CSE 270 (FA24)
Discrete Differential Geometry
Poisson Equations

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Reconstructing function from its derivative

- Poisson reconstruction
- Boundary conditions

Poisson reconstruction

Problem

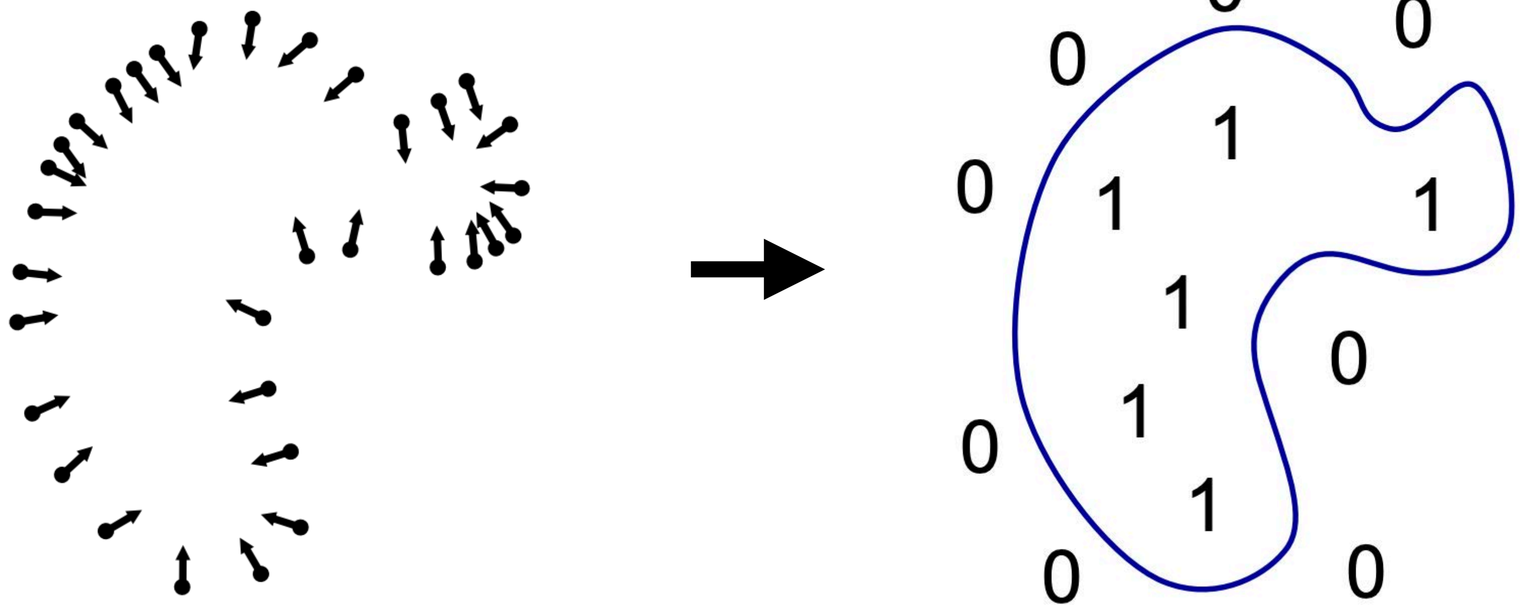
Given $\alpha \in \Omega^1(M)$, find $f \in \Omega^0(M)$ such that

$$df = \alpha$$

as much as possible.

Applications: surface reconstruction

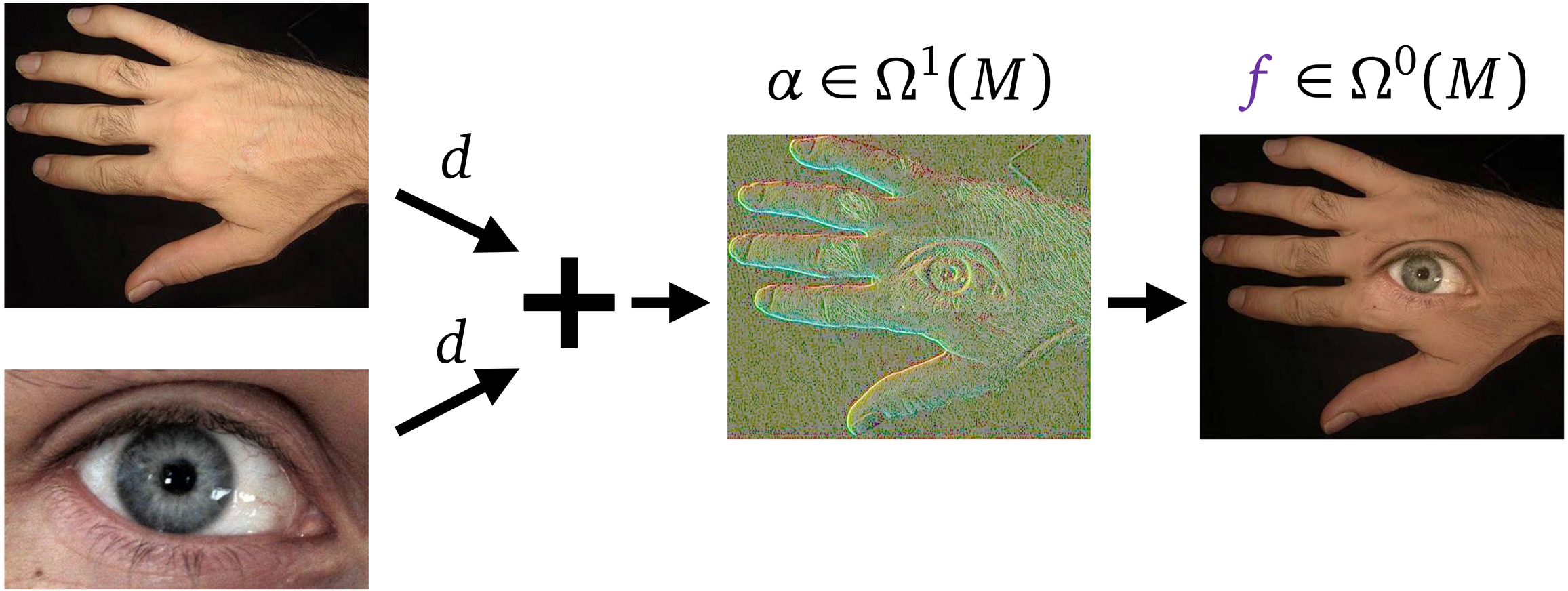
- “Poisson surface reconstruction” (Kazhdan et al 2006)



$\alpha \in \Omega^1(M)$ $\quad f \in \Omega^0(M)$

Applications: image processing

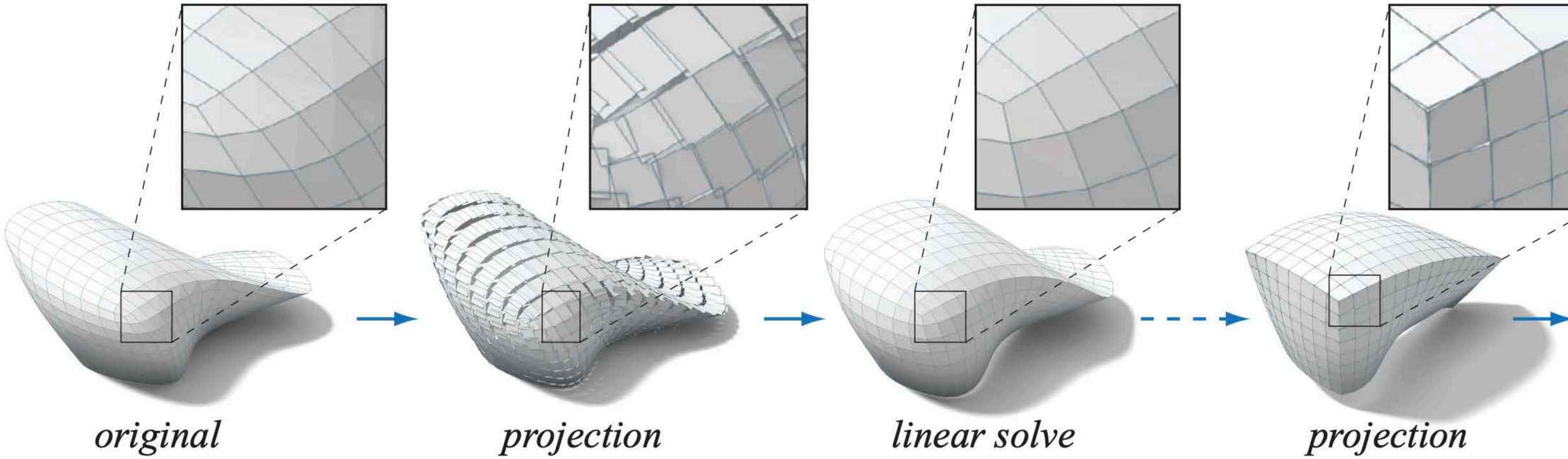
- Gradient-domain image processing



Applications: geometric modeling

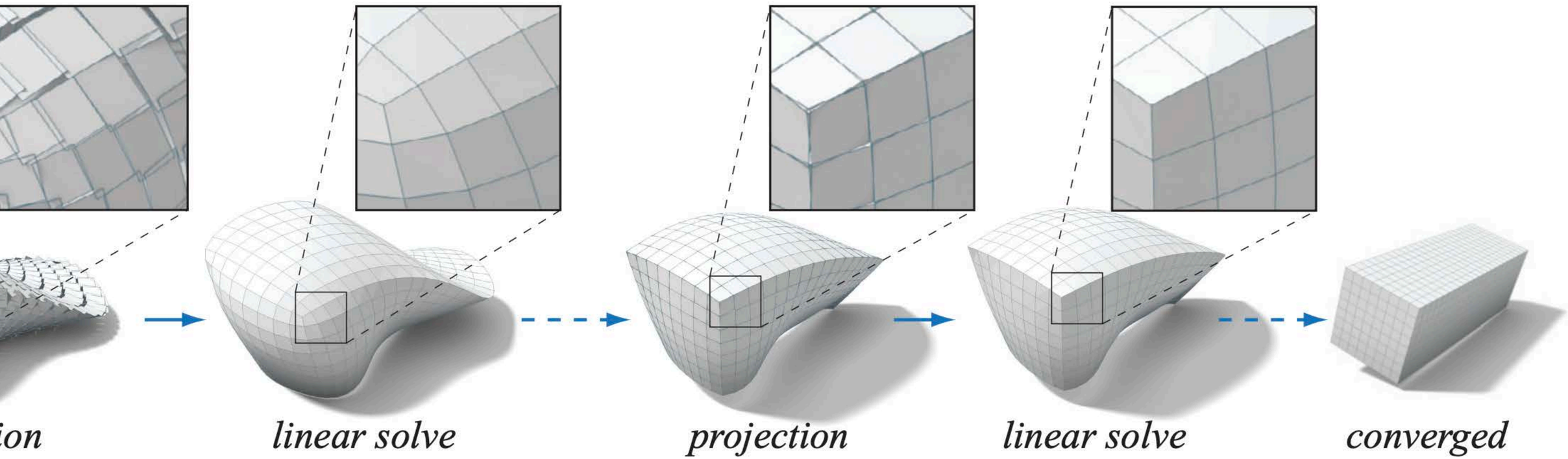
- Surface reconstruction from edge vectors
- “Shape-Up: shaping discrete geometry by projections” (Bouaziz et al 2012)

$$\boldsymbol{\alpha} \in \Omega^1(M;\mathbb{R}^3) \qquad f \in \Omega^0(M;\mathbb{R}^3)$$



Applications: geometric modeling

- Surface reconstruction from edge vectors
- “Shape-Up: shaping discrete geometry by projections” (Bouaziz et al 2012)



Poisson reconstruction

Problem

Given $\alpha \in \Omega^1(M)$, find $f \in \Omega^0(M)$ such that

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as much as possible.

Poisson reconstruction

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Given $\alpha \in \Omega^1(M)$, find $f \in \Omega^0(M)$ such that

$$df = \alpha$$

as much as possible.

$$\underset{f \in \Omega^0(M)}{\text{minimize}} \left\| df - \alpha \right\|^2$$

Poisson reconstruction

Problem

Given $\alpha \in \Omega^1(M)$, find $f \in \Omega^0(M)$ such that

$$df = \alpha$$

as much as possible.

$$\underset{\mathbf{f} \in \Omega^0(M)}{\text{minimize}} \, (\mathbf{d}_0 \mathbf{f} - \boldsymbol{\alpha})^\intercal \star_1 (\mathbf{d}_0 \mathbf{f} - \boldsymbol{\alpha})$$

Poisson reconstruction

$$\underset{\mathbf{f}\in\Omega^0(M)}{\text{minimize}} \underbrace{\boxed{(\mathbf{d}_0\mathbf{f}-\boldsymbol{\alpha})^\mathsf{T} \star_1 (\mathbf{d}_0\mathbf{f}-\boldsymbol{\alpha})}}_{\mathcal{E}(\mathbf{f})}$$

$$0 = d\mathcal{E}|_{\mathbf{f}}[\![\mathring{\mathbf{f}}]\!] = 2(\mathbf{d}_0\mathring{\mathbf{f}})^\mathsf{T} \star_1 (\mathbf{d}_0\mathbf{f}-\boldsymbol{\alpha}) = 2\mathring{\mathbf{f}}^\mathsf{T}\mathbf{d}_0^\mathsf{T} \star_1 (\mathbf{d}_0\mathbf{f}-\boldsymbol{\alpha})$$

for all $\mathring{\mathbf{f}} \in \Omega^0(M)$

$$\Leftrightarrow \boxed{\mathbf{d}_0^\mathsf{T} \star_1 \mathbf{d}_0\mathbf{f} = \mathbf{d}_0^\mathsf{T} \star_1 \boldsymbol{\alpha}} \qquad \mathbf{L}\mathbf{f} = \mathbf{d}_0^\mathsf{T} \star_1 \boldsymbol{\alpha}$$

Poisson reconstruction: smooth theory

$$\underset{\mathbf{f}\in\Omega^0(M)}{\text{minimize}} \boxed{\int_M (df-\alpha)\wedge\star_1(df-\alpha)}$$

$$\mathcal{E}(f)$$

$$0 = d\mathcal{E}|_f[\![\mathring{f}]\!] = 2\int_M d\mathring{f}\wedge\star(df-\alpha)$$

for all $\mathring{f}\in\Omega^0(M)$

$$= 2\oint_{\partial M}\mathring{f}\star(df-\alpha) - 2\int_M \mathring{f}\, d\star(df-\alpha)$$

more unambiguously $\oint_{\partial M} j^*(\mathring{f}\star(df-\alpha))$

where $j:\partial M\hookrightarrow M$ is the embedding map

Poisson reconstruction: smooth theory

$$\underset{\mathbf{f}\in\Omega^0(M)}{\text{minimize}} \boxed{\int_M (df-\alpha)\wedge\star_1(df-\alpha)}$$

$$\mathcal{E}(f)$$

$$0 = d\mathcal{E}|_f[\![\mathring{f}]\!] = 2\int_M d\mathring{f}\wedge\star(df-\alpha)$$

for all $\mathring{f}\in\Omega^0(M)$

$$= 2\oint_{\partial M} j^*(\mathring{f}\star(df-\alpha)) - 2\int_M \mathring{f}\, d\star(df-\alpha)$$

$$= 2\oint_{\partial M} (j^*\mathring{f})(j^*\star(df-\alpha)) - 2\int_M \mathring{f}\, d\star(df-\alpha)$$

Poisson reconstruction: smooth theory

$$0 = d\mathcal{E}|_f[\![\mathring{f}]\!] = 2\int_M d\mathring{f} \wedge \star(df - \alpha)$$

for all $\mathring{f} \in \Omega^0(M)$

$$= 2\oint_{\partial M} j^*(\mathring{f} \star (df - \alpha)) - 2\int_M \mathring{f}\, d \star (df - \alpha)$$

$$= 2\oint_{\partial M} (j^*\mathring{f})(j^* \star (df - \alpha)) - 2\int_M \mathring{f}\, d \star (df - \alpha)$$

$$\Leftrightarrow \quad \begin{cases} d \star df = d \star \alpha, & \text{in } M \\ j^* \star df = j^* \star \alpha, & \text{on } \partial M \end{cases}$$

natural boundary condition

Poisson reconstruction: smooth theory

$$\begin{cases} d \star d f = d \star \alpha, & \text{in } M \\ j^* \star d f = j^* \star \alpha, & \text{on } \partial M \end{cases}$$

natural boundary condition

$$\alpha = (\mathbf{a})_{1\text{-form}} = \mathbf{a}^\flat$$

$$\begin{cases} \nabla \cdot \nabla f = \nabla \cdot \alpha, & \text{in } M \\ \frac{\partial f}{\partial \mathbf{n}} = \mathbf{a} \cdot \mathbf{n}, & \text{on } \partial M \end{cases}$$

Poisson reconstruction: smooth theory

$$\begin{cases} \nabla \cdot \nabla f = \nabla \cdot \alpha, & \text{in } M \\ \frac{\partial f}{\partial \mathbf{n}} = \mathbf{a} \cdot \mathbf{n}, & \text{on } \partial M \end{cases}$$

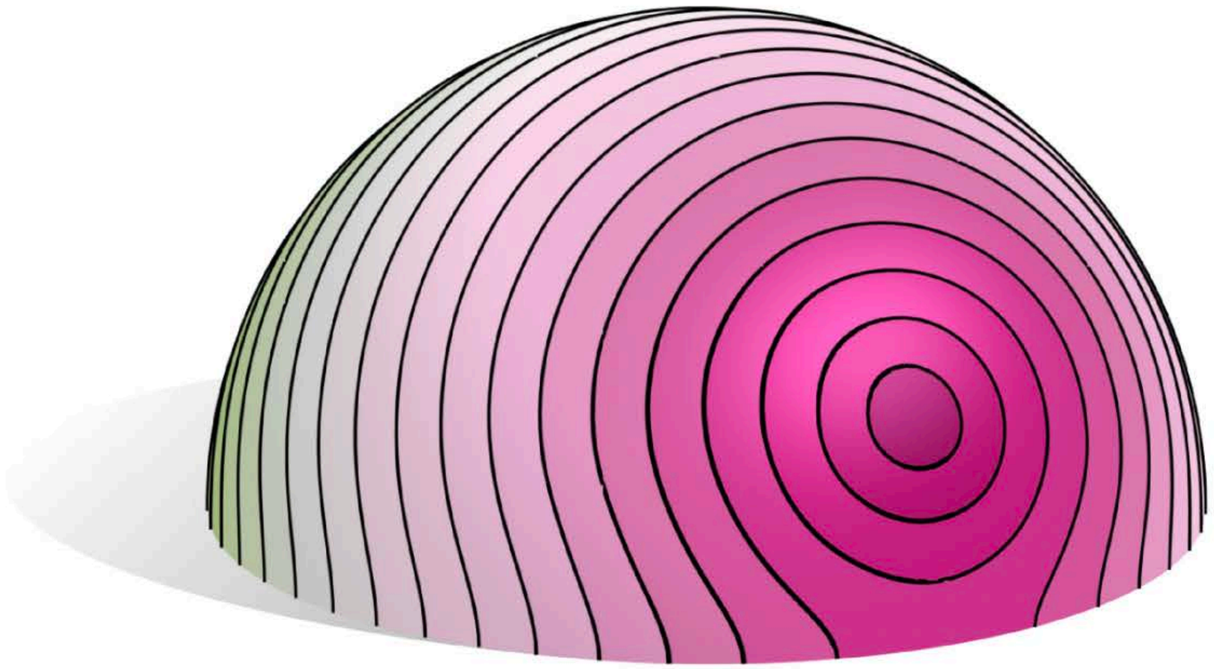
interior PDE

boundary condition

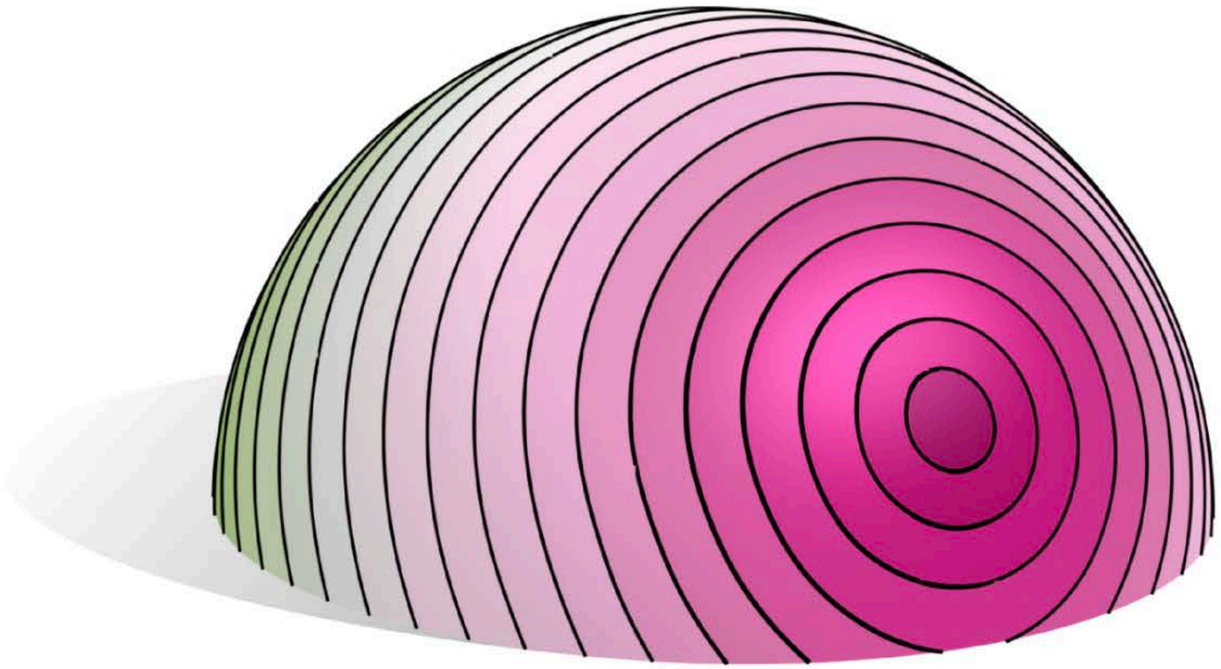
$$\mathbf{d}_0^\intercal \star_1 \mathbf{d}_0 \mathbf{f} = \mathbf{d}_0^\intercal \star_1 \boldsymbol{\alpha}$$

solving this linear system automatically takes care of both the interior PDE and the natural boundary condition

Natural boundary conditions



$\frac{\partial f}{\partial \mathbf{n}} = 0$ as the emerged natural BC



Choose other loss function, giving more suitable natural BC

“**Natural boundary condition for smoothing in geometry processing**” (Stein, Wardetzky, Jacobson 2018)

Boundary condition for Poisson equations

- Poisson reconstruction
- Boundary conditions

No boundary: equation

- If M has no boundary

temperature in equilibrium *heat source*

$$\Delta u = f \quad \text{in } M$$

no need for specifying BC

  - The equation has a solution only when

$$\int_M f \star 1 = 0$$

  - The equation is unique up to adding a constant
  - this is because the only solution to $\Delta v = 0$ is a constant function. (Why?)
  - If sol'n is non-unique, choose the one that has least norm (what condition do you get?)

No boundary: variational form

- If M has no boundary

$$\Delta u = f \quad \text{in } M$$

no need for specifying BC

- It is the optimality condition for the minimization problem:

$$\underset{u \in \Omega^0(M)}{\text{minimize}} \; \frac{1}{2} \int_M |du|^2 \star 1 - \int_M uf \star 1$$

  - or in terms of L^2 inner product $\underset{u \in \Omega^0(M)}{\text{minimize}} \; \frac{1}{2} \|du\|^2 - \langle\langle f, u \rangle\rangle$
  - The term $\frac{1}{2}\|du\|^2$ is called the **Dirichlet energy**.

No boundary: discrete

- If M has no boundary

$$\Delta u = f \quad \text{in } M$$

no need for specifying BC

- Discrete version:

$$\mathbf{d}_0^\intercal \star_1 \mathbf{d}_0 \mathbf{u} = \star_0 \mathbf{f} \iff \underset{\mathbf{u} \in \Omega^0(M)}{\text{minimize}} \tfrac{1}{2} (\mathbf{d}_0 \mathbf{u})^\intercal \star_1 (\mathbf{d}_0 \mathbf{u}) - \mathbf{f}^\intercal \star_0 \mathbf{u}$$

  - It is a self-adjoint equation (system matrix is symmetric)
  - Solutions exist only when $\mathbb{1}^\intercal \star_0 \mathbf{f} = 0 \qquad \mathbb{1} = (1, \ldots, 1)^\intercal$
  - Kernel of $\mathbf{d}_0^\intercal \star_1 \mathbf{d}_0$ is spanned by $\mathbb{1}$ (constant functions).

Neumann boundary condition

$$\begin{cases} \Delta u = f & \text{in } M \\ \frac{\partial u}{\partial \mathbf{n}} = g & \text{on } \partial M \end{cases}$$

Neumann boundary condition

(heat source from boundary)

- Variational form $\underset{u \in \Omega^0(M)}{\text{minimize}} \frac{1}{2}\|du\|^2 - \langle\langle f, u \rangle\rangle - \langle\langle g, u \rangle\rangle_{\partial M}$

- Existence condition $\int_M f \star 1 + \oint_{\partial M} g \star_{\partial M} 1 = 0$

- Solution unique up to an additive constant

- Discrete version $\mathbf{d}_0^\intercal \star_1 \mathbf{d}_0 \mathbf{u} = \star_0 \mathbf{f} + \underbrace{\star_0^{\partial M} \mathbf{g}}_{\text{on boundary points}}$

Neumann boundary condition

$$\begin{cases} \Delta u = f & \text{in } M \\ \frac{\partial u}{\partial \mathbf{n}} = g & \text{on } \partial M \end{cases}$$

Neumann boundary condition

(heat source from boundary)

- Variational form $\underset{u \in \Omega^0(M)}{\text{minimize}} \frac{1}{2}\|du\|^2 - \langle\langle f, u \rangle\rangle - \langle\langle g, u \rangle\rangle_{\partial M}$
- Existence condition $\int_M f \star 1 + \oint_{\partial M} g \star_{\partial M} 1 = 0$
- Solution unique up to an additive constant
- Discrete version $\mathbf{d}_0^\mathsf{T} \star_1 \mathbf{d}_0 \mathbf{u} = \star_0 \mathbf{f} + \begin{bmatrix} \mathbf{0} \\ \star_0^{\partial M} \mathbf{g} \end{bmatrix}$ interior points / boundary points

Robin boundary condition

$$\begin{cases} \Delta u = f & \text{in } M \\ \frac{\partial u}{\partial \mathbf{n}} + au = g & \text{on } \partial M \end{cases} \qquad a : \partial M \to \mathbb{R}, \quad a > 0$$

heat bath

- Variational form

$$\underset{u \in \Omega^0(M)}{\text{minimize}} \ \frac{1}{2}\|du\|^2 + \int_{\partial M} au^2 \star_{\partial M} 1 - \langle\!\langle f, u \rangle\!\rangle - \langle\!\langle g, u \rangle\!\rangle_{\partial M}$$

- Discrete version $\left(\mathbf{d}_0^{\mathsf{T}} \star_1 \mathbf{d}_0 + \begin{bmatrix} \mathbf{0} & \\ & \star_0^{\partial M} \operatorname{diag}(\mathbf{a}) \end{bmatrix}\right)\mathbf{u} = \star_0 \mathbf{f} + \begin{bmatrix} \mathbf{0} \\ \star_0^{\partial M} \mathbf{g} \end{bmatrix}$

- Solution always exist and is unique.

Dirichlet boundary condition

$$\begin{cases} \Delta u = f & \text{in } M \\ u = g & \text{on } \partial M \end{cases}$$

Dirichlet boundary condition

- Write $\mathbf{d}_0^\intercal \star_1 \mathbf{d}_0 = \mathbf{L} = \begin{bmatrix} \mathbf{L}_{II} & \mathbf{L}_{IB} \\ \mathbf{L}_{BI} & \mathbf{L}_{BB} \end{bmatrix}$
- Equation $$\begin{bmatrix} \mathbf{L}_{II} & \mathbf{L}_{IB} \\ \mathbf{L}_{BI} & \mathbf{L}_{BB} \end{bmatrix} \begin{bmatrix} \mathbf{u}_I \\ \mathbf{u}_B = \mathbf{g} \end{bmatrix} = \begin{bmatrix} (\star_0 \mathbf{f})_I \\ \sim \end{bmatrix}$$

the source term is not satisfied at the boundary, traded out from enforcing values of u.

- Simplifies to $\mathbf{L}_{II} \mathbf{u}_I = (\star_0 \mathbf{f})_I - \mathbf{L}_{IB} \mathbf{g}$
- Solution always exist and is unique.

One-point Dirichlet boundary

- For Poisson systems with non-unique solution by a constant, like

$$\Delta u = f \quad \text{in } M$$

no boundary

$$\begin{cases} \Delta u = f & \text{in } M \\ \frac{\partial u}{\partial \mathbf{n}} = g & \text{on } \partial M \end{cases}$$

- Pick an arbitrary point and set a Dirichlet boundary $\mathbf{u} = 0$ at that point
- The effective equation $\mathbf{L}_{II}\mathbf{u}_I = (\star_0 \mathbf{f})_I$ is non-singular
- One may shift the solution $\begin{bmatrix} \mathbf{u}_I \\ 0 \end{bmatrix}$ to have zero-mean afterward.