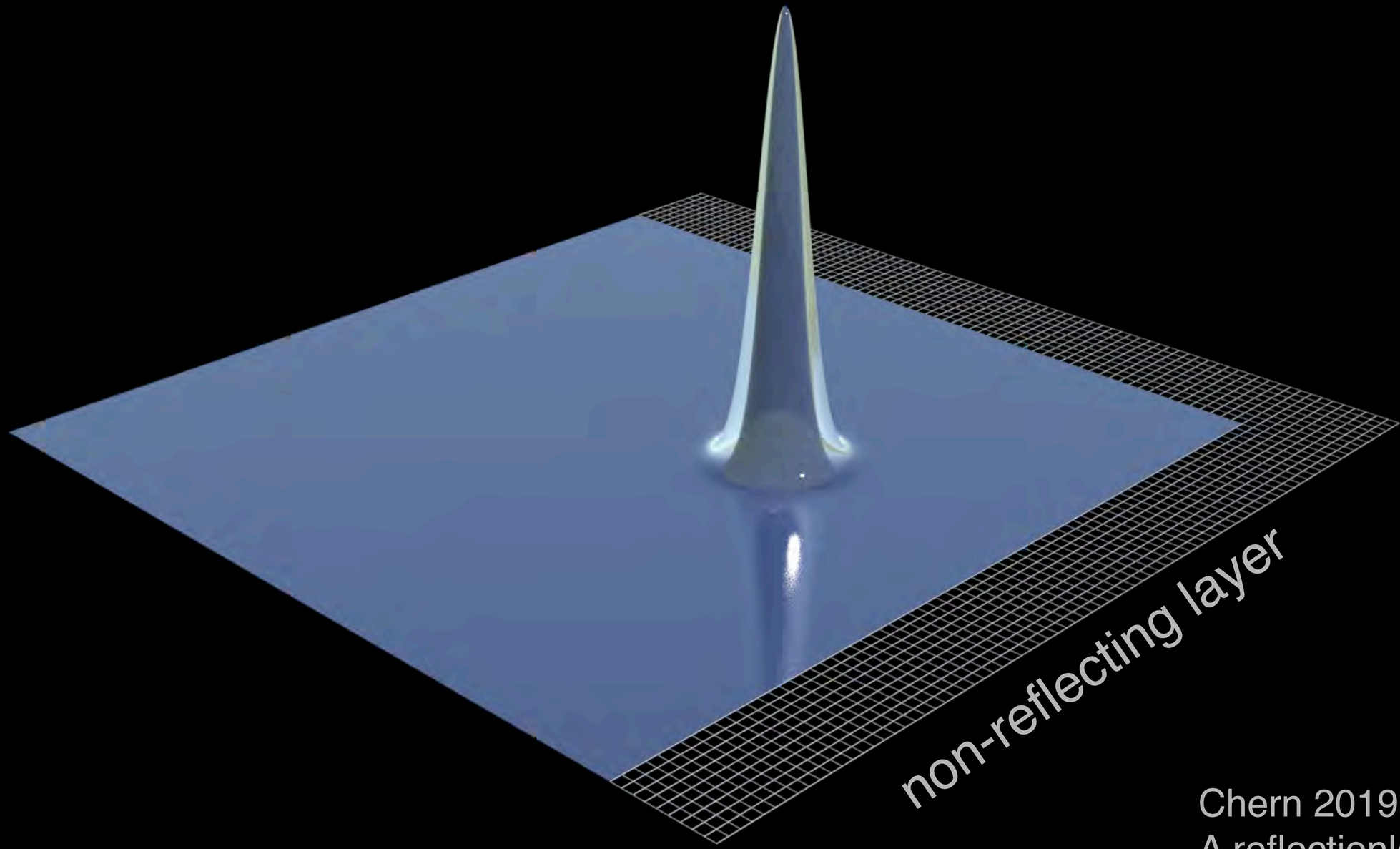


Liu et al 2023 “Deployable strip structures”

Discretely perfect absorbing material



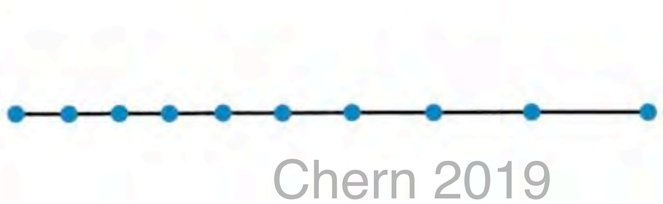
“A reflectionless discrete perfectly matched layer”
Journal of Computational Physics, 2019



Chern 2019
A reflectionless discrete perfectly matched layer
J. Comp. Phys.

Discrete complex analysis





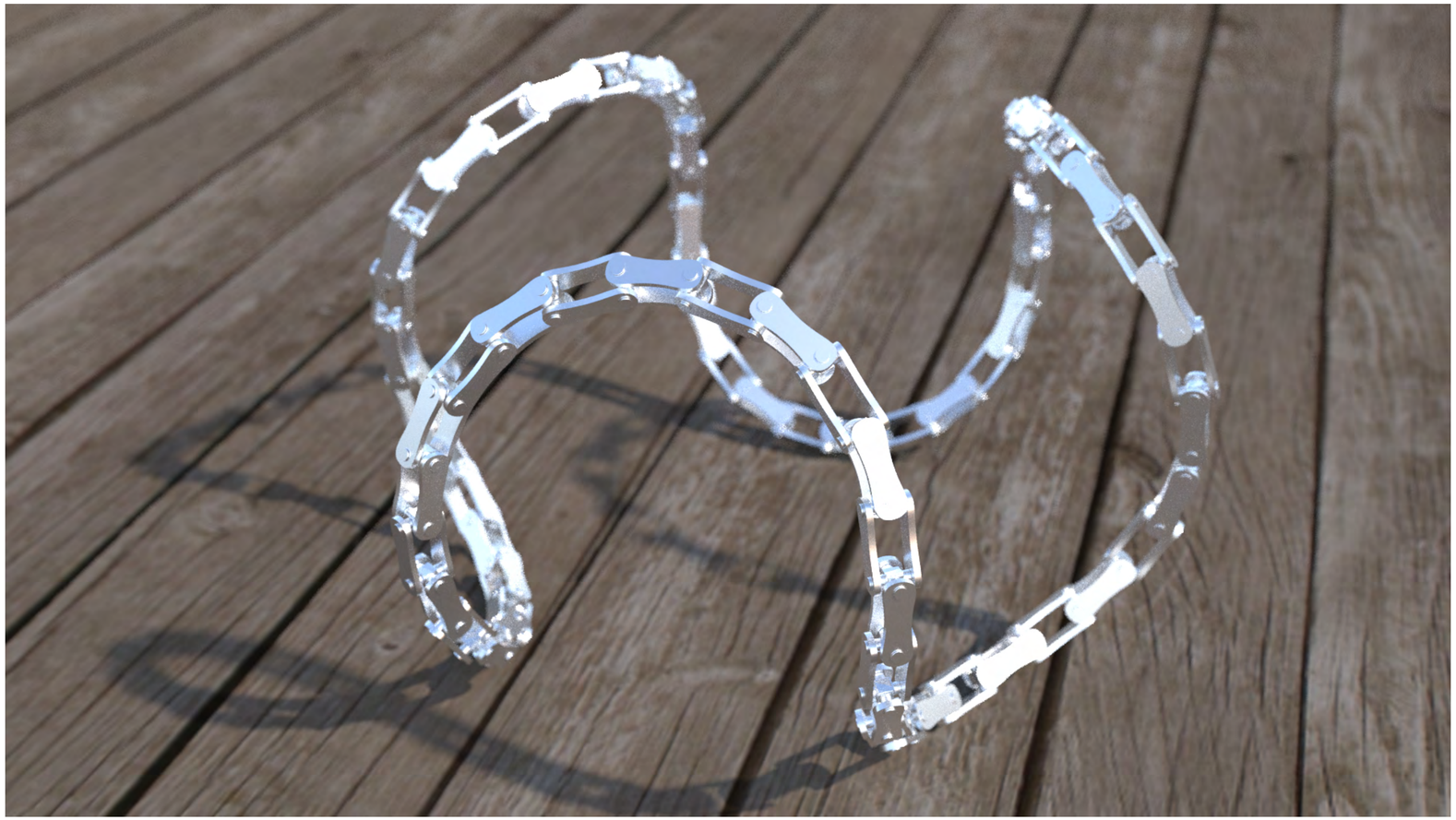
Chern 2019
A reflectionless discrete perfectly matched layer
J. Comp. Phys.

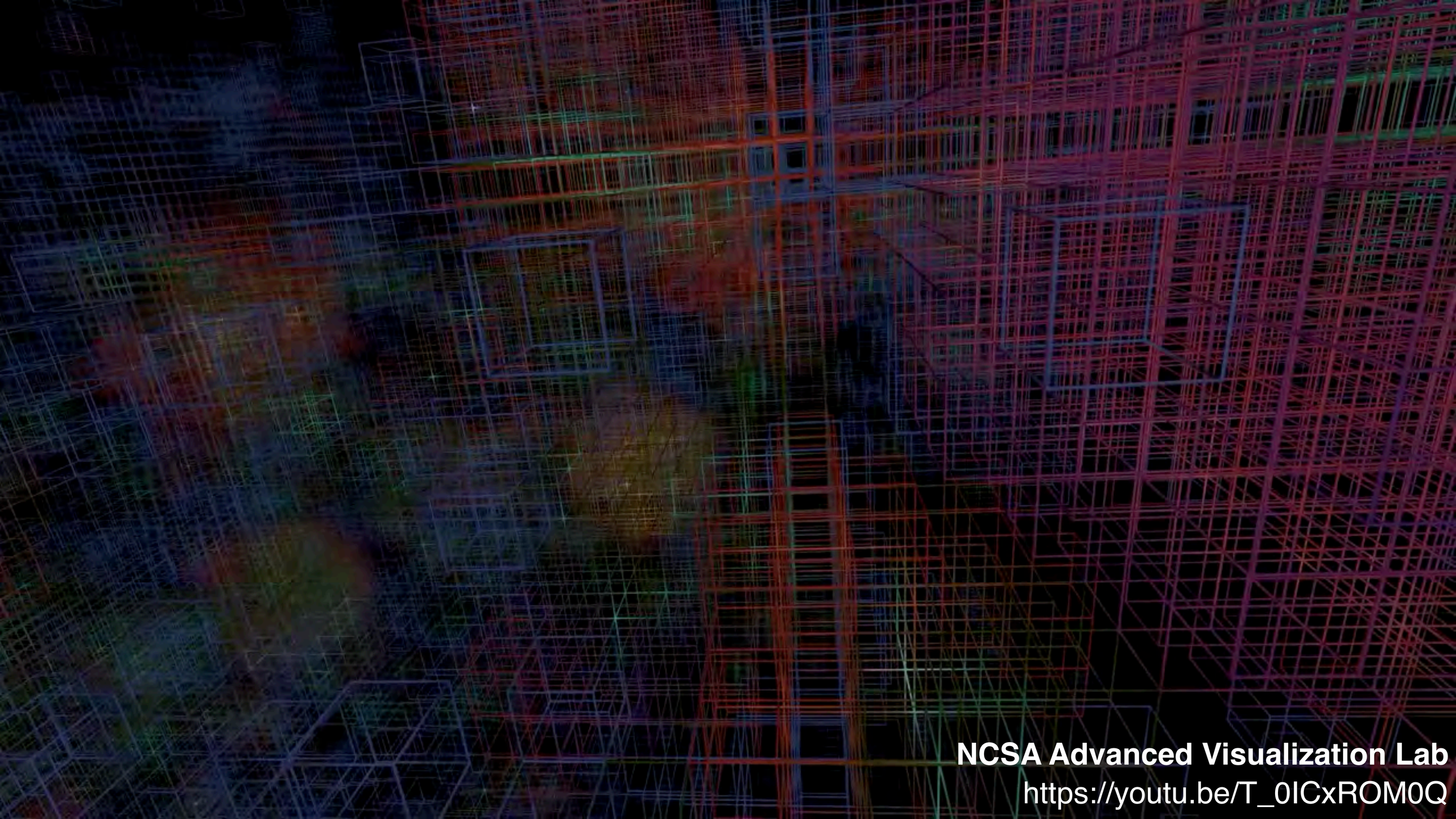
Roadmap of the Course

- Geometry in computations
- What is geometry?
- What is computing?
- Discrete Differential Geometry
- Structure preserving examples
- Roadmap of the course

Roadmap of the course

- We will use Houdini software for implementation

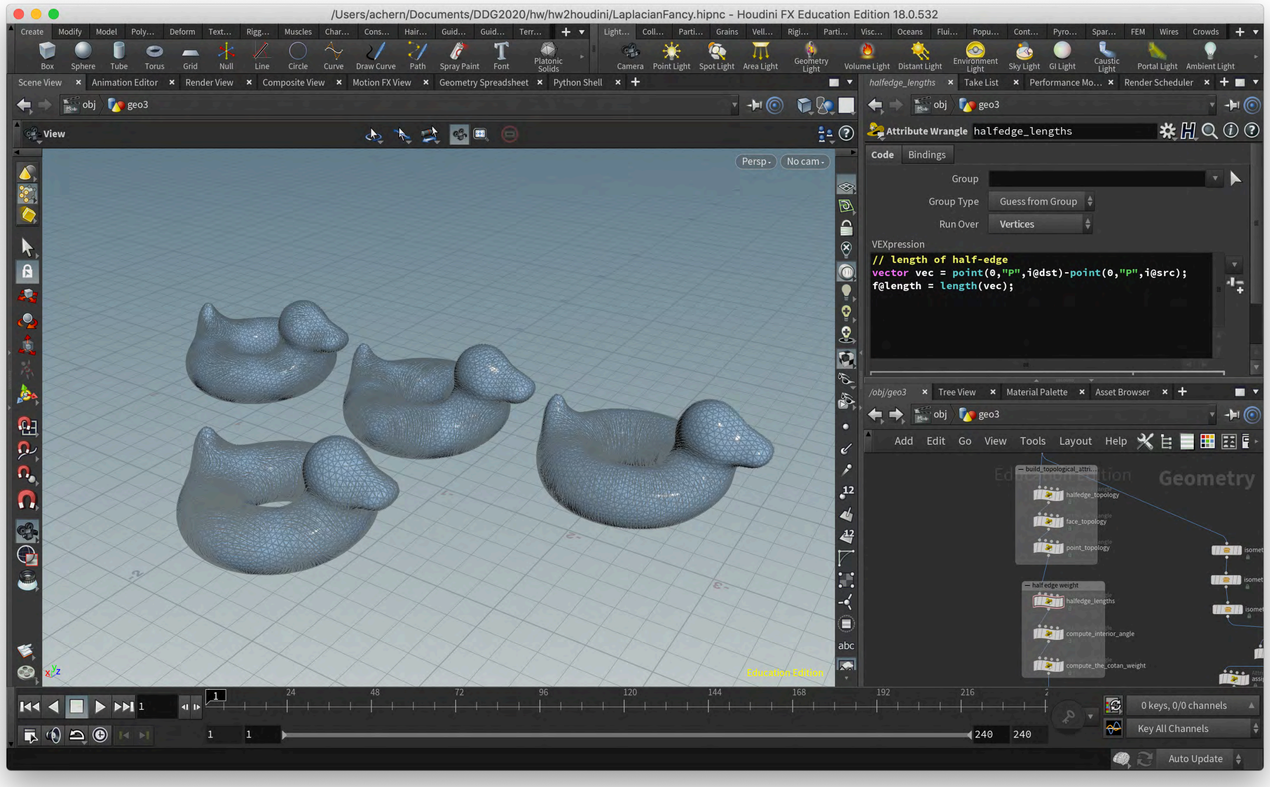




NCSA Advanced Visualization Lab
https://youtu.be/T_0ICxROM0Q

Homework 0

- Implementation part of the homework: **Houdini Software**



- Watch our tutorial (link in the syllabus on the course website)
- Part of HW1: upload a screenshot/ render of some cool mathematical visualization.

Install SciPy for Houdini (link in syllabus)

- Install Houdini & SciPy (numerical linear algebra for sparse matrices)

Houdini FX

Houdini is a 3D animation software by SideFX that is broadly used in filming industry. Its built-in shader language (VEX), geometric data structures, and its integration with Python (where one can call numerical linear algebra library such as SciPy) make it a powerful tool for scientific computing and geometry processing. Moreover Houdini comes with industry standard renderer allowing stunning visualization. Its free apprentice license is fully functional, with only a few limitations in the resolution of rendering preventing commercial uses. Houdini has become a frequently used software for not only research demonstration, but also for teaching and generating illustrations for lectures and talks.



A terrain with a shape of Mandelbrot set created in Houdini.

Lecture Note

- Houdini Introduction Lecture Note
- Houdini Tech Blog

Getting Started

To get started, first we recommend rummaging around the SideFX website and searching for tutorials on YouTube.

We also recommend that you work through some of the tutorials on the Houdini Tech Blog, a course blog for "Visualization in Mathematics" in TU Berlin. For example you can work through all the tutorials in the "Introduction" category. There are more tutorials on that site and if you feel inspired check them out. All come with Houdini files that allows you to load the completed tutorials if you wish.

Our lecture note listed above teaches you how to create simple geometry inside Houdini using the VEX language; basic scene setup with simple lighting and camera as well as manipulating surface appearance for rendering.

The complete documentation is also available on the SideFX website.

A note on installing Houdini (and SciPy)

1. *Find the Python directory*. Open Houdini. On any dock (subwindow) add a tab "Python Shell." In the Python shell, type

```
>>> import os
>>> path=os.path.abspath(os.__file__)
>>> print(path)
```

   The result should be something like
   `/Applications/Houdini/Houdini18.0.499/Frameworks/Python.framework/Versions/Current/lib/python2.7/os.pyc`
2. Open Terminal, and goto the bin directory ".../Versions/Current/bin" (according to the path shown on Step 1.)

```
$ cd /Applications/Houdini/Houdini18.0.499/Frameworks/Python.framework/Versions/Current/bin
```

3. *Use the local python to install the Python Package Installer (pip)*. In that directory execute (you will need to enter your root password for the superuser do (sudo) command)

```
$ sudo curl https://bootstrap.pypa.io/get-pip.py -o get-pip.py
$ sudo ./python get-pip.py
```

4. *Use the local pip to install SciPy*.

```
$ sudo ./pip install scipy
```

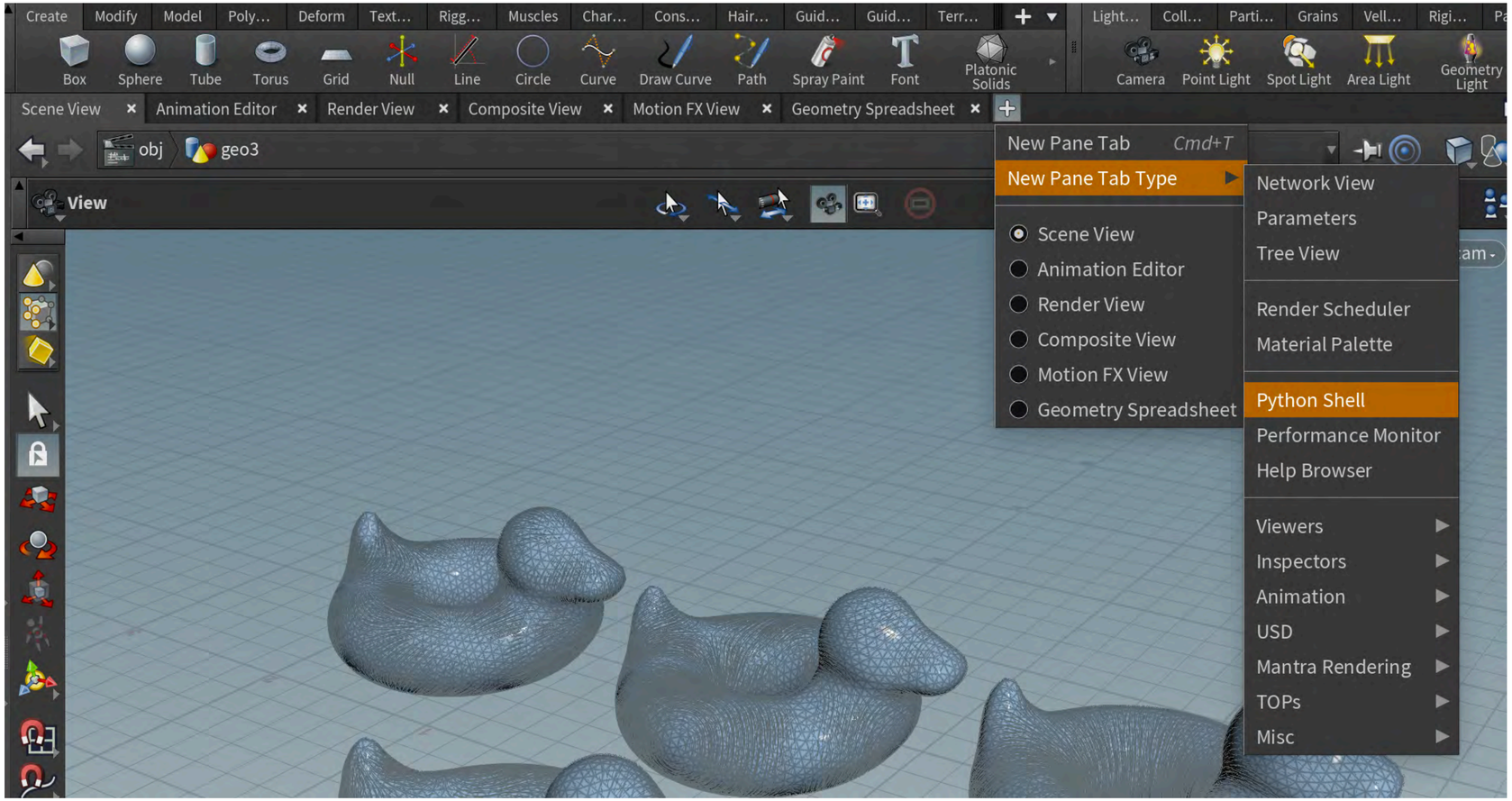
5. Back to the Houdini Python shell (no need to restart Houdini). Now

```
>>> import scipy
```

   should give no error.

Install SciPy for Houdini (link in syllabus)

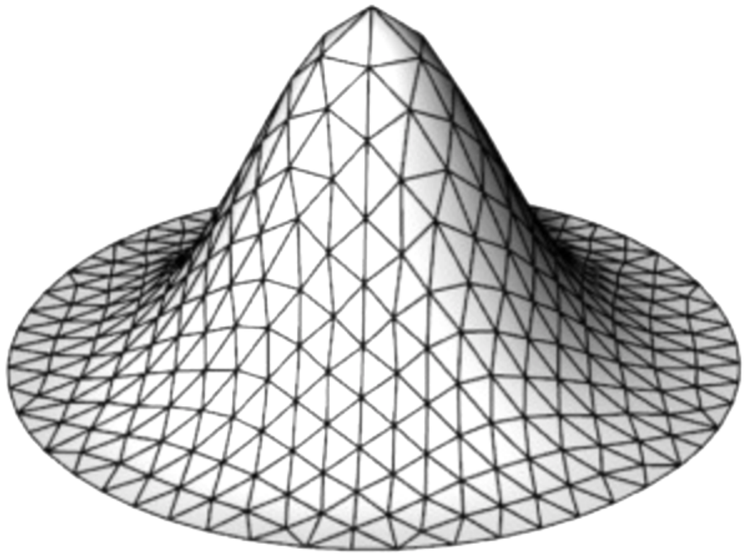
- Install Houdini & SciPy (numerical linear algebra for sparse matrices)



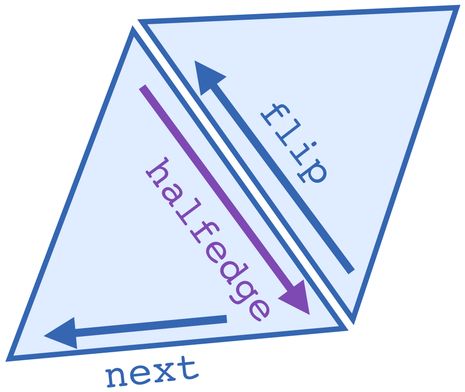


Roadmap of the course

- We will use Houdini software for implementation
- **Topic 1** Manifolds and Triangle meshes



$$|V| - |E| + |F| = \chi$$



half-edge
data structure

Roadmap of the course

- We will use Houdini software for implementation
- **Topic 1** Manifolds and Triangle meshes
- **Topic 2** Exterior calculus, PDE on surfaces

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$$



$$d\star df$$

Roadmap of the course

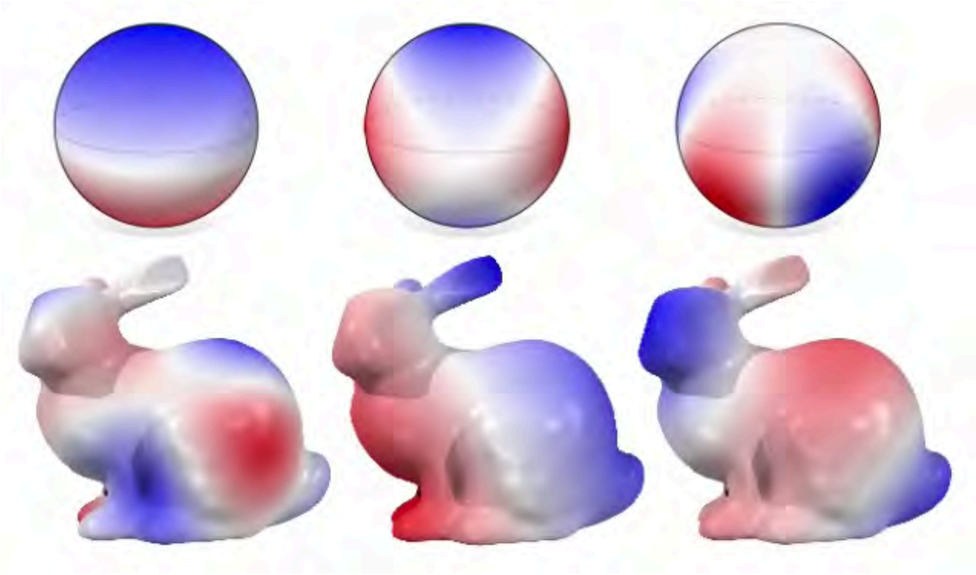
- We will use Houdini software for implementation
- **Topic 1** Manifolds and Triangle meshes
- **Topic 2** Exterior calculus, PDE on surfaces

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Laplace filter on images

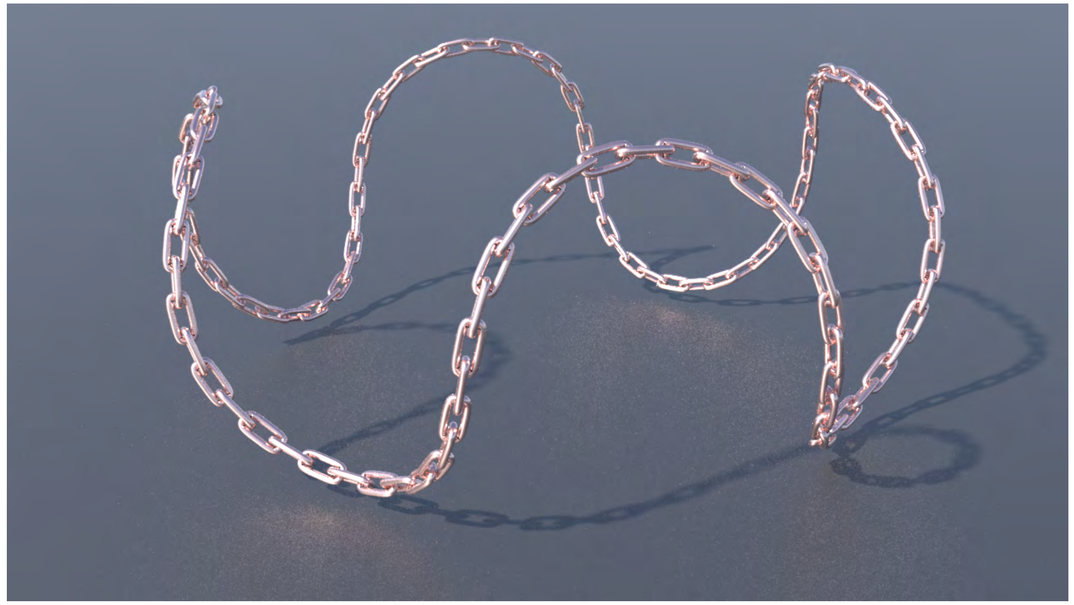
On triangle meshes?

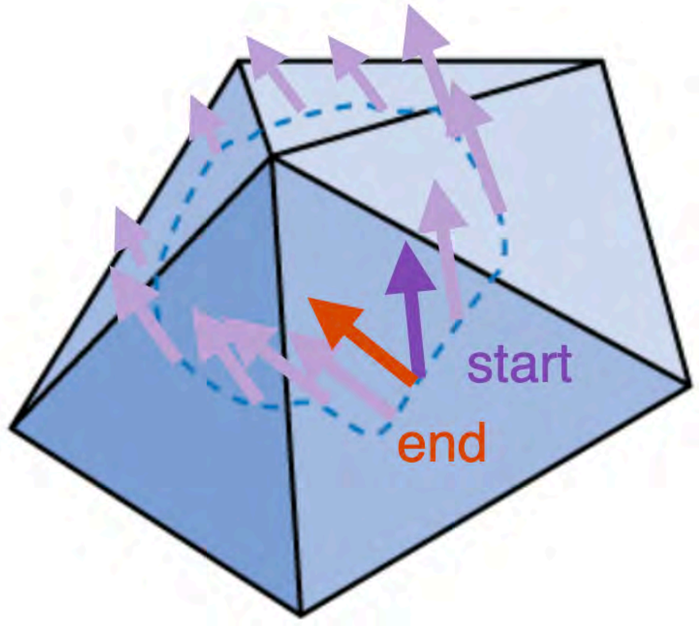
$$d^{\mathsf{T}} \star d$$



Roadmap of the course

- We will use Houdini software for implementation
- **Topic 1** Manifolds and Triangle meshes
- **Topic 2** Exterior calculus, PDE on surfaces
- **Topic 3** Differential geometry of curves and surfaces





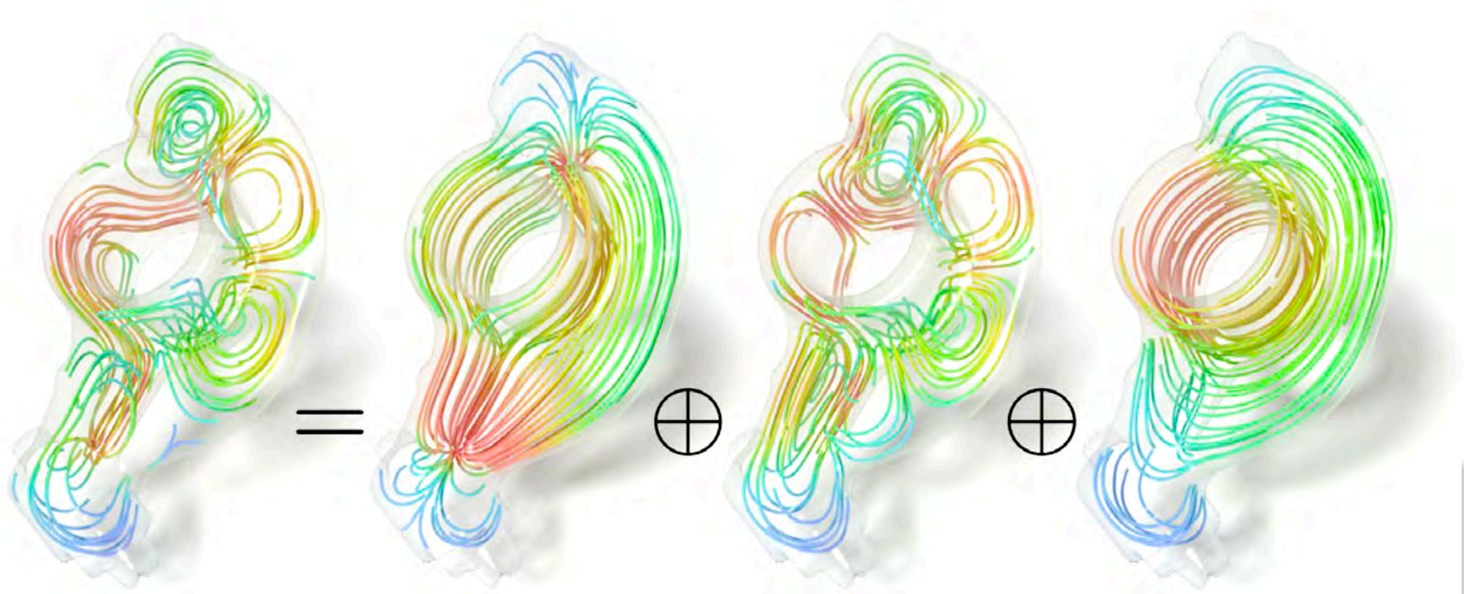
Roadmap of the course

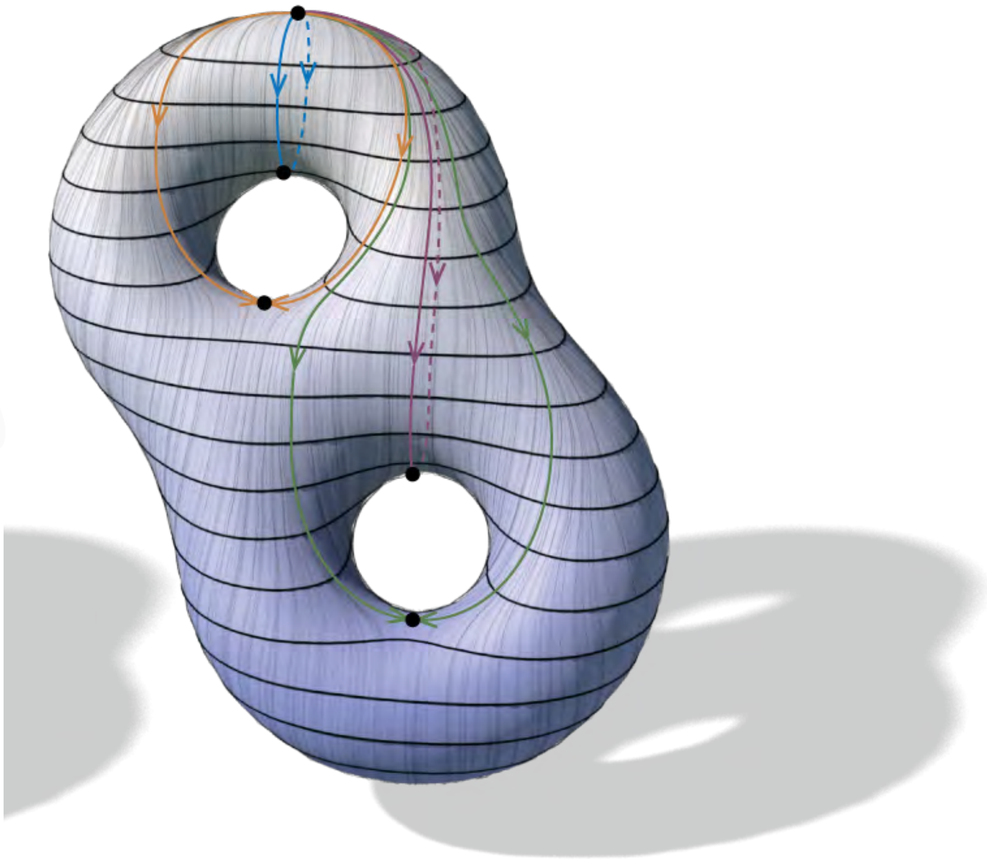
- We will use Houdini software for implementation
- **Topic 4** Conformal mapping



Roadmap of the course

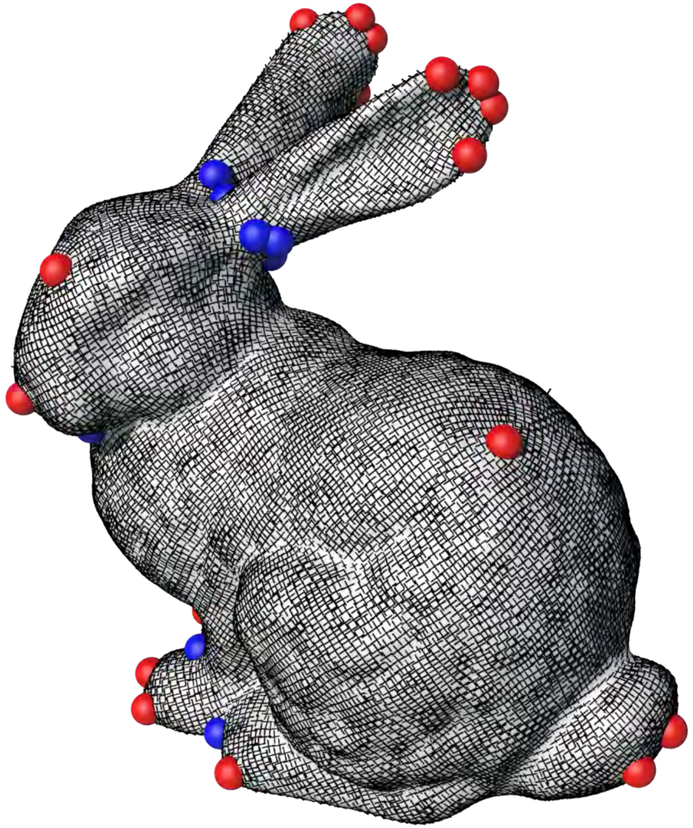
- We will use Houdini software for implementation
- **Topic 4** Linear algebraic topology

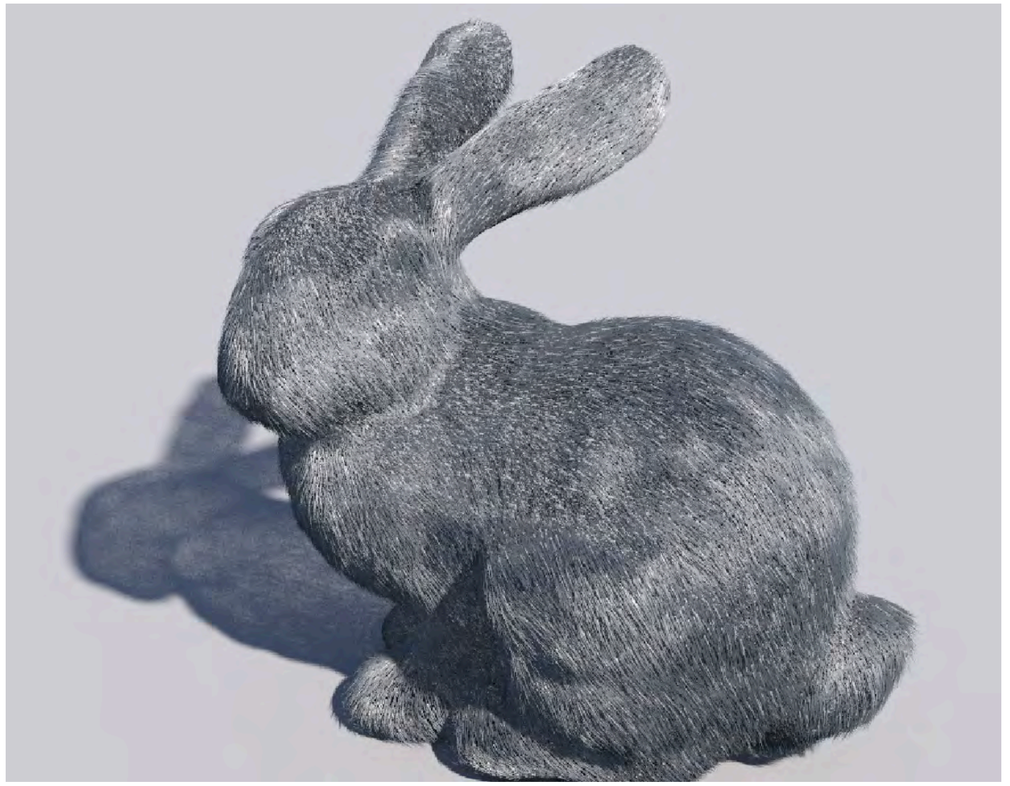




Roadmap of the course

- We will use Houdini software for implementation
- **Topic 4** Linear algebraic topology
- **Topic 5** Vector field design, gauge theory





Roadmap of the course

- We will use Houdini software for implementation

- **Topic 4** Linear algebraic topology
- **Topic 5** Vector field design, gauge theory
- **Additional topics** Lie algebra, finite elements, integral geometry, Riemannian geometry, general relativity

