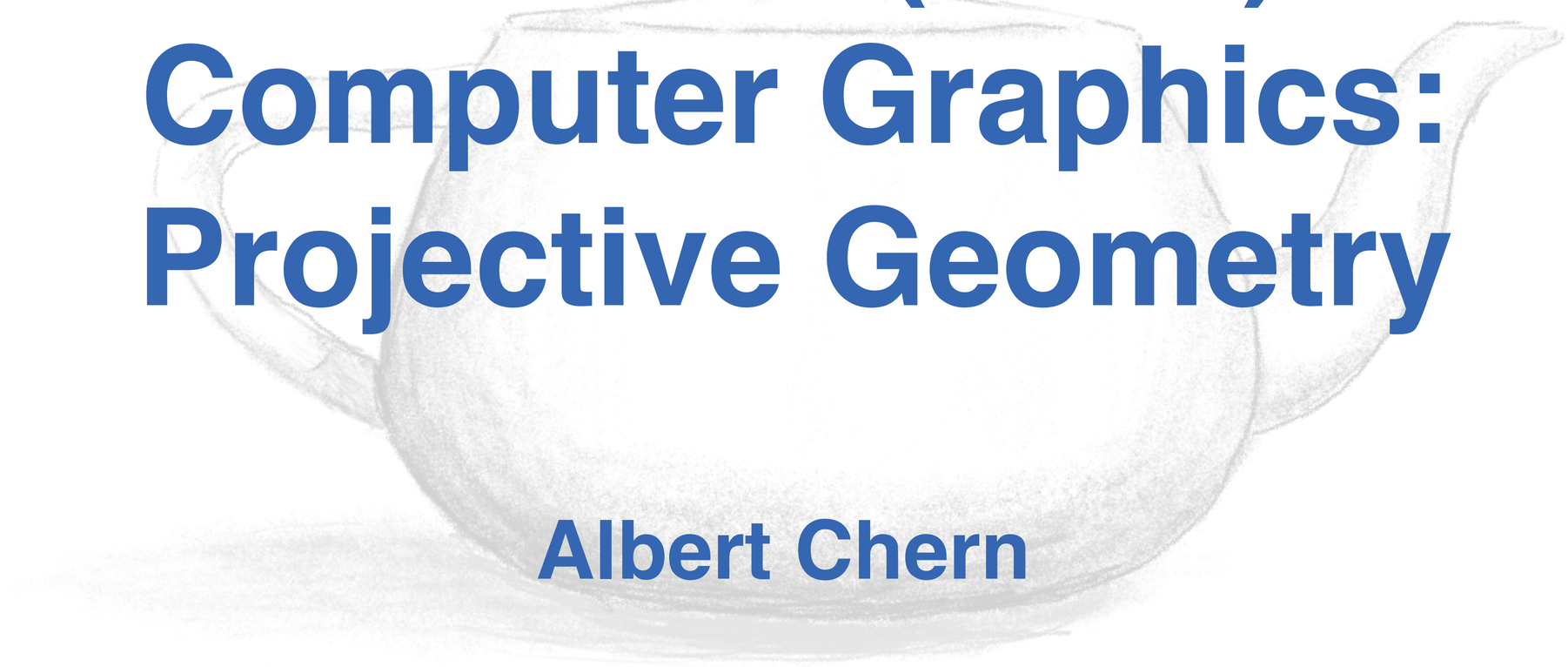
CSE 167 (WI26)
Computer Graphics:
Projective Geometry



Albert Chern

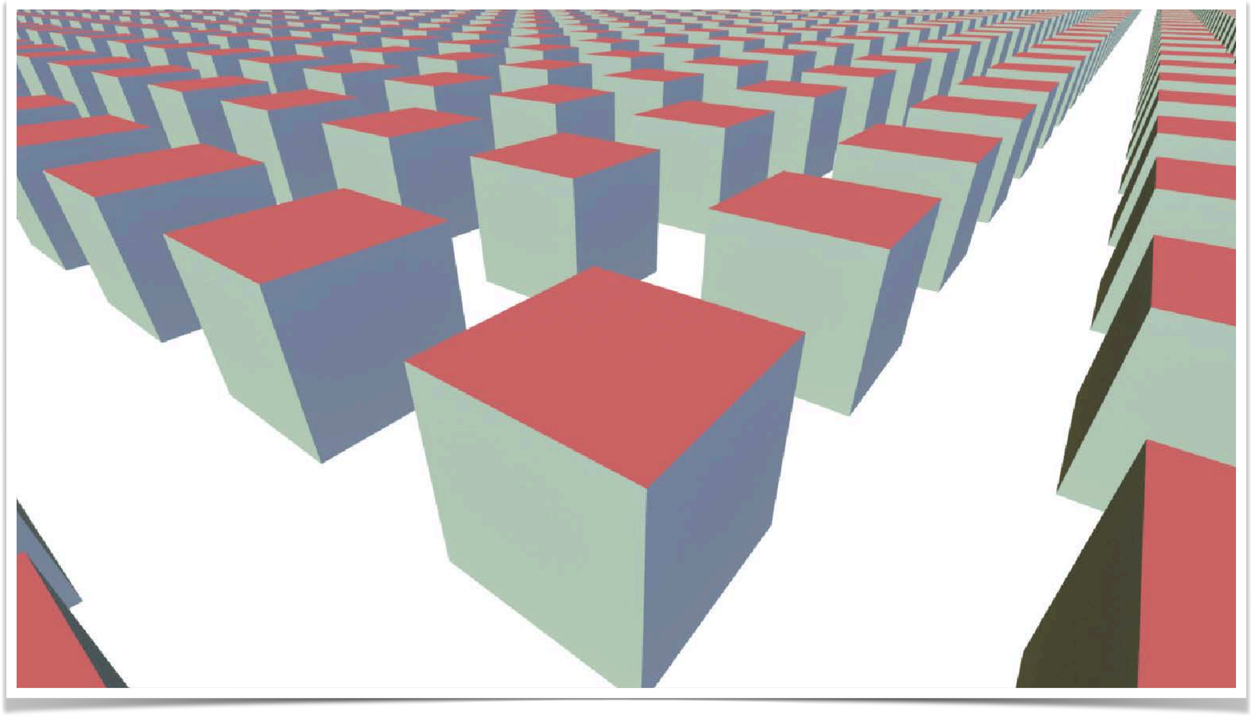
UC San Diego
JACOBS SCHOOL OF ENGINEERING

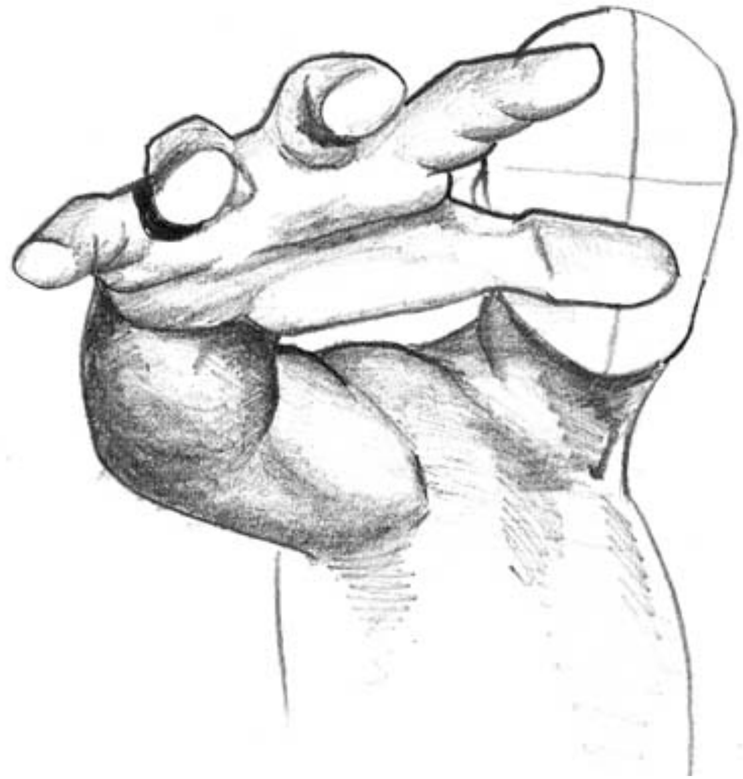
What are Projective Transformations?

- What are projective transforms?
- How homogeneous coordinates work
- Derivation of the projection matrix
- Art of projective geometry

Perspective

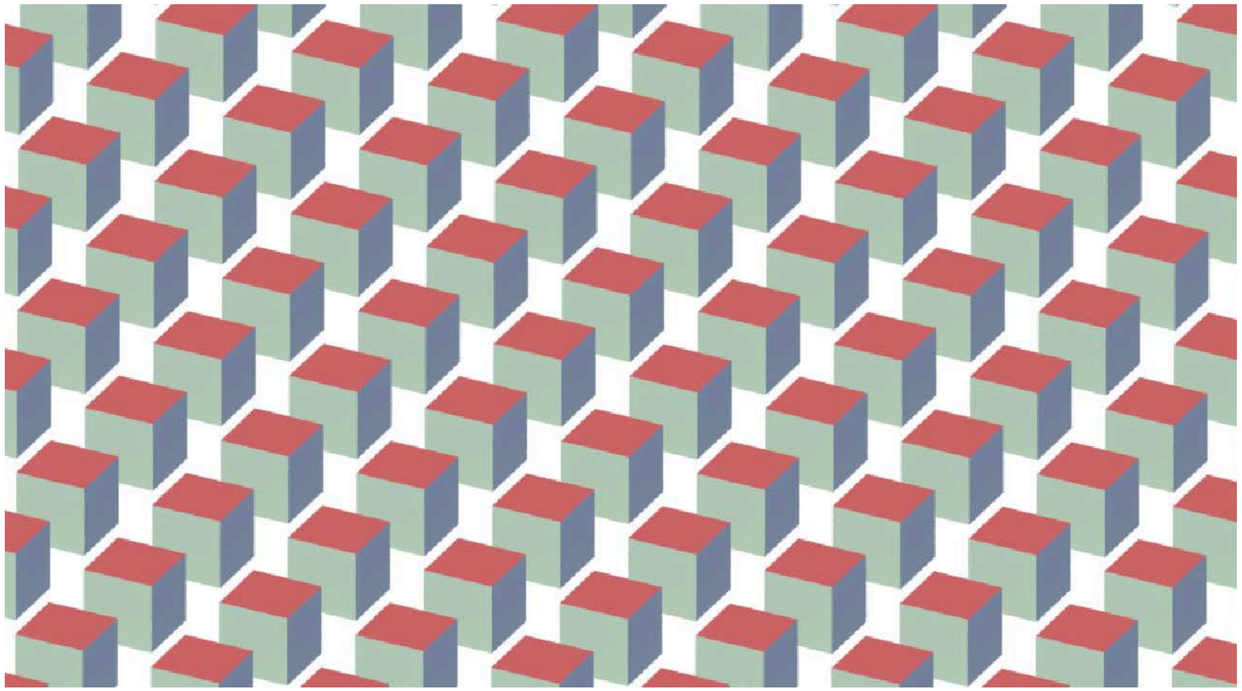
- Foreshortening
  - Further objects look smaller
  - The scaling is non-uniform

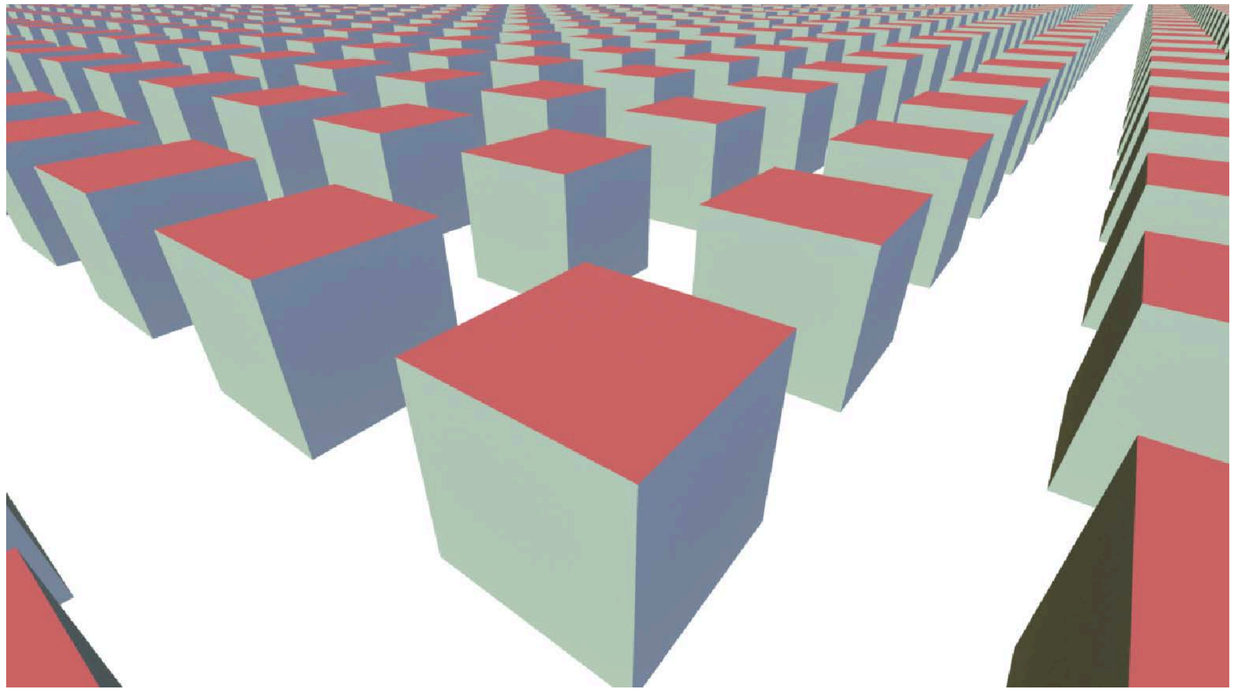




Perspective

- Orthographic view
  - Telescopic view standing infinitely far away.
  - A view preserving parallelism, proportions
  - Lack a sense of depth
- Perspective view
  - The vantage point is immersed in the scene.
  - Does not preserve parallelism, proportions
  - Provides a sense of depth

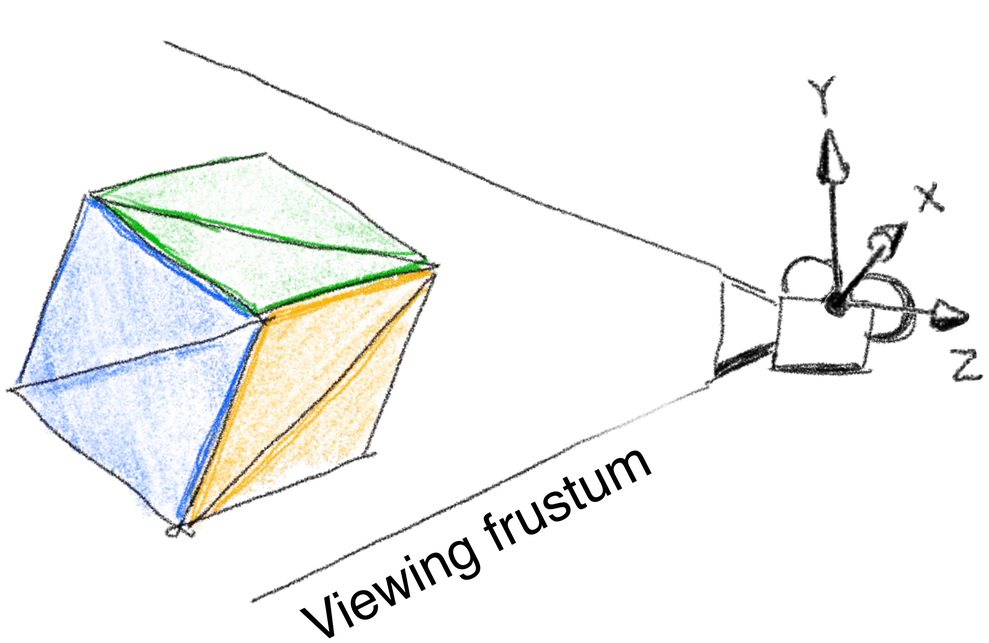


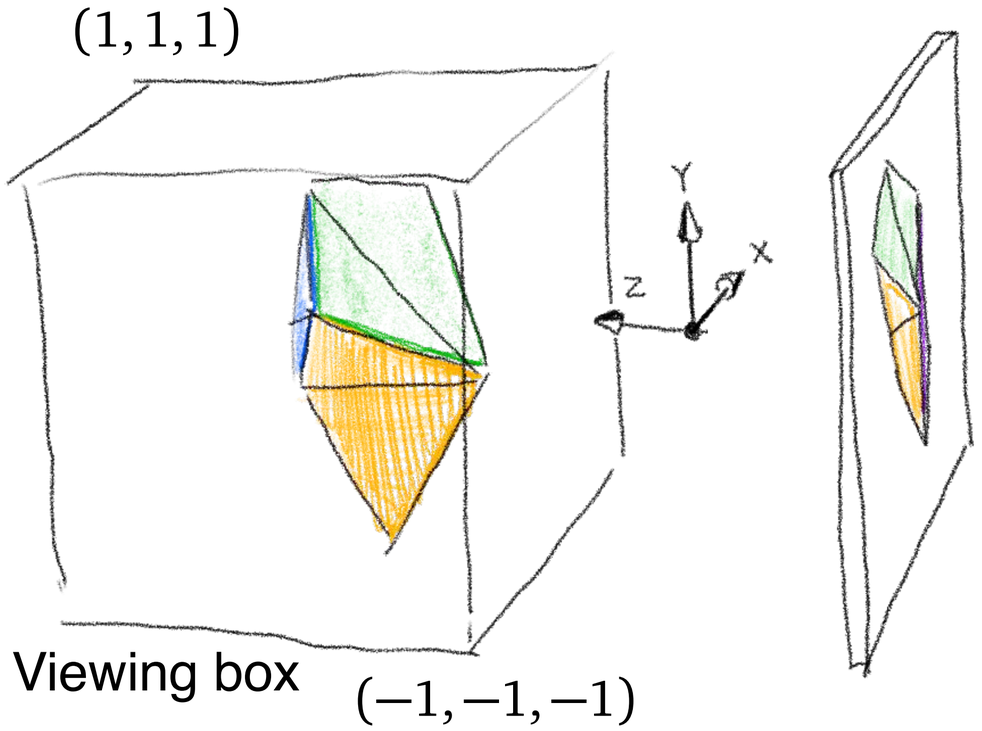


Goal

Map points from the **camera coordinate** to the **Normalized Device Coordinate** (NDC) so that

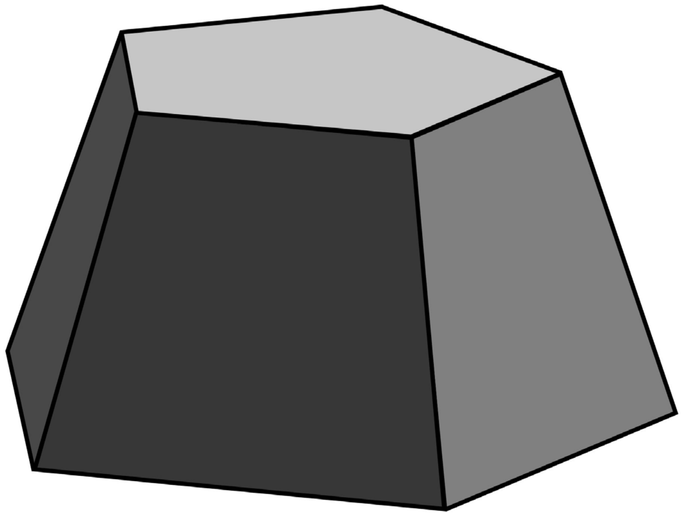
- The **viewing frustum** is mapped to the viewing box in NDC;
- The map is preserves collinearity (**collineation**.)

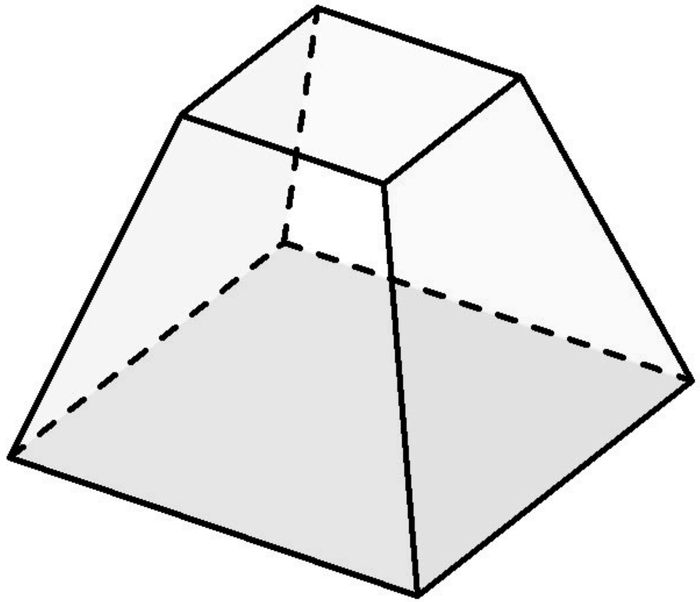




Frustum

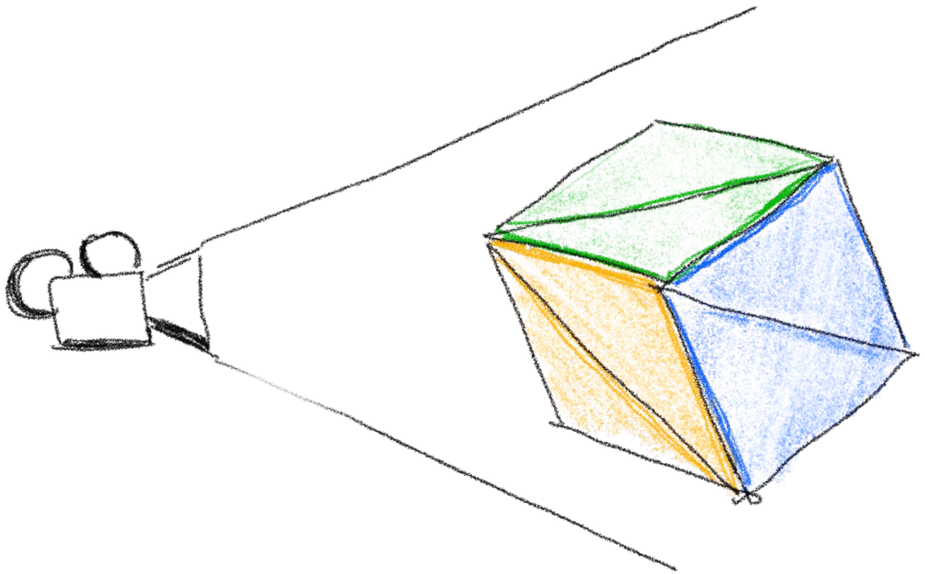
In geometry, a *frustum* is the portion of a pyramid (or cone) that lies between two parallel planes cutting it.

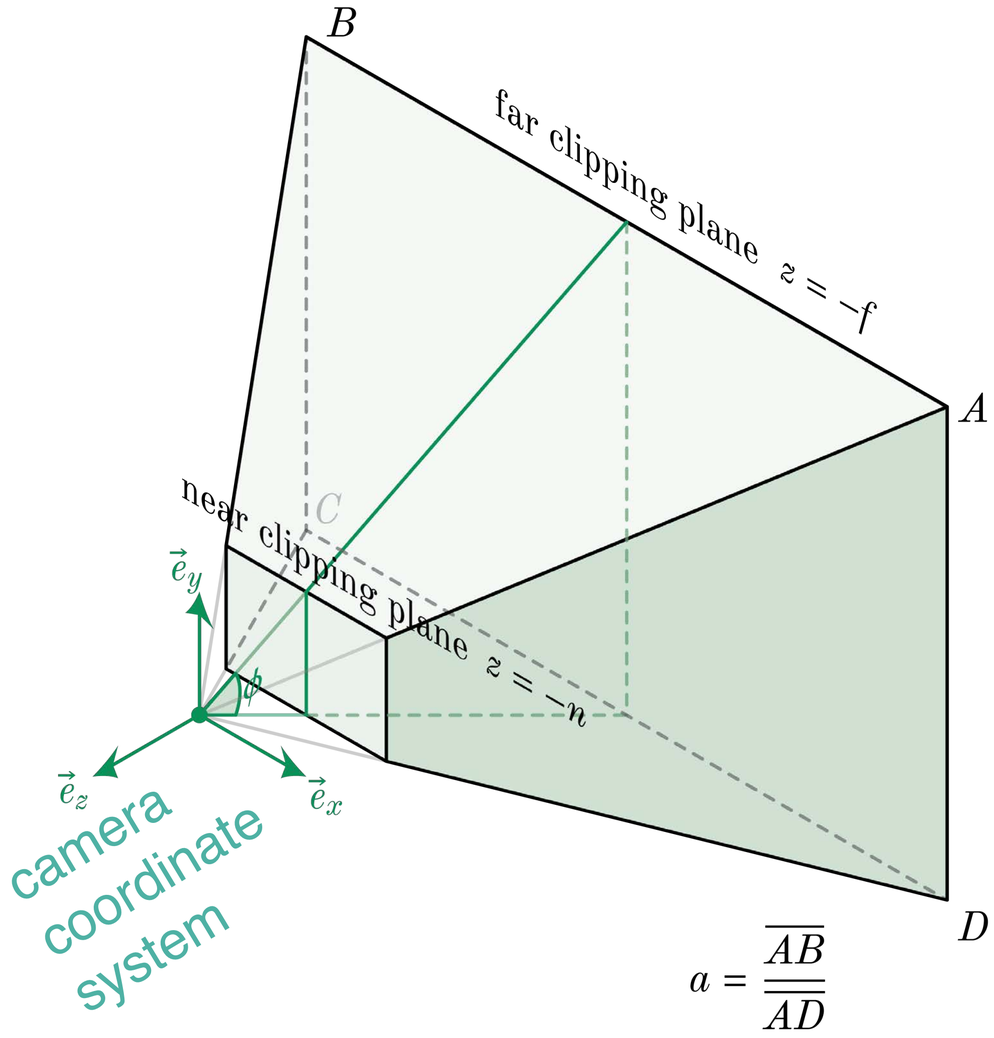




Viewing frustum

In graphics, a *viewing frustum* is the 3 dimensional region that is visible on the screen.

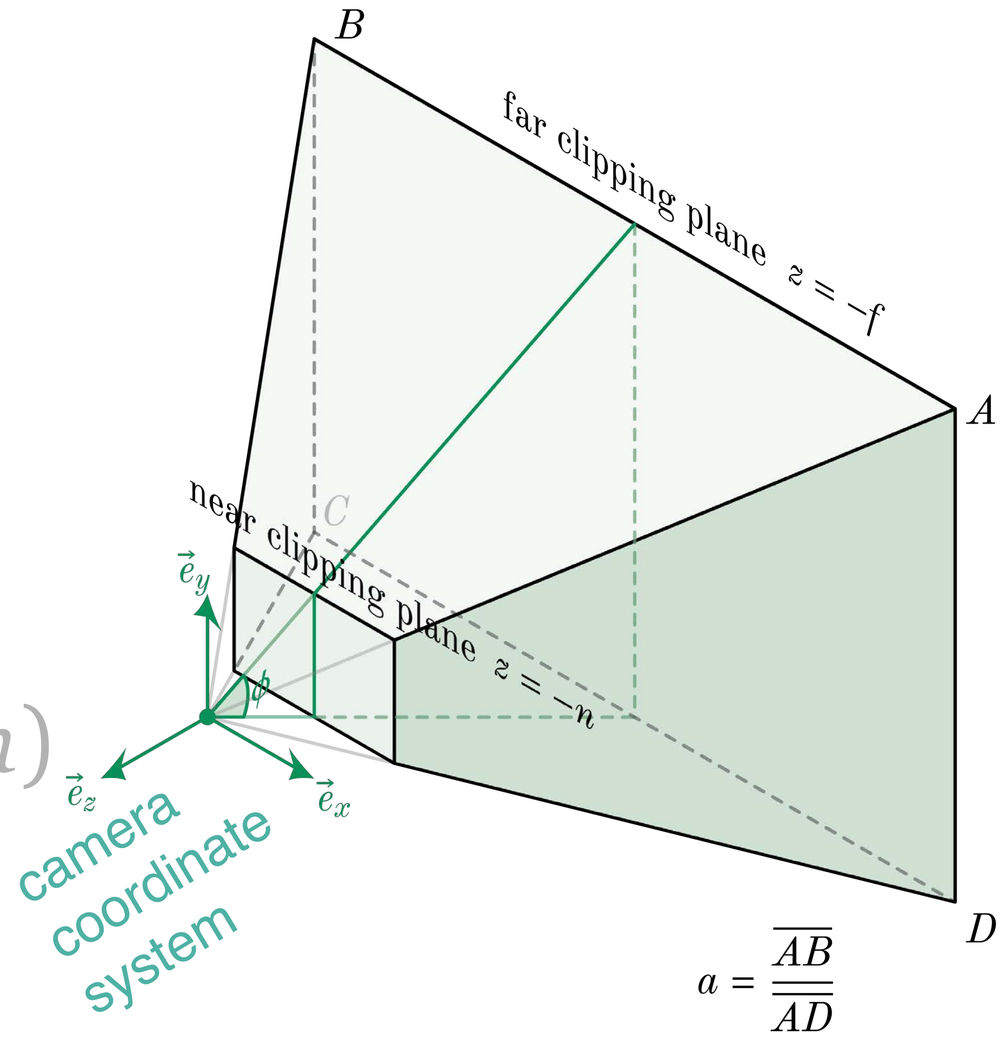




Viewing frustum

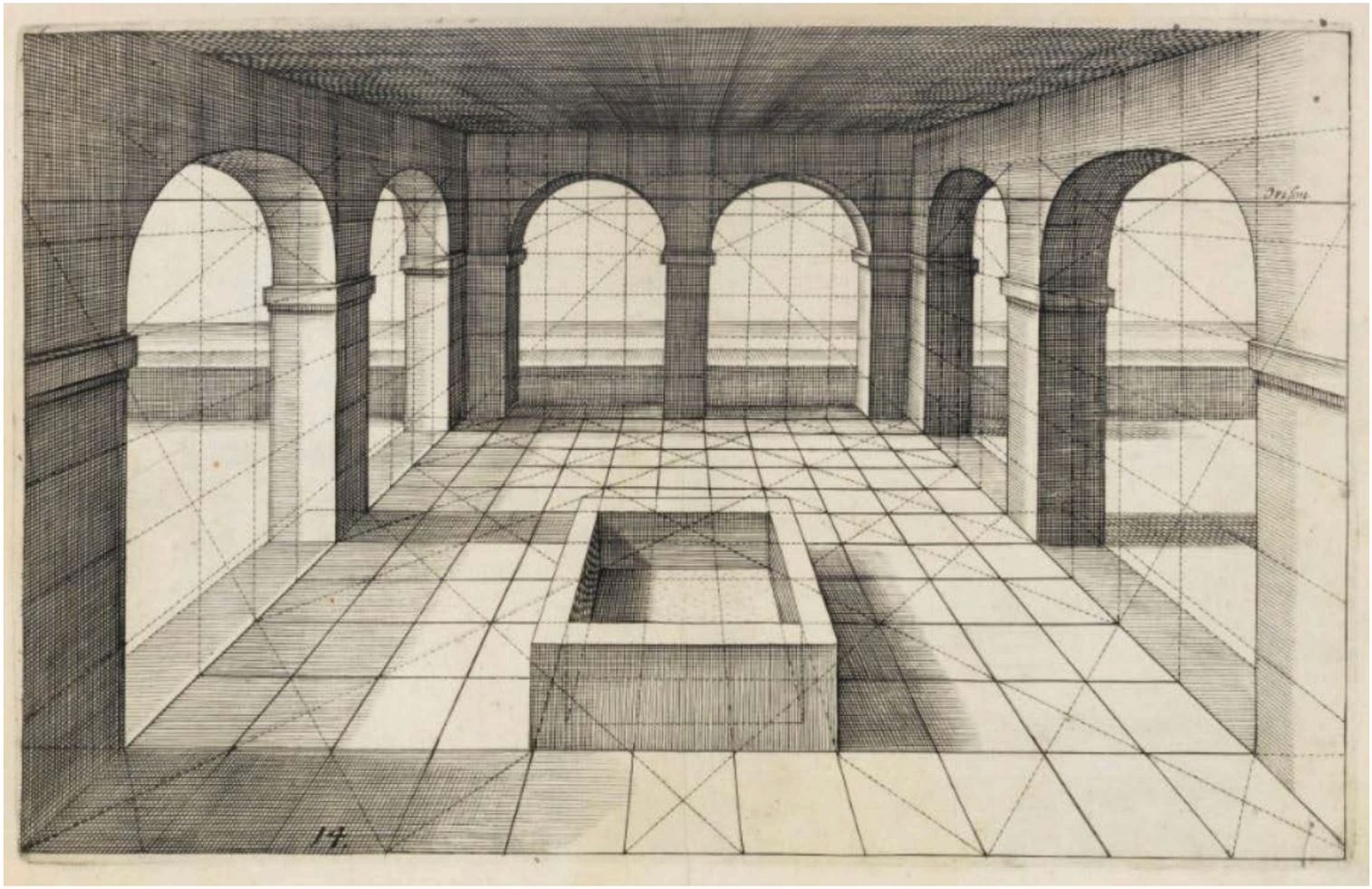
In graphics, a *viewing frustum* is the 3 dimensional region that is visible on the screen.

- Parameters for a viewing frustum:
  - **field of view:** angle ϕ
  - **aspect ratio:** $a = \frac{\text{width}}{\text{height}}$
  - **near clipping distance** $n > 0$
  - **far clipping distance** f $(f > n)$



Collineation

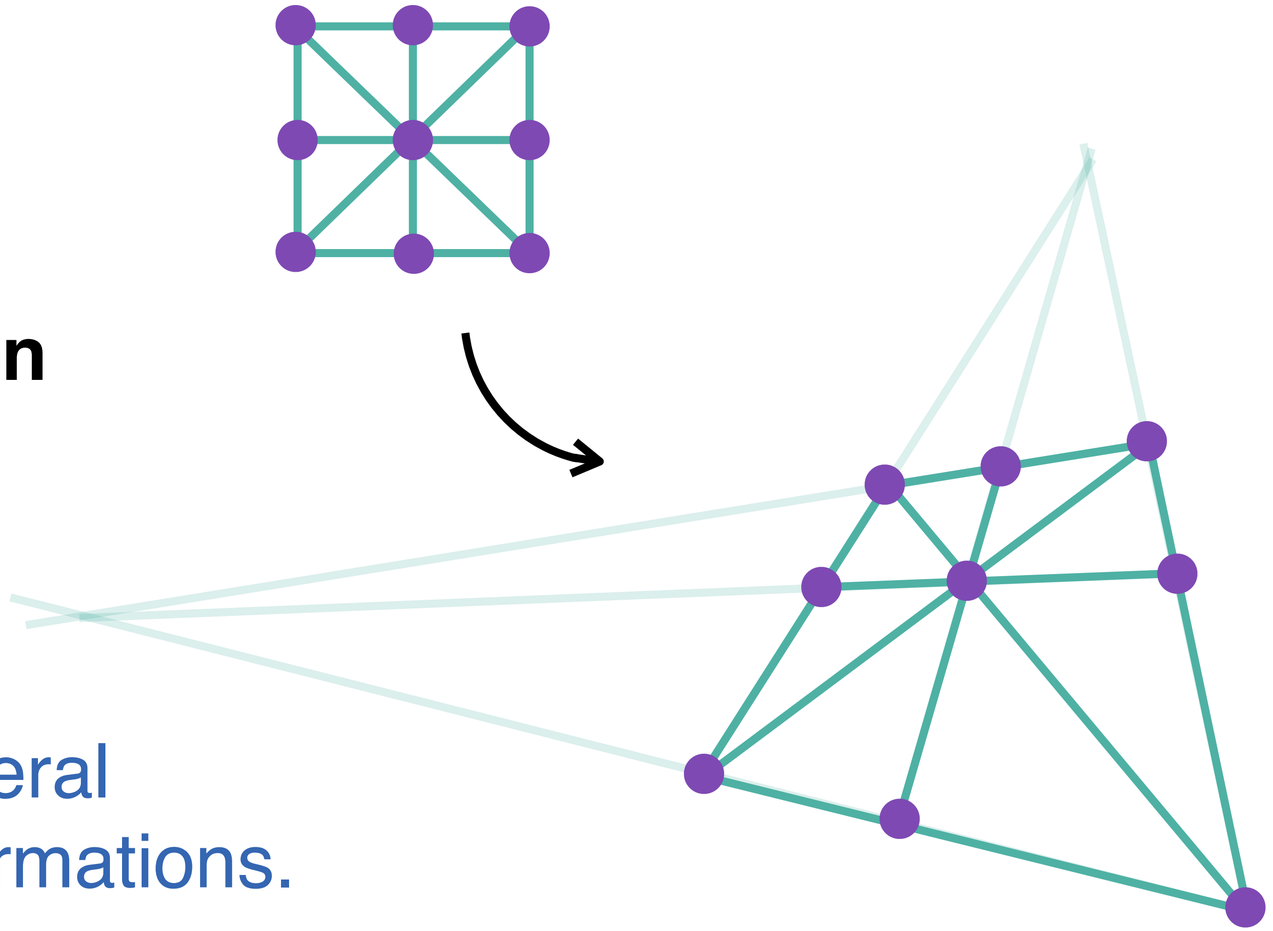
A straight line in the physical space should still be a straight line in the viewing box or on the screen.



Collineation

A transformation is called a collineation if straight lines are mapped to straight lines.

- Synonyms
  - **Collineation**
  - **Projective transformation**
  - **Homography**



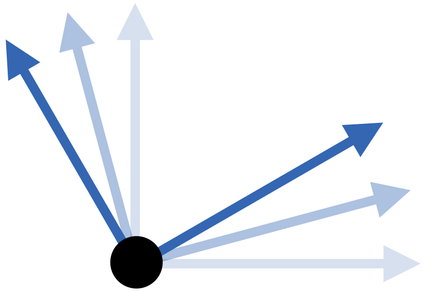
- Parallelism is generally not preserved! This is more general than linear and affine transformations.

Linear, affine and projective

Multiplying 3D vectors by a 3x3 matrix

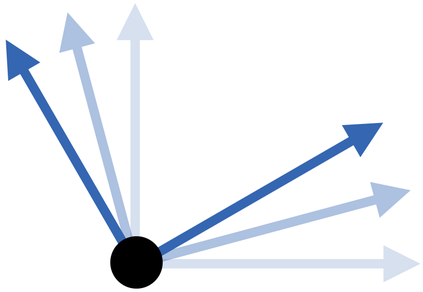
$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

Linear transformation



Linear, affine and projective

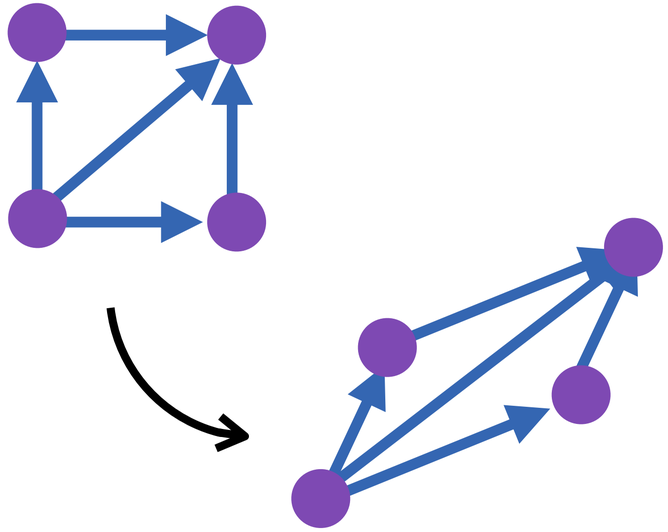
$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$



Multiplying 3D positions (in 4D hom. coord.) by a 4x4 matrix with 0,0,0,1 as the last row.

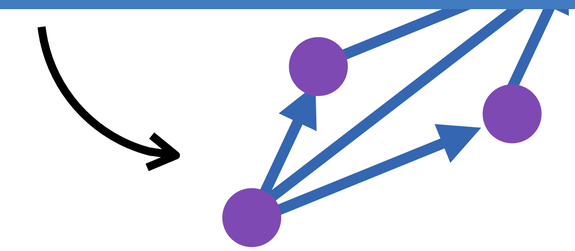
$$\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_x \\ a_{21} & a_{22} & a_{23} & b_y \\ a_{31} & a_{32} & a_{33} & b_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

Affine transformation



Linear, affine and projective

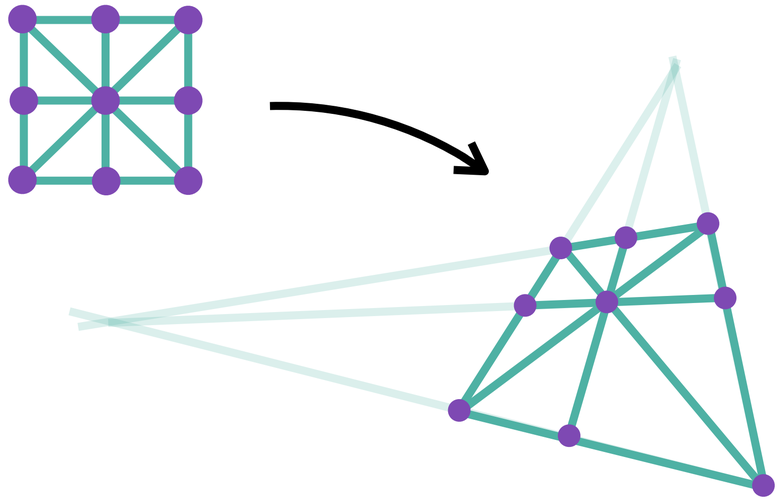
$$\begin{bmatrix} p_z \\ 1 \end{bmatrix} \quad \begin{bmatrix} a_{31} & a_{32} & a_{33} & b_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_z \\ 1 \end{bmatrix}$$



Multiplying 3D positions (in 4D hom. coord.) by a general 4x4 matrix.

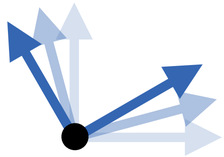
$$\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

Projective transformation (collineation)



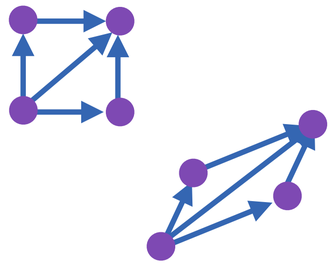
Linear, affine and projective

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$



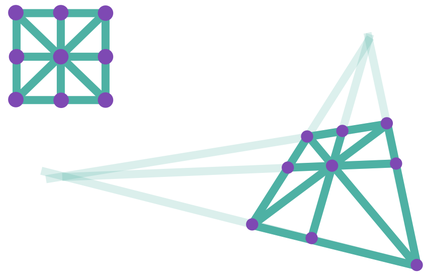
Linear transformations
are affine transformations
that preserve the origin.

$$\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_x \\ a_{21} & a_{22} & a_{23} & b_y \\ a_{31} & a_{32} & a_{33} & b_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$



Affine transformations
are projective transformations
that preserve parallelism.

$$\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

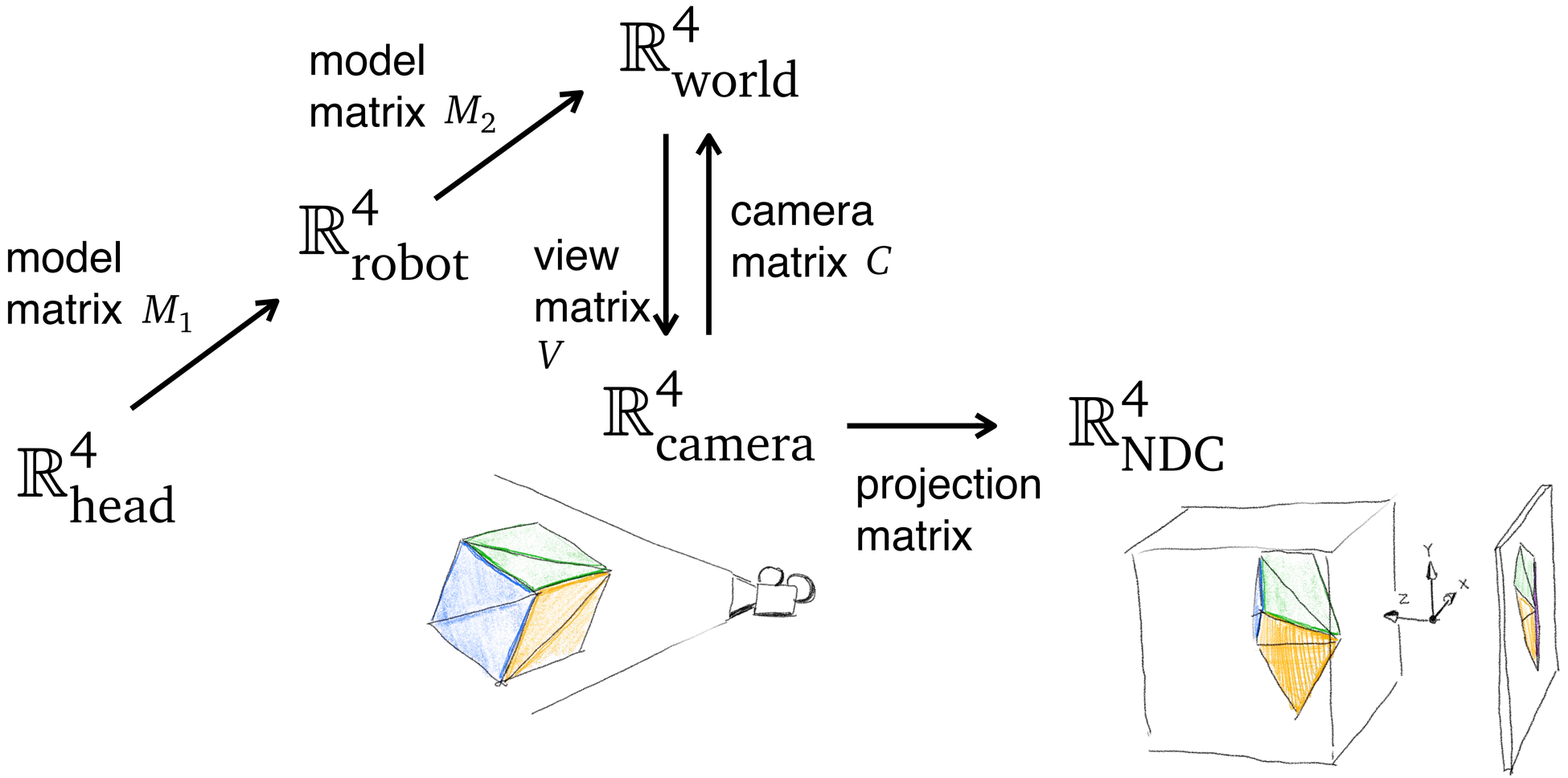


Projective transformations
preserve straight lines.

Final projection after modelview

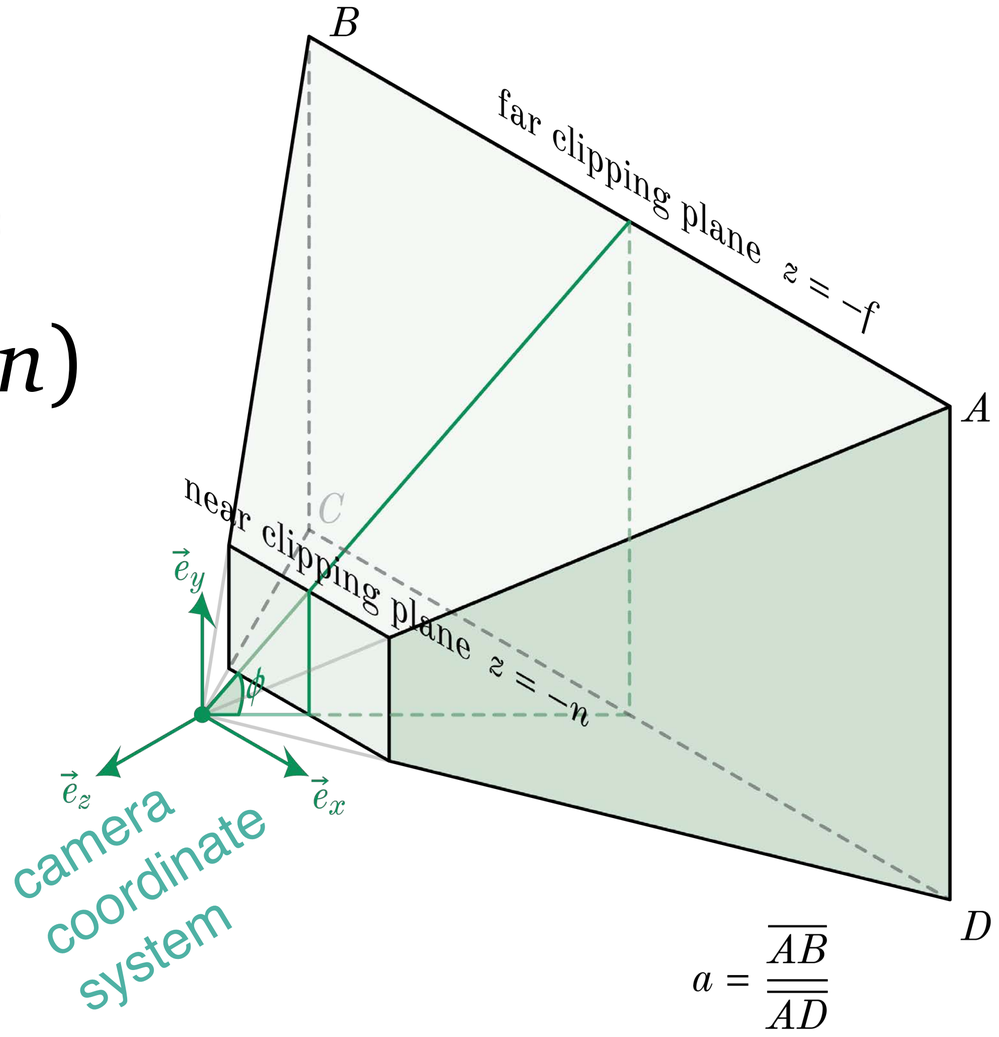
In the vertex shader:

```
gl_Position = projection * modelview * vec4( vertex_position, 1.0f );
```

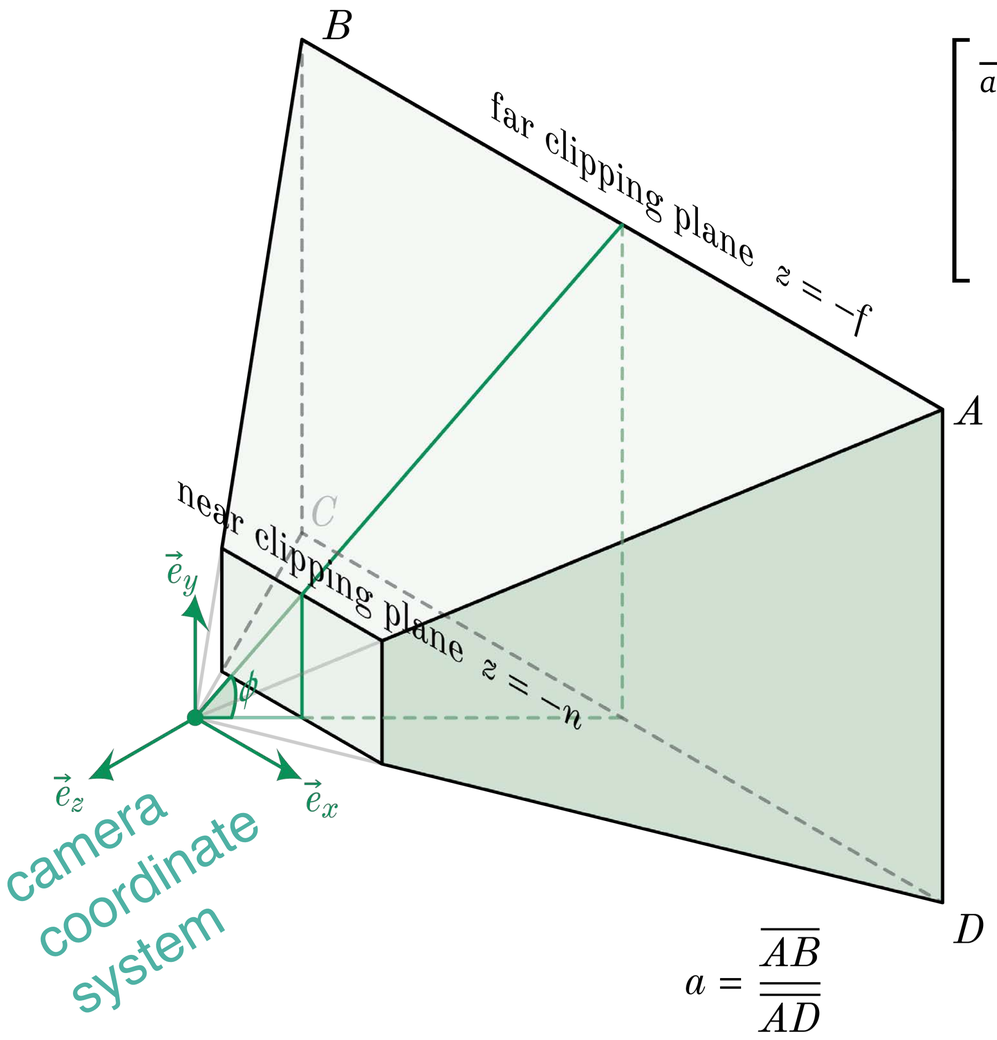


Projection matrix

- **field of view:** angle ϕ
- **aspect ratio:** $a = \frac{\text{width}}{\text{height}}$
- **near clipping distance** $n > 0$
- **far clipping distance** f $(f > n)$



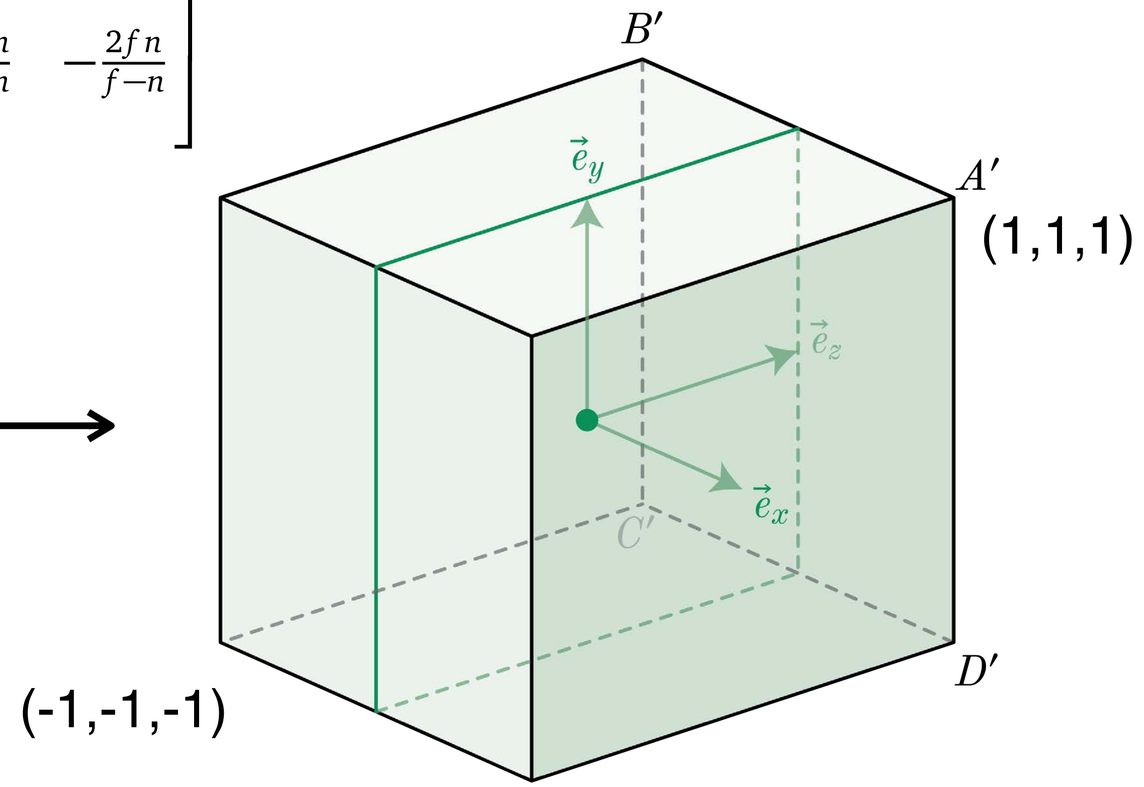
Projection matrix



camera coordinate system

$$a = \frac{\overline{AB}}{\overline{AD}}$$

$$\begin{bmatrix} \frac{1}{a\tan(\phi/2)} & & & \\ & \frac{1}{\tan(\phi/2)} & & \\ & & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ & & -1 & \end{bmatrix}$$



normalized device coordinate

How Homogeneous Coordinates actually work

- What are projective transforms?
- How homogeneous coordinates work
- Derivation of the projection matrix
- Art of projective geometry

How does it work?

Homogeneous coordinates

Homogeneous coordinates

In the homogeneous coordinate, we view each 4D vector as a 4-proportion.

We write $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \sim \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \\ \lambda w \end{bmatrix}$ to indicate that they have the *same proportion*.

Homogeneous coordinates

$$\begin{bmatrix} 2 \\ 6 \\ -4 \\ 5 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 3 \\ -2 \\ 2.5 \end{bmatrix} \sim \begin{bmatrix} -100 \\ -300 \\ 200 \\ -250 \end{bmatrix}$$

$$2 : 6 : -4 : 5$$

$$= 1 : 3 : -2 : 2.5$$

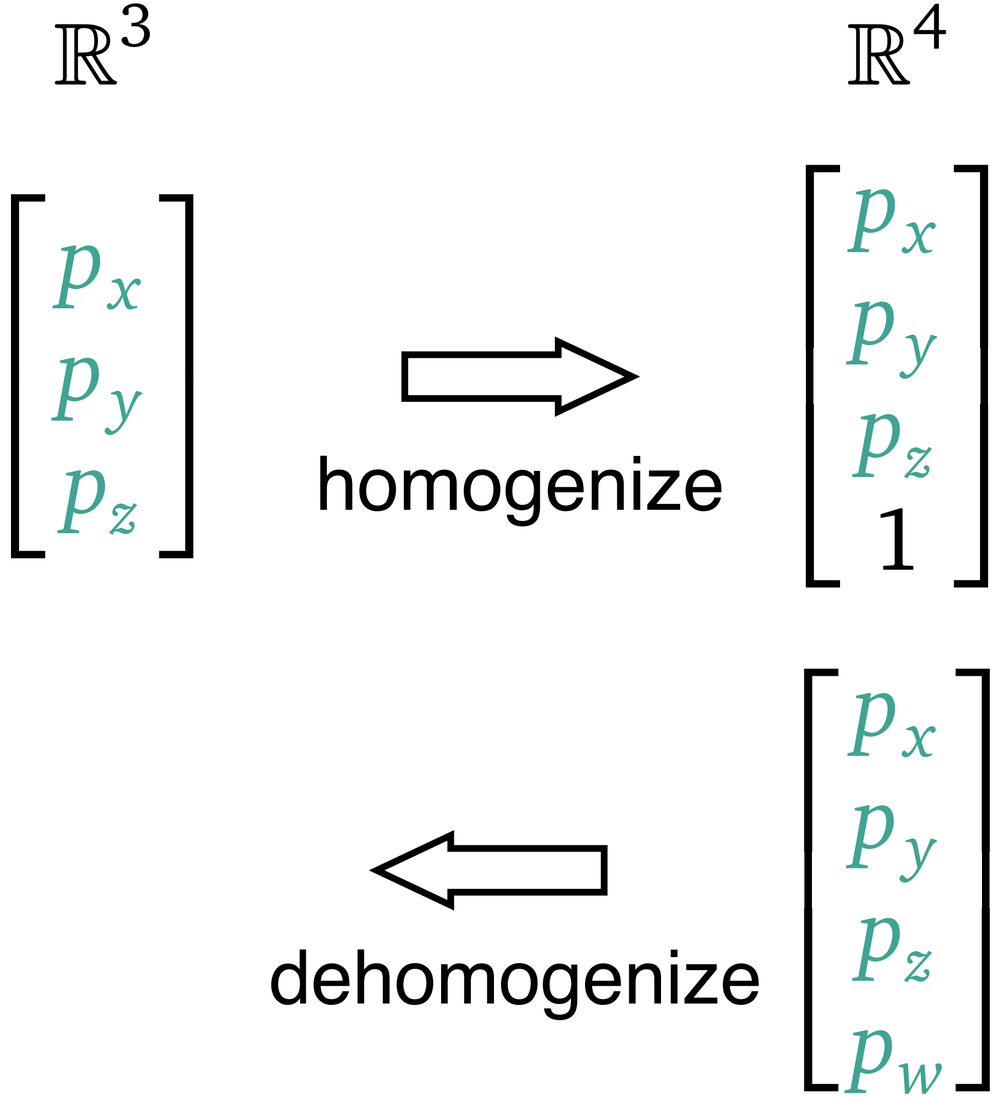
$$= -100 : -300 : 200 : -250$$

Between 3D position & 4D homo. coord.

$\mathbb{R}^3$ $\mathbb{R}^4$

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \Rightarrow \text{homogenize} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

Between 3D position & 4D homo. coord.



Between 3D position & 4D homo. coord.

$\mathbb{R}^3$ $\mathbb{R}^4$

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \Longrightarrow \text{homogenize} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} p_x/p_w \\ p_y/p_w \\ p_z/p_w \end{bmatrix} \Longleftarrow \text{dehomogenize} \begin{bmatrix} p_x/p_w \\ p_y/p_w \\ p_z/p_w \\ 1 \end{bmatrix} \sim \begin{bmatrix} p_x \\ p_y \\ p_z \\ p_w \end{bmatrix}$$

Between 3D position & 4D homo. coord.

$\mathbb{R}^3$ $\mathbb{R}^4$

$$\begin{bmatrix} p_x/p_w \\ p_y/p_w \\ p_z/p_w \end{bmatrix} \Leftarrow \text{dehomogenize} \quad \begin{bmatrix} p_x/p_w \\ p_y/p_w \\ p_z/p_w \\ 1 \end{bmatrix} \sim \begin{bmatrix} p_x \\ p_y \\ p_z \\ p_w \end{bmatrix}$$

(It is a good idea to keep positions represented in 4D, so that we can talk about points at infinity at ease)

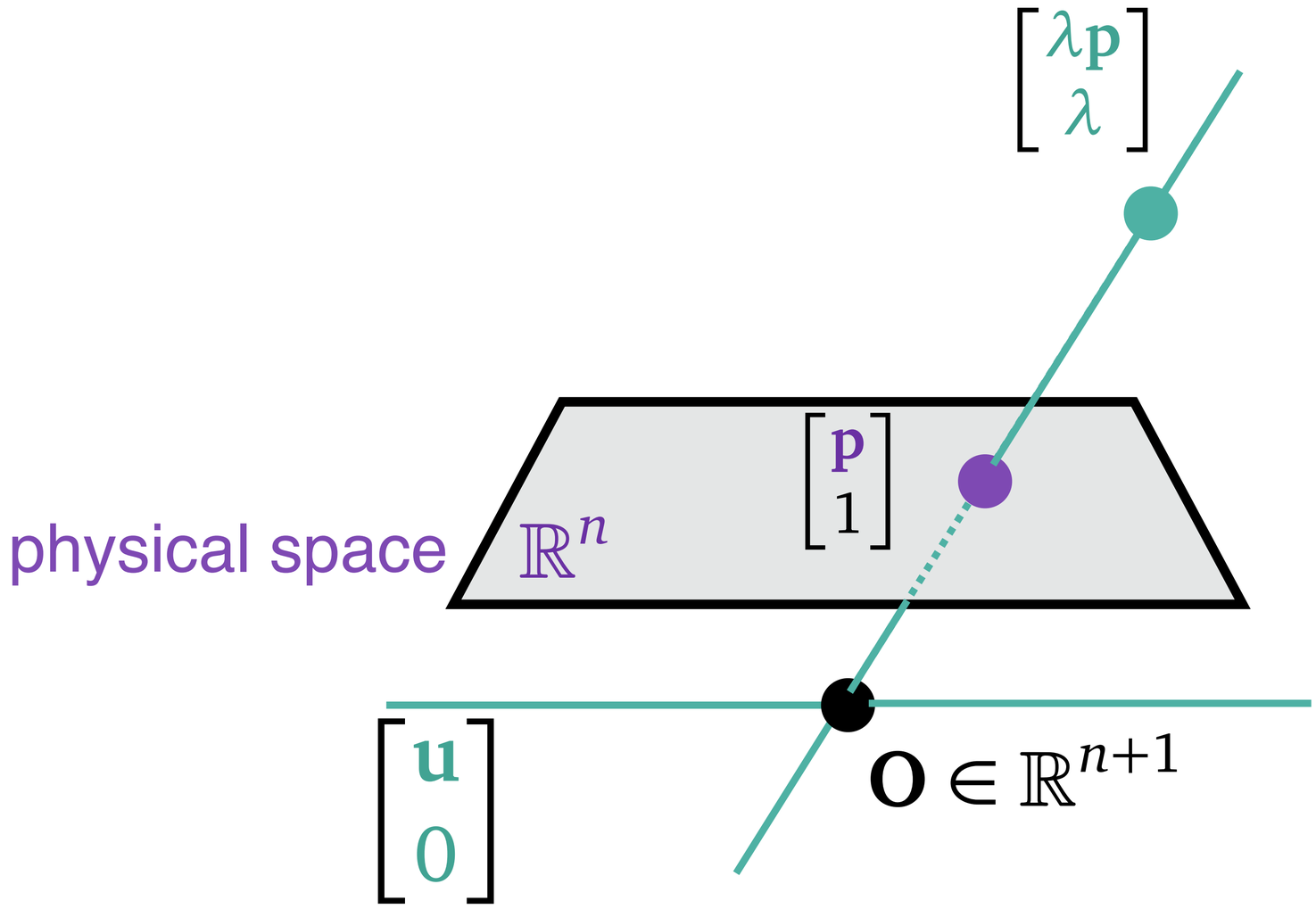
$$\begin{bmatrix} p_x \\ p_y \\ p_z \\ 0 \end{bmatrix}$$

represents “points at infinity,” (or a pure direction like a vector)

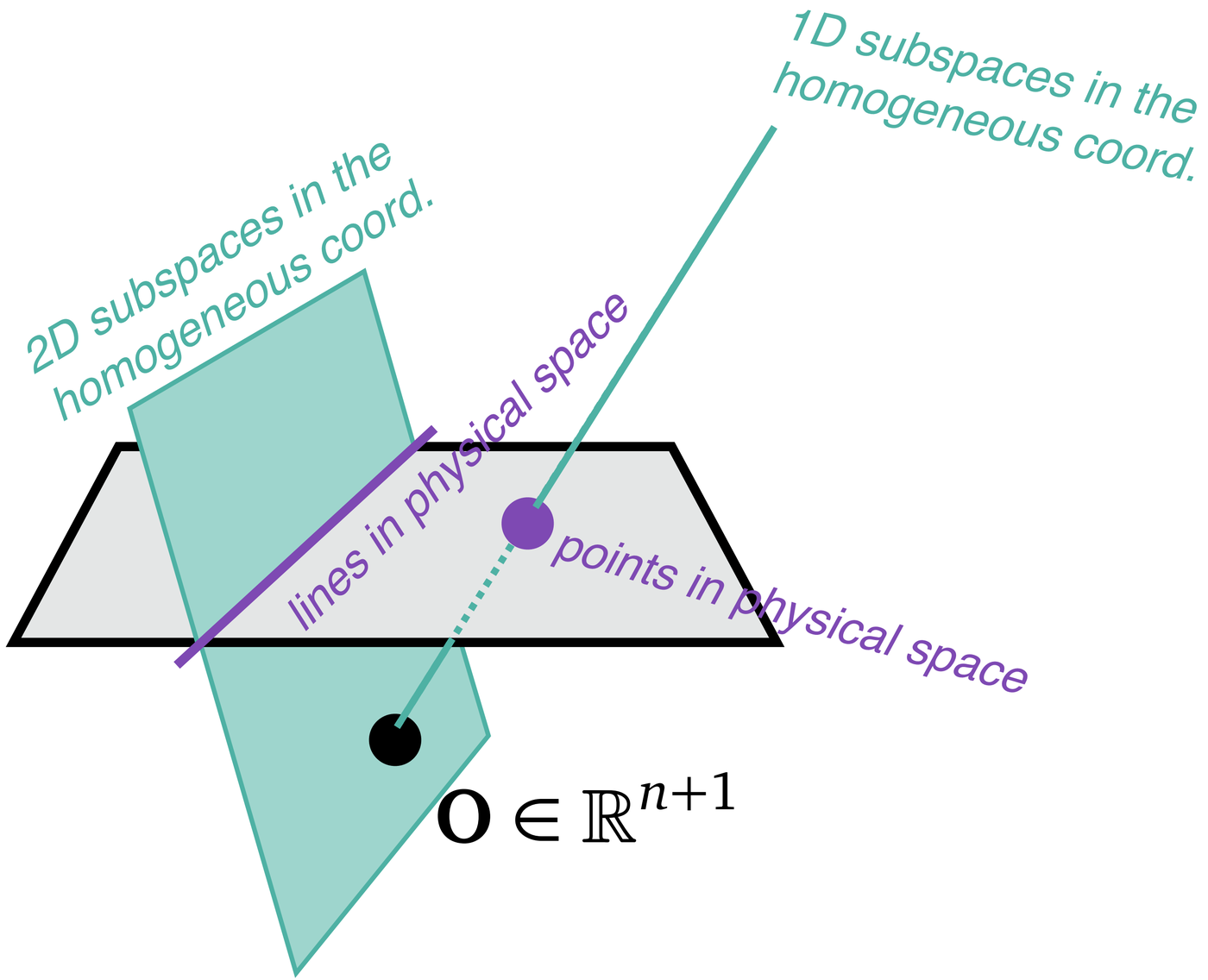
Between 3D position & 4D homo. coord.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \xrightarrow{\text{dehomogenize}} \begin{bmatrix} 0.25 \\ 0.5 \\ 0.75 \end{bmatrix}$$

Between 3D position & 4D homo. coord.



Between 3D position & 4D homo. coord.



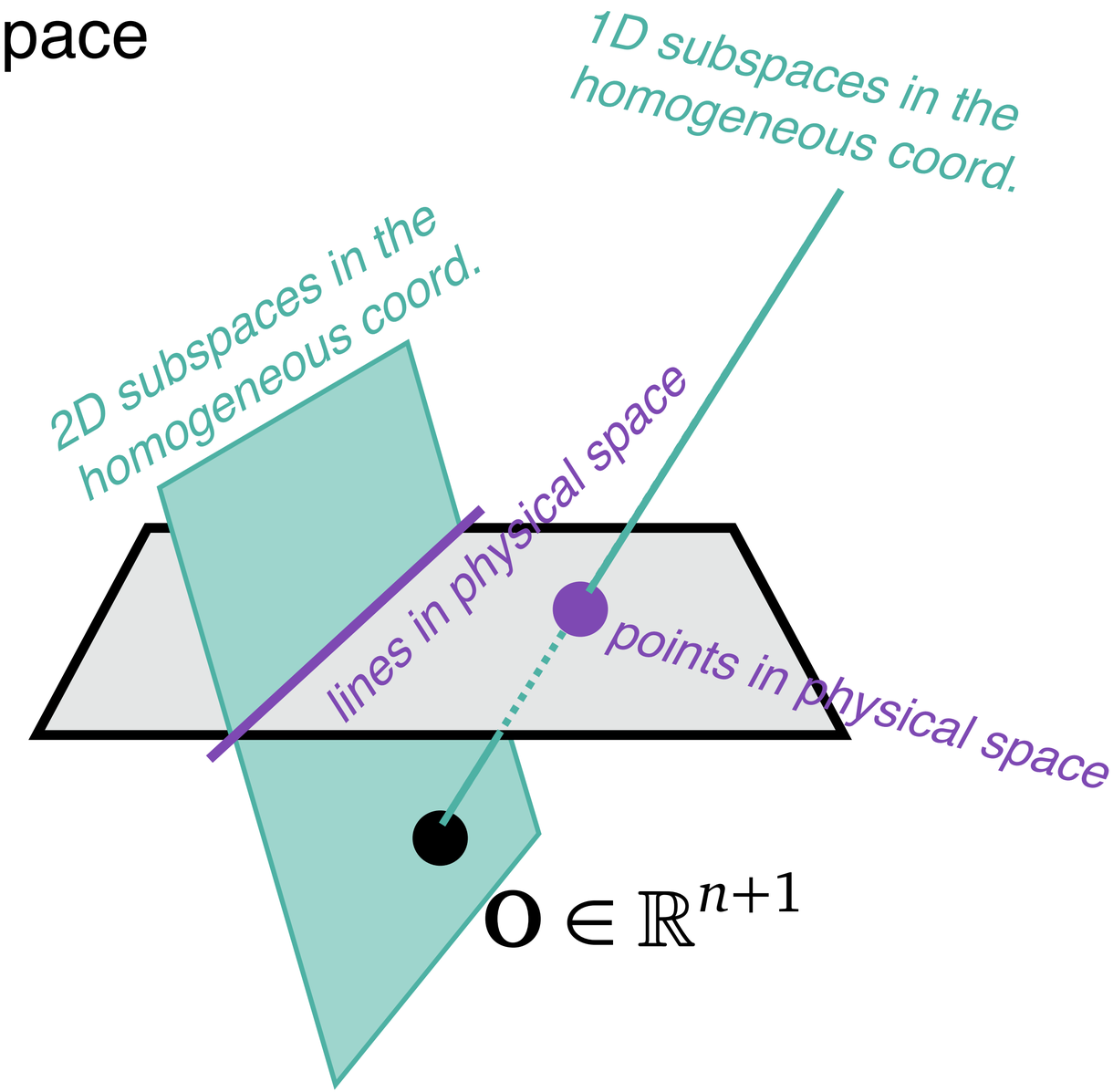
Between 3D position & 4D homo. coord.

Every collineation on the physical space

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \mapsto \begin{bmatrix} p_x^{\text{new}} \\ p_y^{\text{new}} \\ p_z^{\text{new}} \end{bmatrix}$$

can be written as the result of

- Step1: homogenize
- Step2: 4D linear transformation
- Step3: dehomogenize



An example projective transformation

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{bmatrix}$$ will perform the following transformation:

Each point $\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$ is mapped to $\begin{bmatrix} p_x^{\text{new}} \\ p_y^{\text{new}} \\ p_z^{\text{new}} \end{bmatrix} = ?$

An example projective transformation

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ 1 \\ p_z \end{bmatrix} \sim \begin{bmatrix} p_x/p_z \\ p_y/p_z \\ 1/p_z \\ 1 \end{bmatrix}$$

dehomogenize

$$\Rightarrow \quad \begin{bmatrix} p_x/p_z \\ p_y/p_z \\ 1/p_z \end{bmatrix}$$

An example projective transformation

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{bmatrix}$$

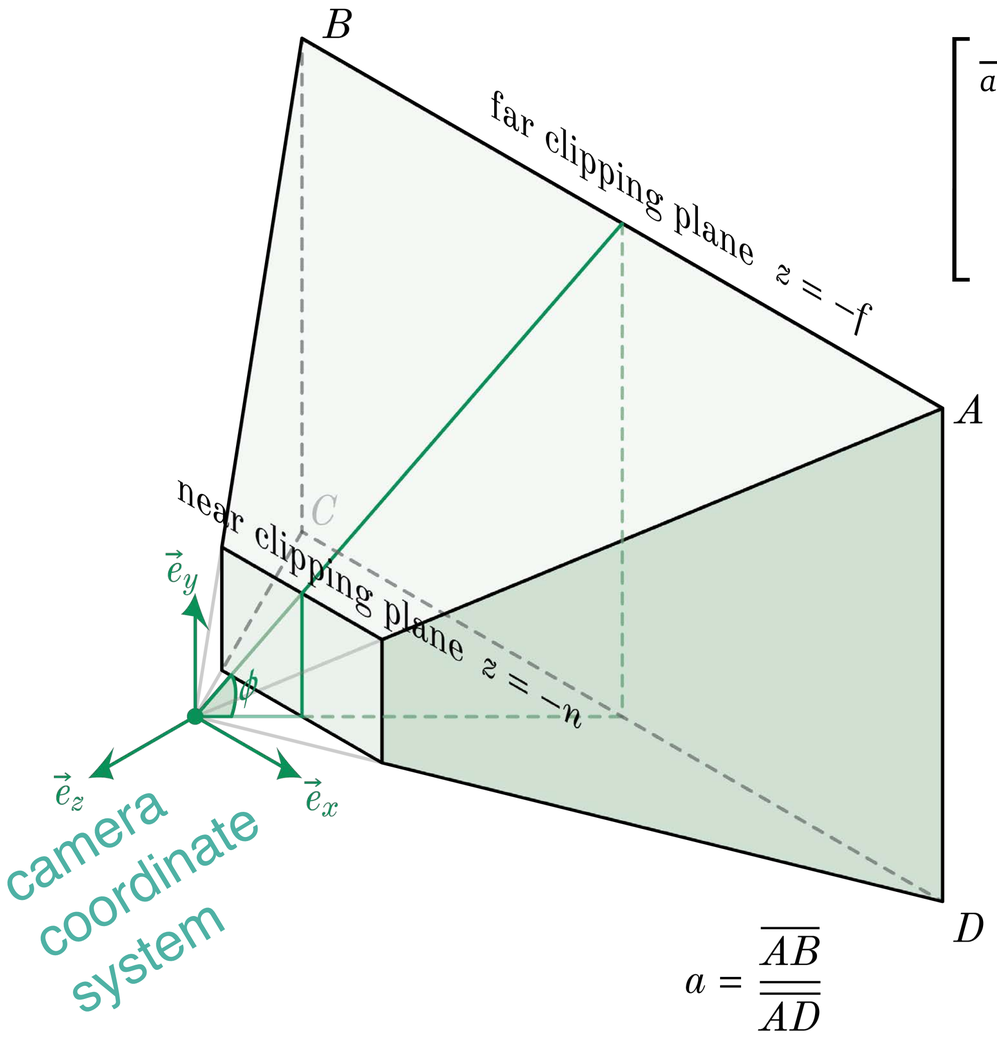
will perform the following transformation:

Each point $\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$ is mapped to $\begin{bmatrix} p_x^{\text{new}} \\ p_y^{\text{new}} \\ p_z^{\text{new}} \end{bmatrix} = \begin{bmatrix} p_x/p_z \\ p_y/p_z \\ 1/p_z \end{bmatrix}$

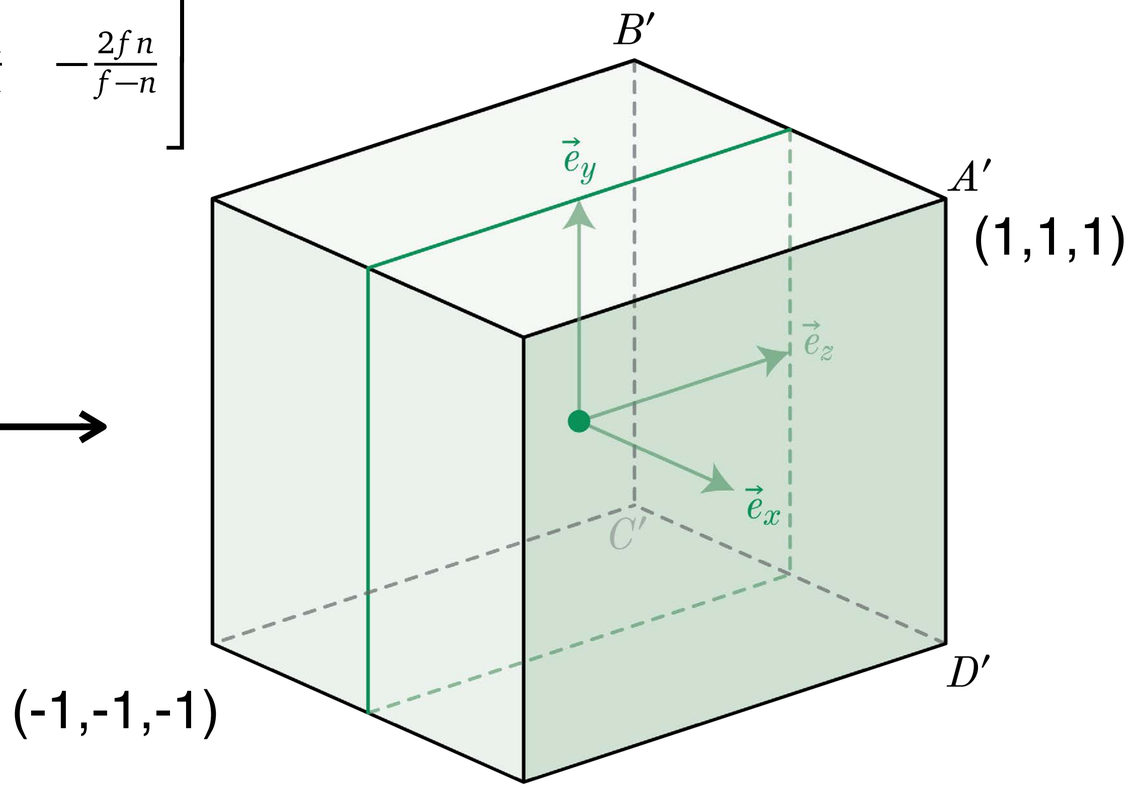
Derivation of the Projection Matrix

- What are projective transforms?
- How homogeneous coordinates work
- Derivation of the projection matrix
- Art of projective geometry

Projection matrix

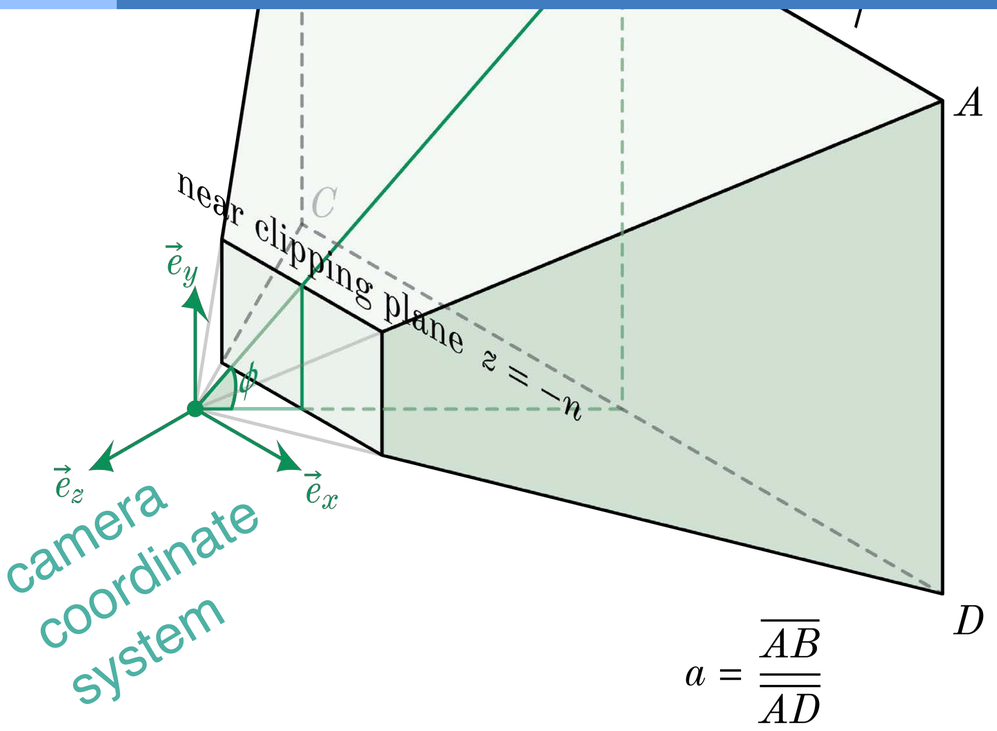


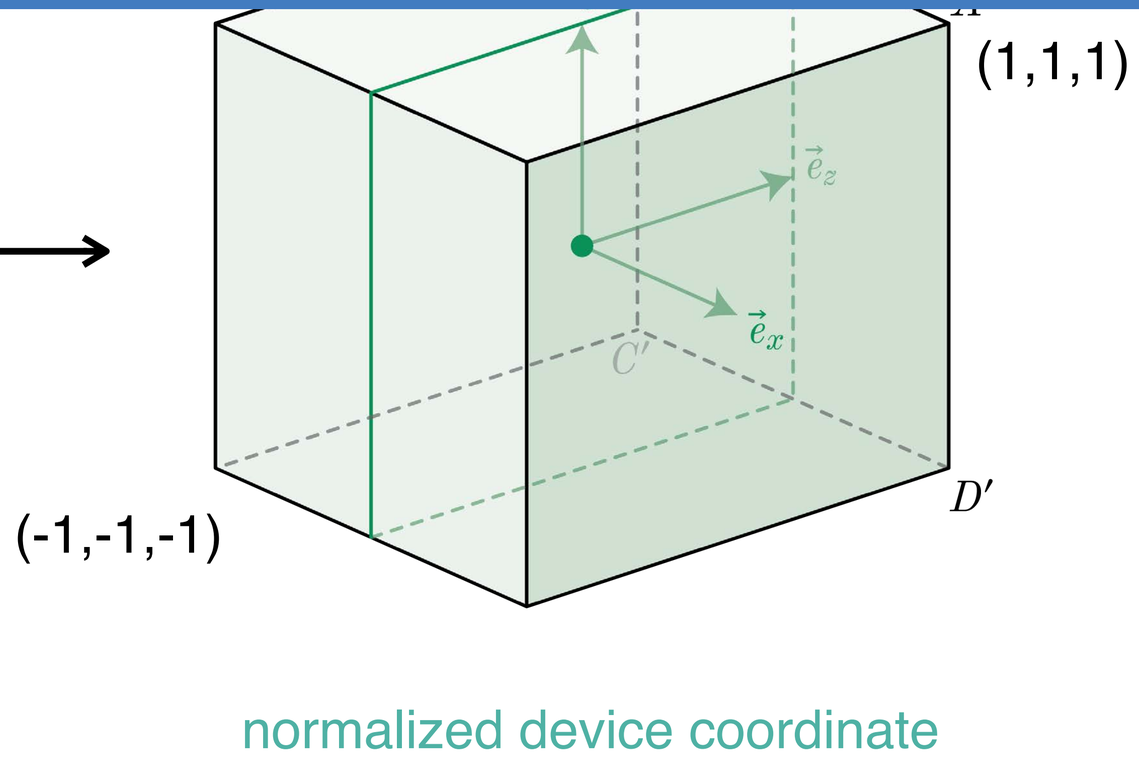
$$\begin{bmatrix} \frac{1}{a\tan(\phi/2)} & & & \\ & \frac{1}{\tan(\phi/2)} & & \\ & & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ & & -1 & \end{bmatrix}$$



normalized device coordinate

Projection matrix





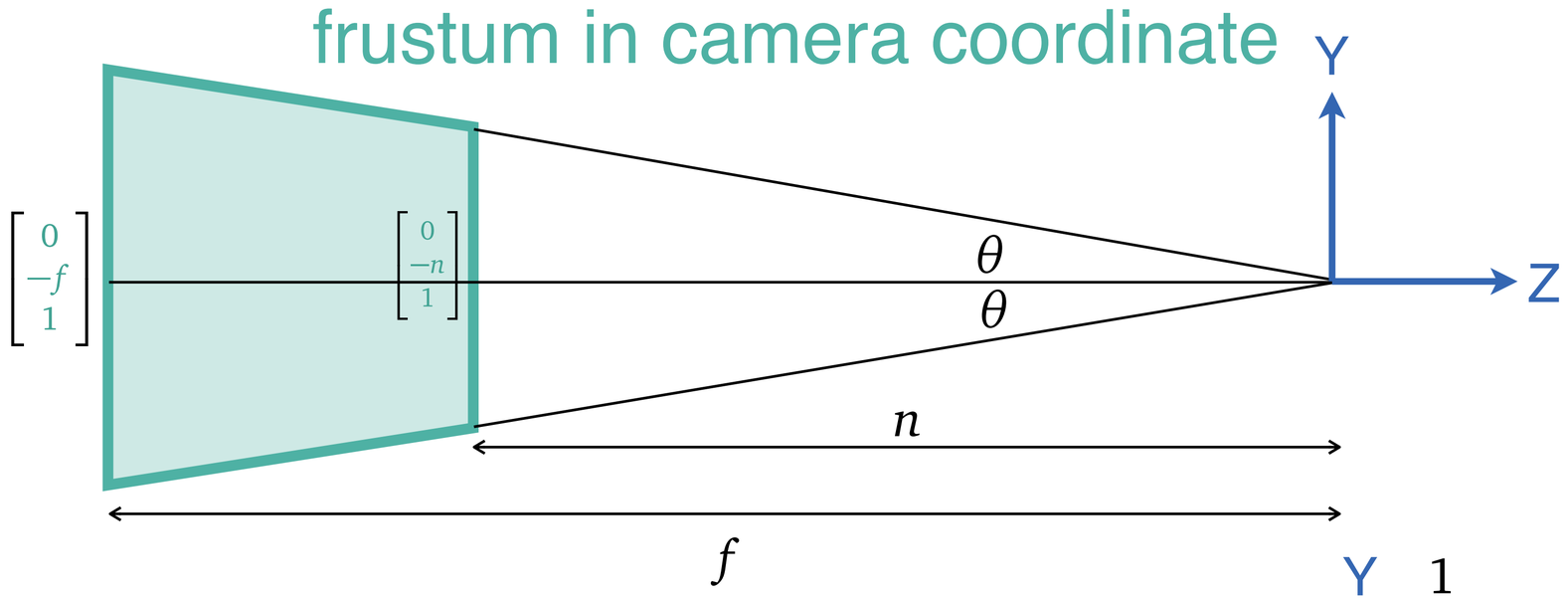
normalized device coordinate

- We will derive it in 2D for visual simplicity
- Notice: in camera coordinate, z points to the back of camera
  in NDC, z represents depth and points away from the viewer

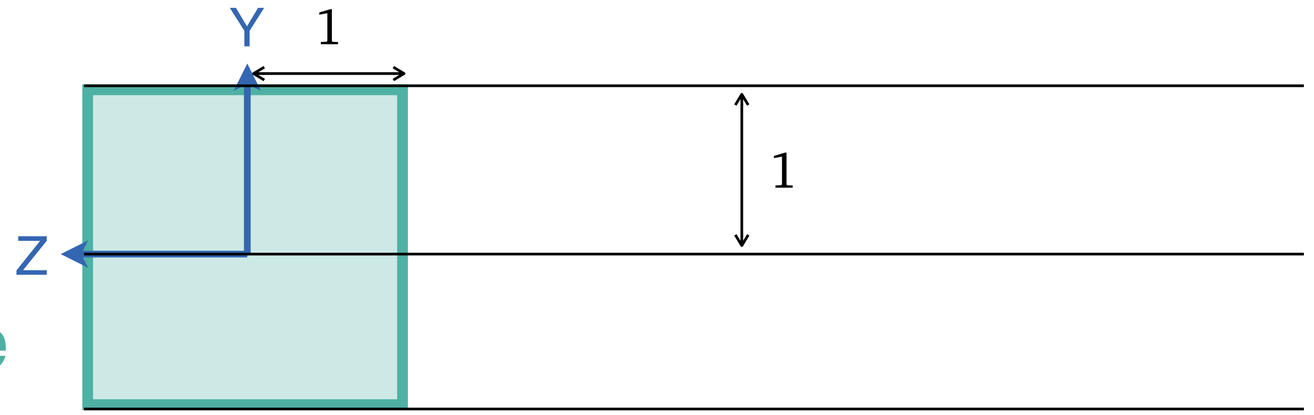
  (The basis in NDC is left-hand oriented.)

Trapezoid to square

- Design a 3x3 matrix (applied to homogeneous coordinates) that deforms the trapezoid to the square.



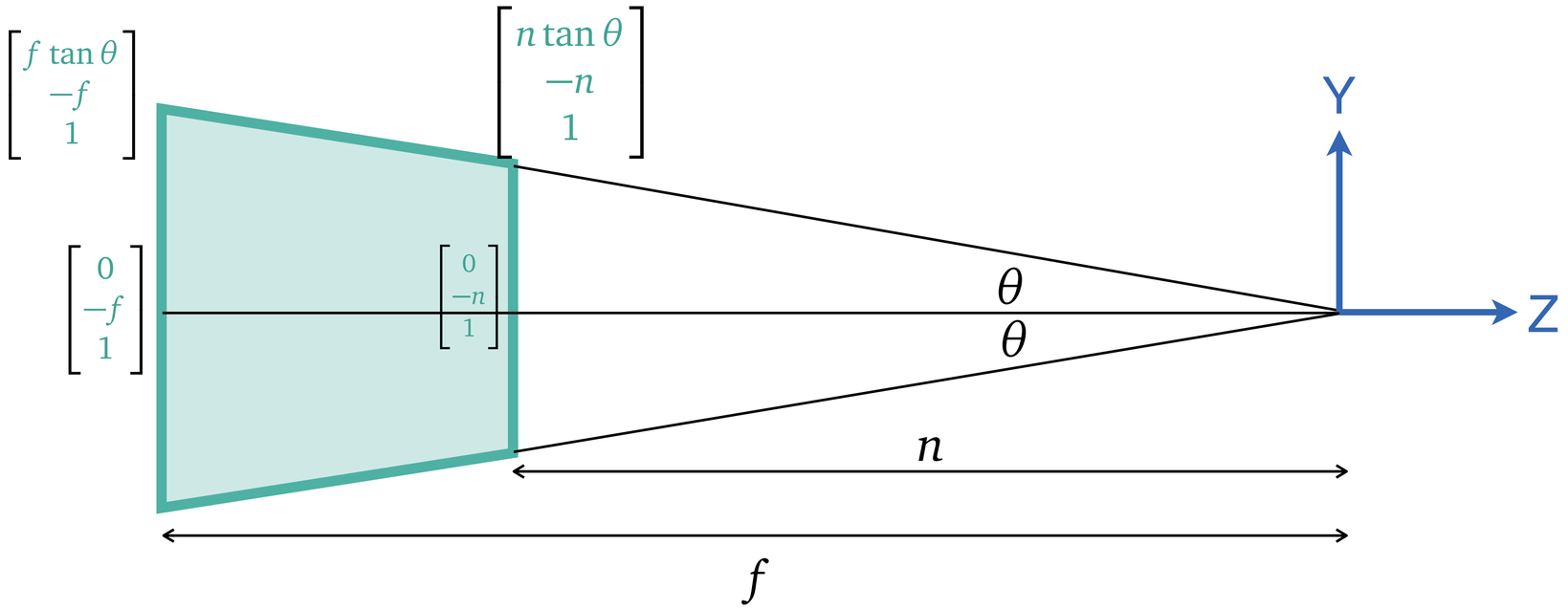
$$\theta = \frac{\phi}{2}$$

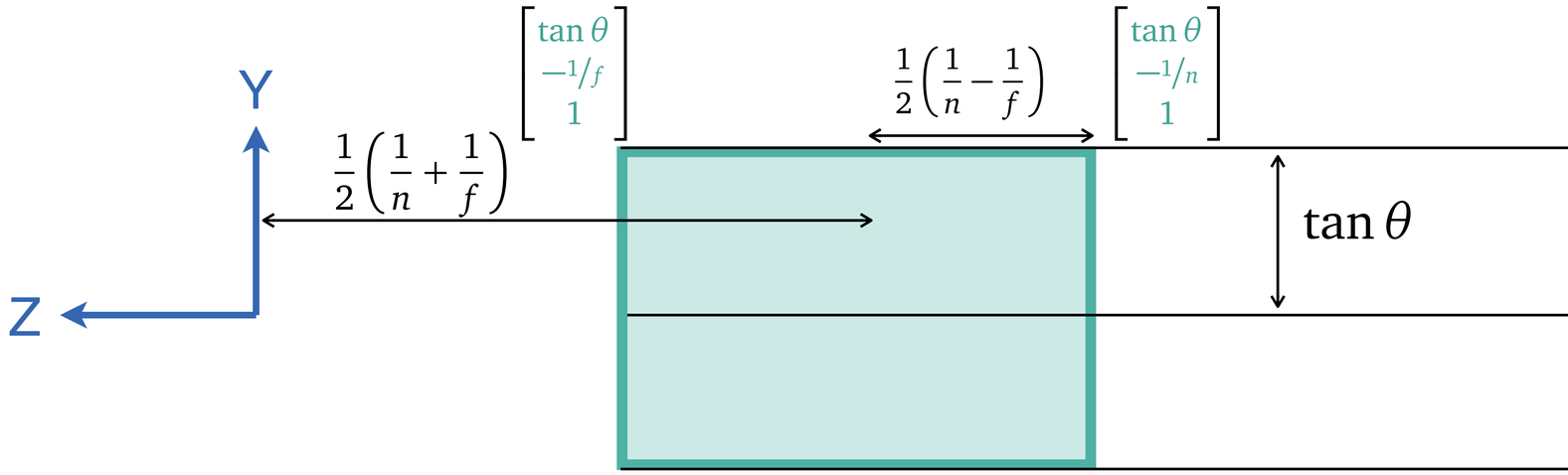


Trapezoid to square

- Inversion

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} y \\ -1 \\ -z \end{bmatrix} \sim \begin{bmatrix} -y/z \\ 1/z \\ 1 \end{bmatrix}$$



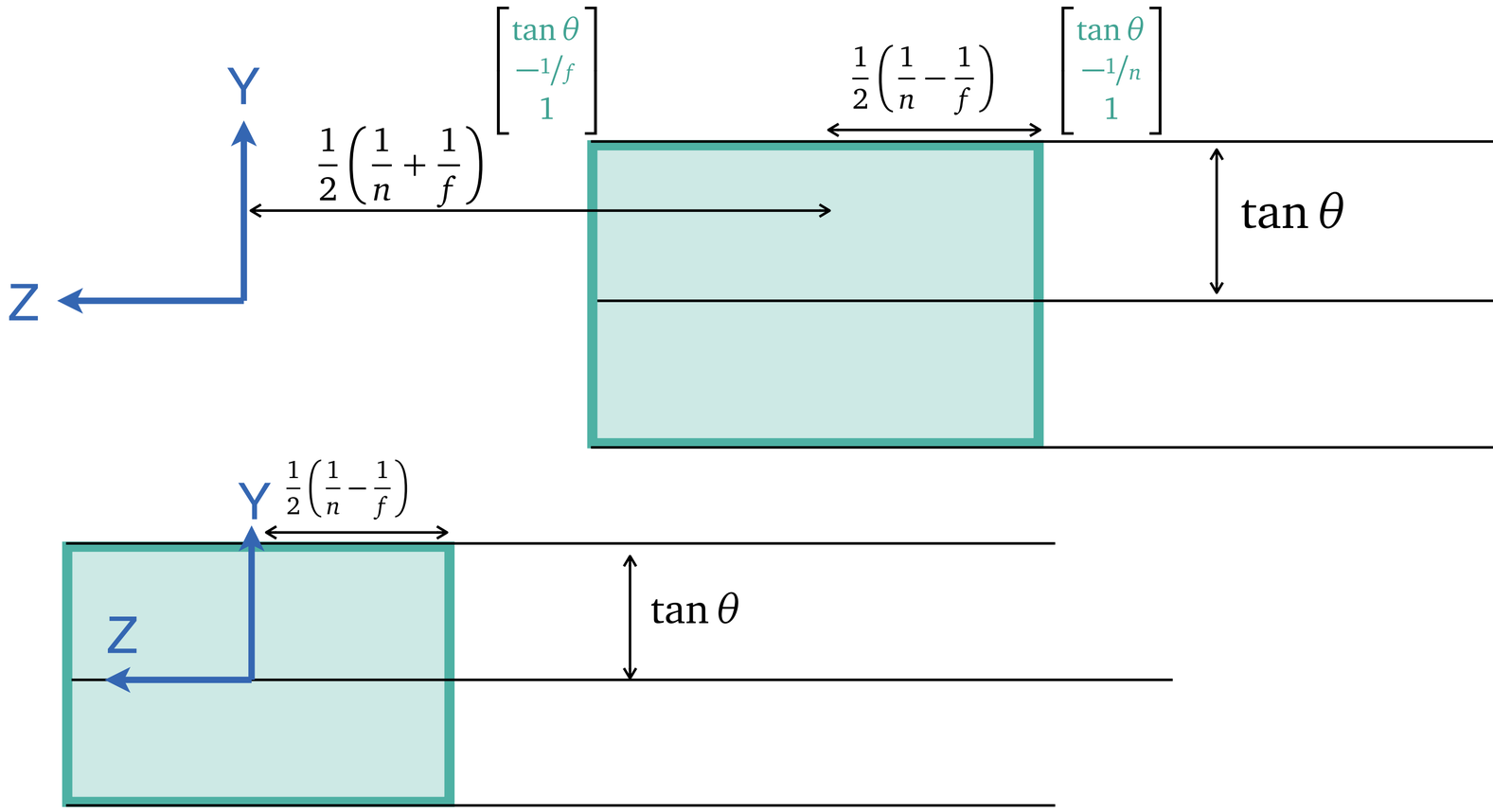


Trapezoid to square

- Inversion

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} y \\ -1 \\ -z \end{bmatrix} \sim \begin{bmatrix} -y/z \\ 1/z \\ 1 \end{bmatrix}$$

- Translation $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{f+n}{2fn} \\ 0 & 0 & 1 \end{bmatrix}$



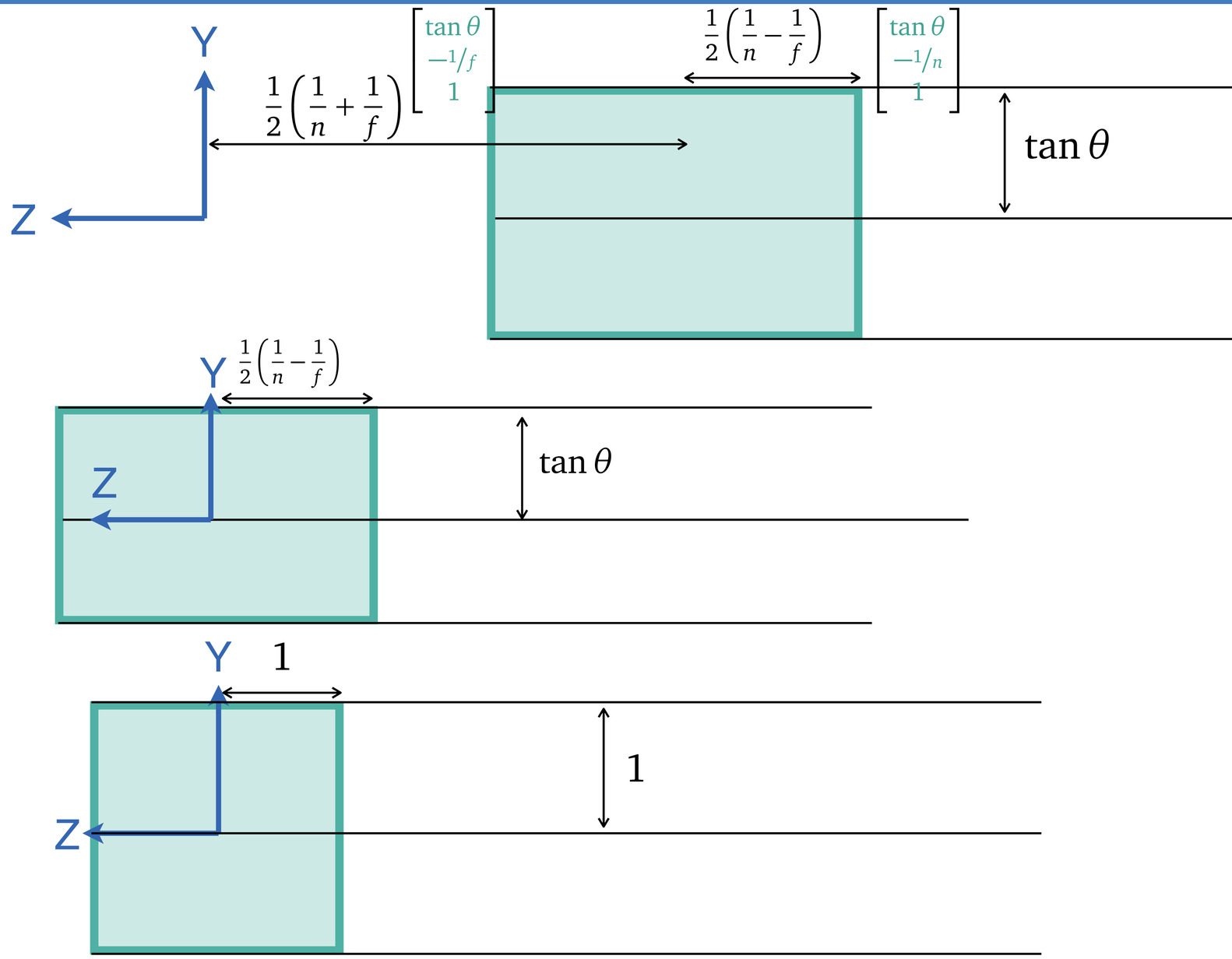
Trapezoid to square

- Inversion

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} y \\ -1 \\ -z \end{bmatrix} \sim \begin{bmatrix} -y/z \\ 1/z \\ 1 \end{bmatrix}$$

- Translation $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{f+n}{2fn} \\ 0 & 0 & 1 \end{bmatrix}$

- Scaling $\begin{bmatrix} \frac{1}{\tan\theta} & 0 & 0 \\ 0 & \frac{2fn}{f-n} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

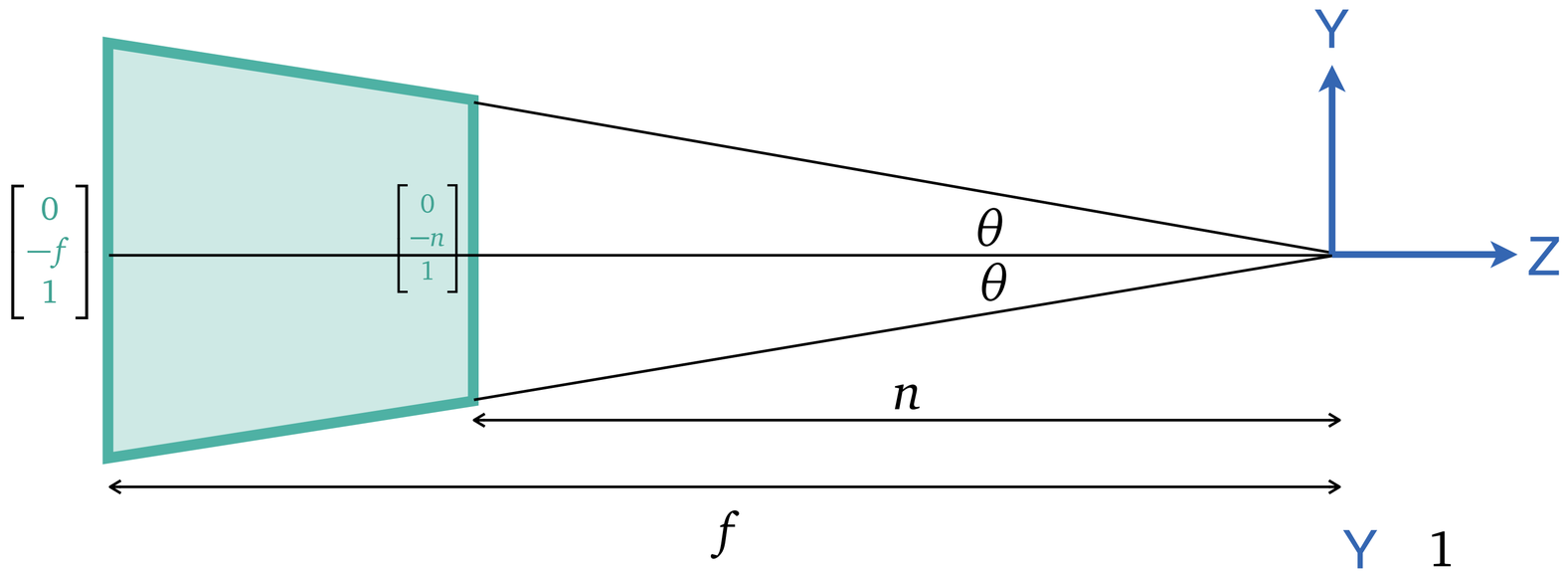


Trapezoid to square

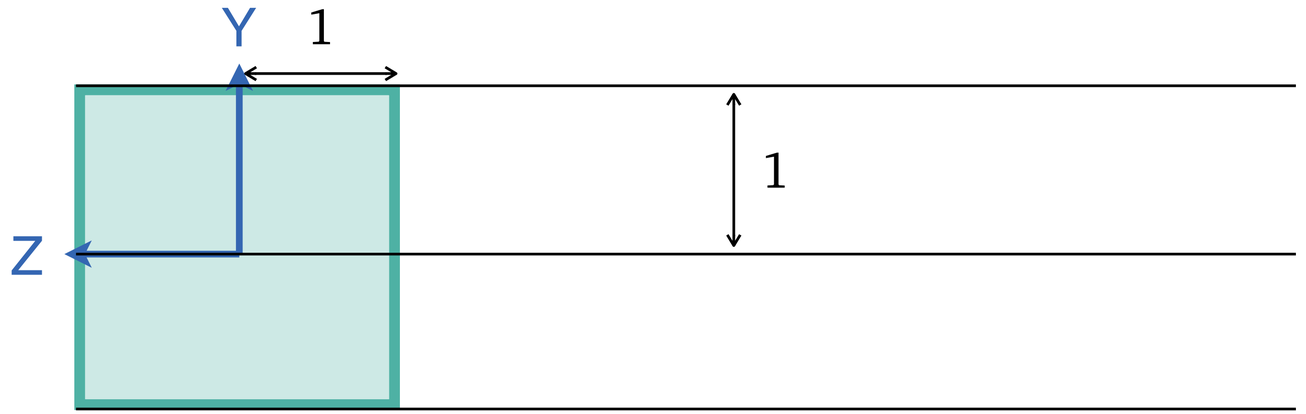
$$\begin{bmatrix} \frac{1}{\tan\theta} & 0 & 0 \\ 0 & \frac{2fn}{f-n} & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{f+n}{2fn} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}}_{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{f+n}{2fn} & -1 \\ 0 & -1 & 0 \end{bmatrix}} = \begin{bmatrix} \frac{1}{\tan\theta} & 0 & 0 \\ 0 & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ 0 & -1 & 0 \end{bmatrix}$$

Trapezoid to square

- Design a 3x3 matrix (applied to homogeneous coordinates) that deforms the trapezoid to the square.



$$\begin{bmatrix} \frac{1}{\tan\theta} & 0 & 0 \\ 0 & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ 0 & -1 & 0 \end{bmatrix}$$



Add x coordinate

$$\begin{bmatrix} \frac{1}{\tan\theta} & 0 & 0 \\ 0 & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ 0 & -1 & 0 \end{bmatrix}$$

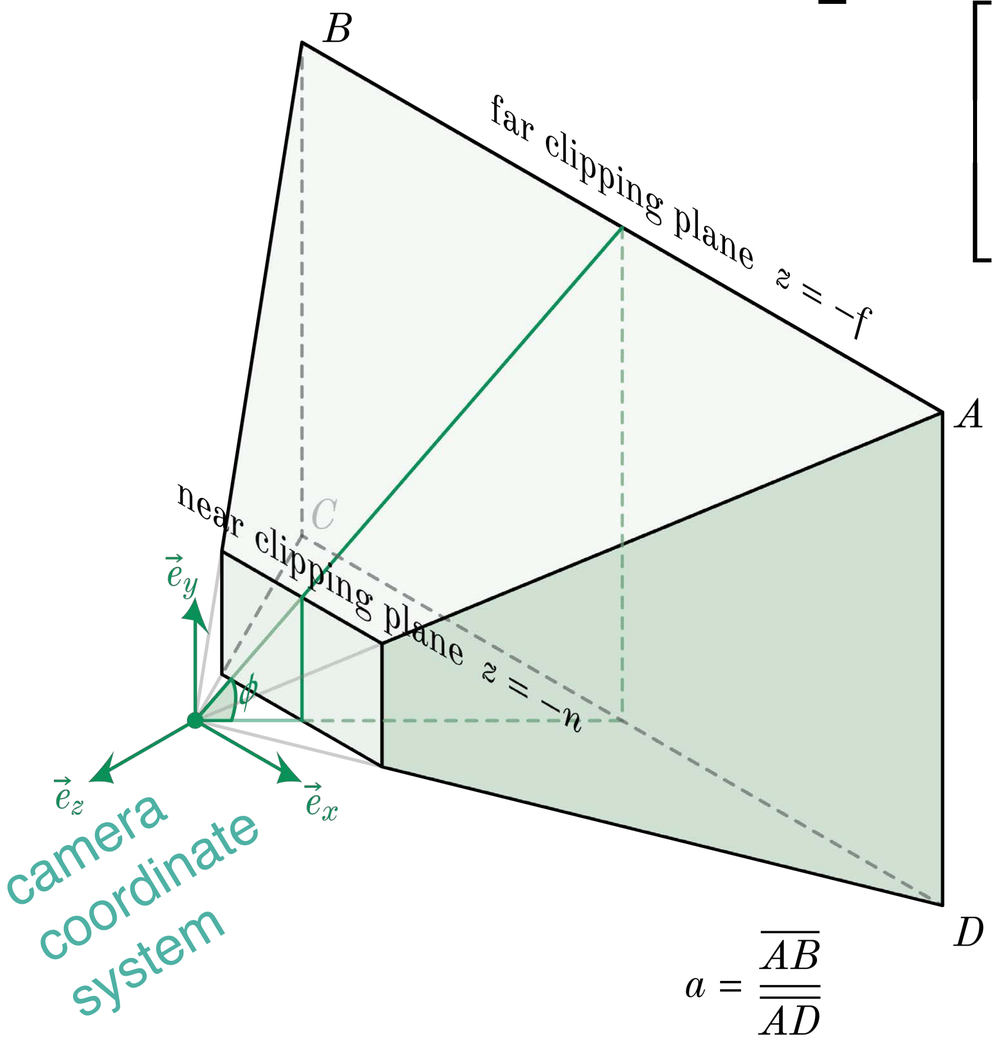
Add x coordinate

$$\begin{bmatrix} \frac{1}{a\tan\theta} & 0 & 0 & 0 \\ 0 & \frac{1}{\tan\theta} & 0 & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

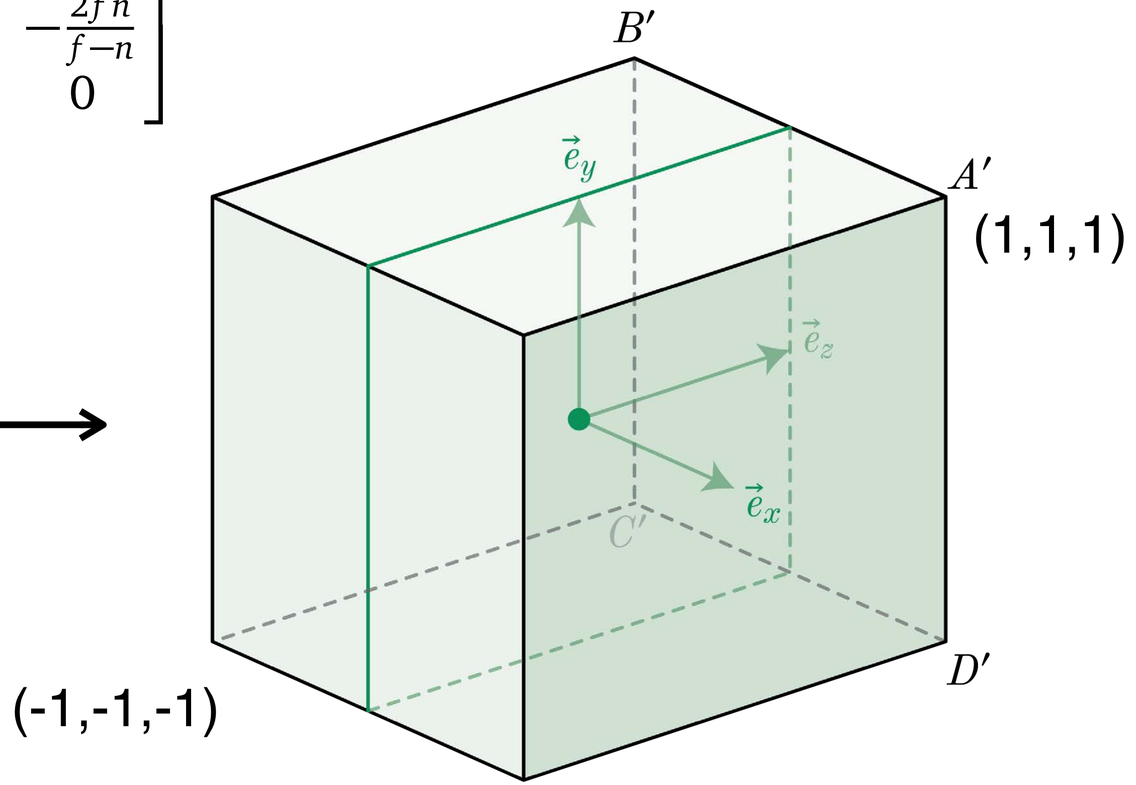
Projection matrix

$$\theta = \frac{\phi}{2}$$

$$\begin{bmatrix} \frac{1}{a\tan\theta} & 0 & 0 & 0 \\ 0 & \frac{1}{\tan\theta} & 0 & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$



$$a = \frac{\overline{AB}}{\overline{AD}}$$



normalized device coordinate

Art of Projective Geometry

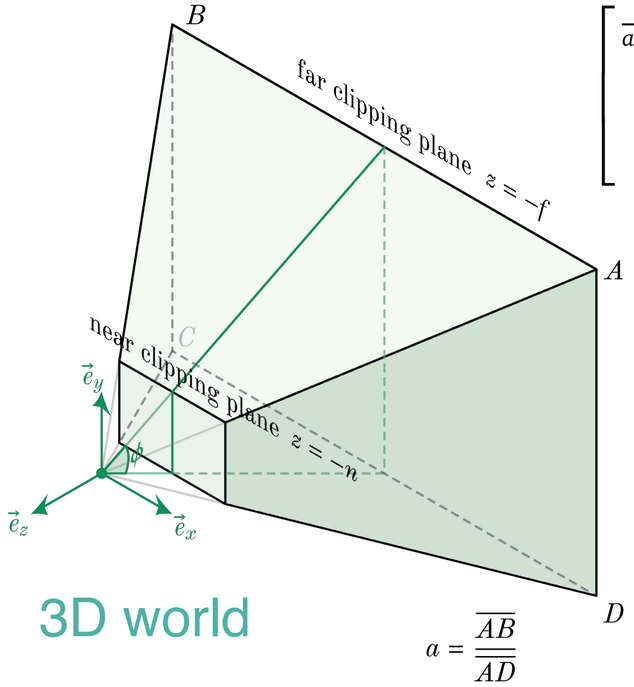
- What are projective transforms?
- How homogeneous coordinates work
- Derivation of the projection matrix
- Art of projective geometry

Art of Projective Geometry

- Art of projective geometry
  - Perspective drawings
  - Cross ratio

2D/3D projective transformations

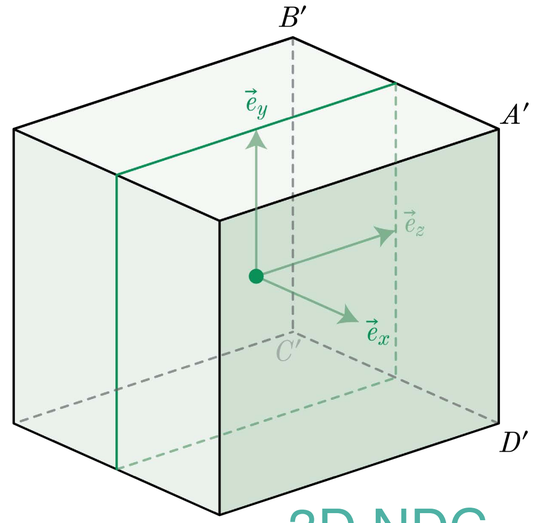
Computer graphics



3D world

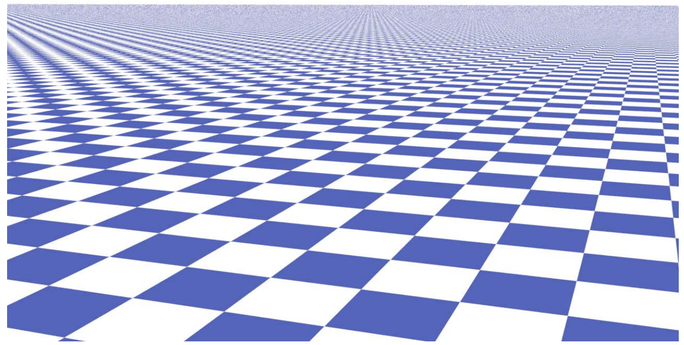
$$\begin{bmatrix} \frac{1}{a\tan(\phi/2)} & & & \\ & \frac{1}{\tan(\phi/2)} & & \\ & & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ & & -1 & \end{bmatrix}$$

linear transform in 4D



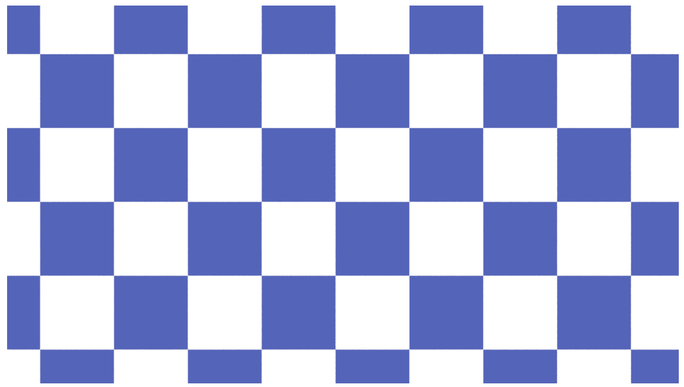
3D NDC

Renaissance artists:



2D canvas

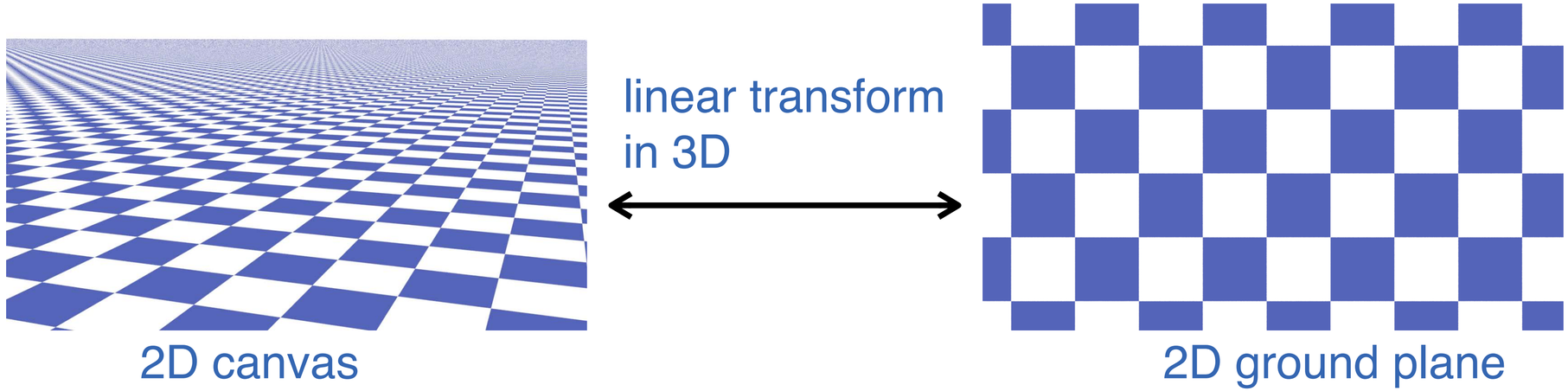
linear transform in 3D



2D ground plane

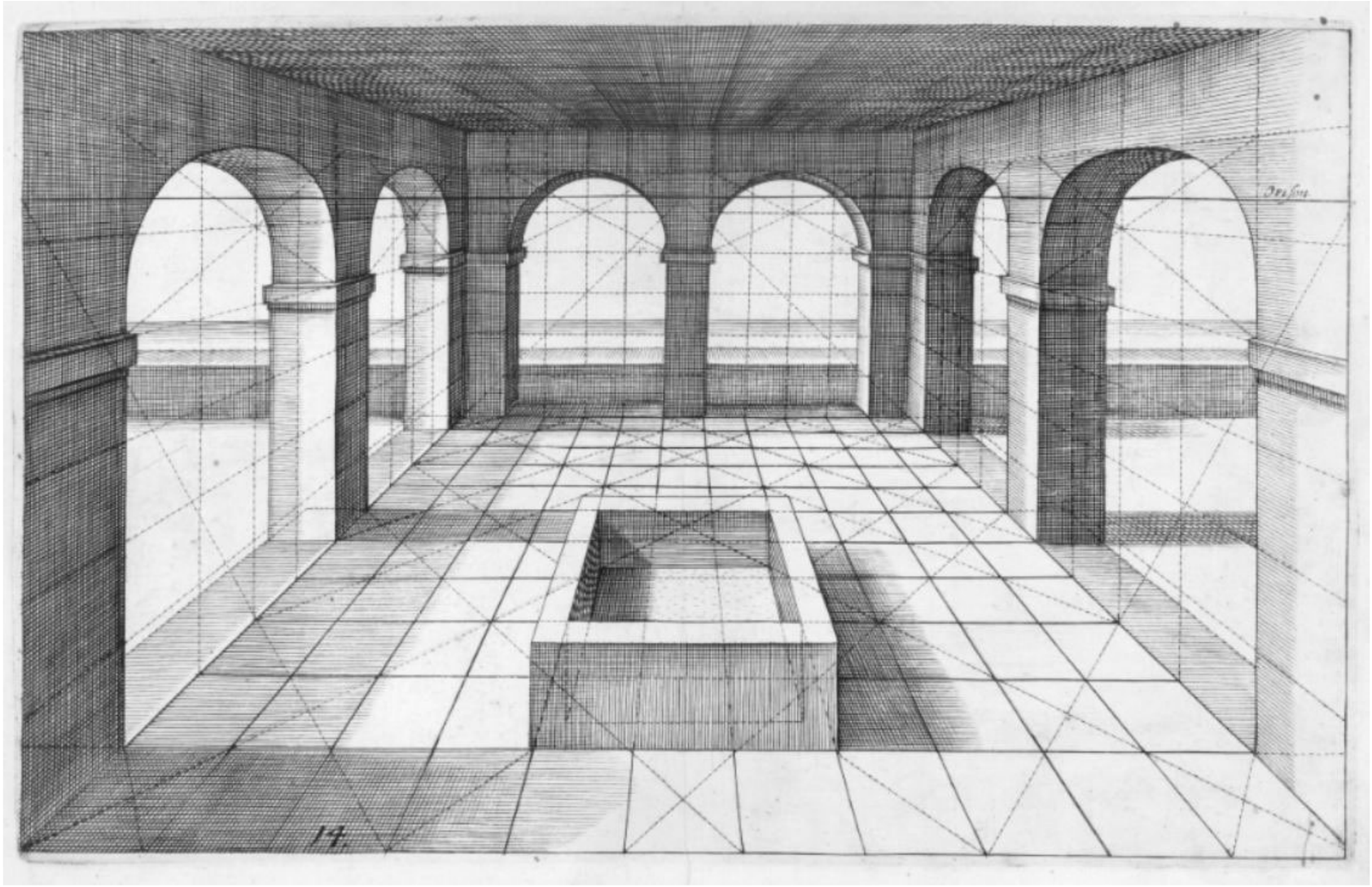
2D/3D projective transformations

Renaissance artists:



- Case study the 2D projective geometry
- One dimension lower than frustum-to-NDC (easier to grasp)
- Important for
  - History (tools developed for graphics purposes in 15th century)
  - Descriptive/engineering illustration
  - Computer graphics / vision / surveying

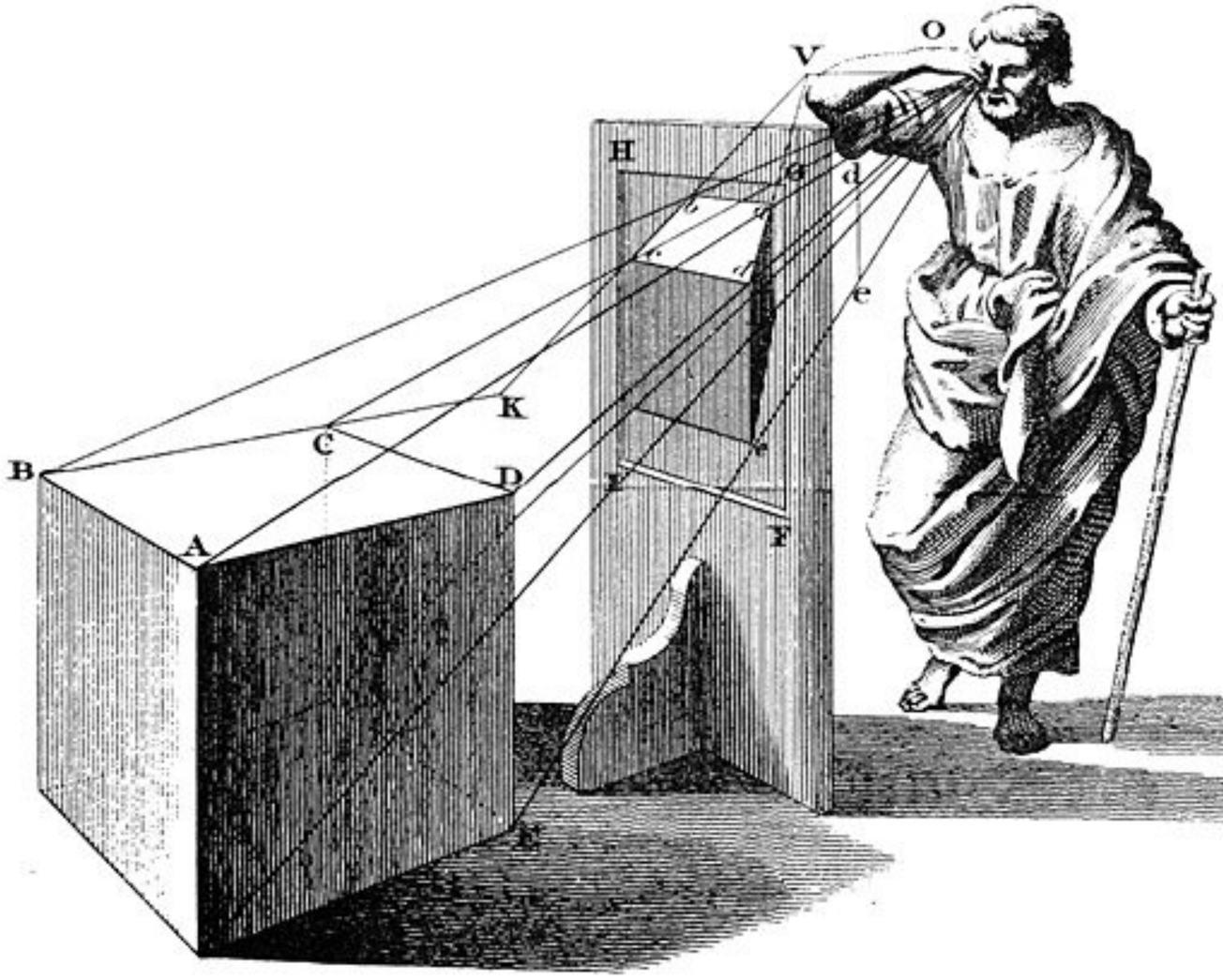
Renaissance perspective drawing



Linear perspective

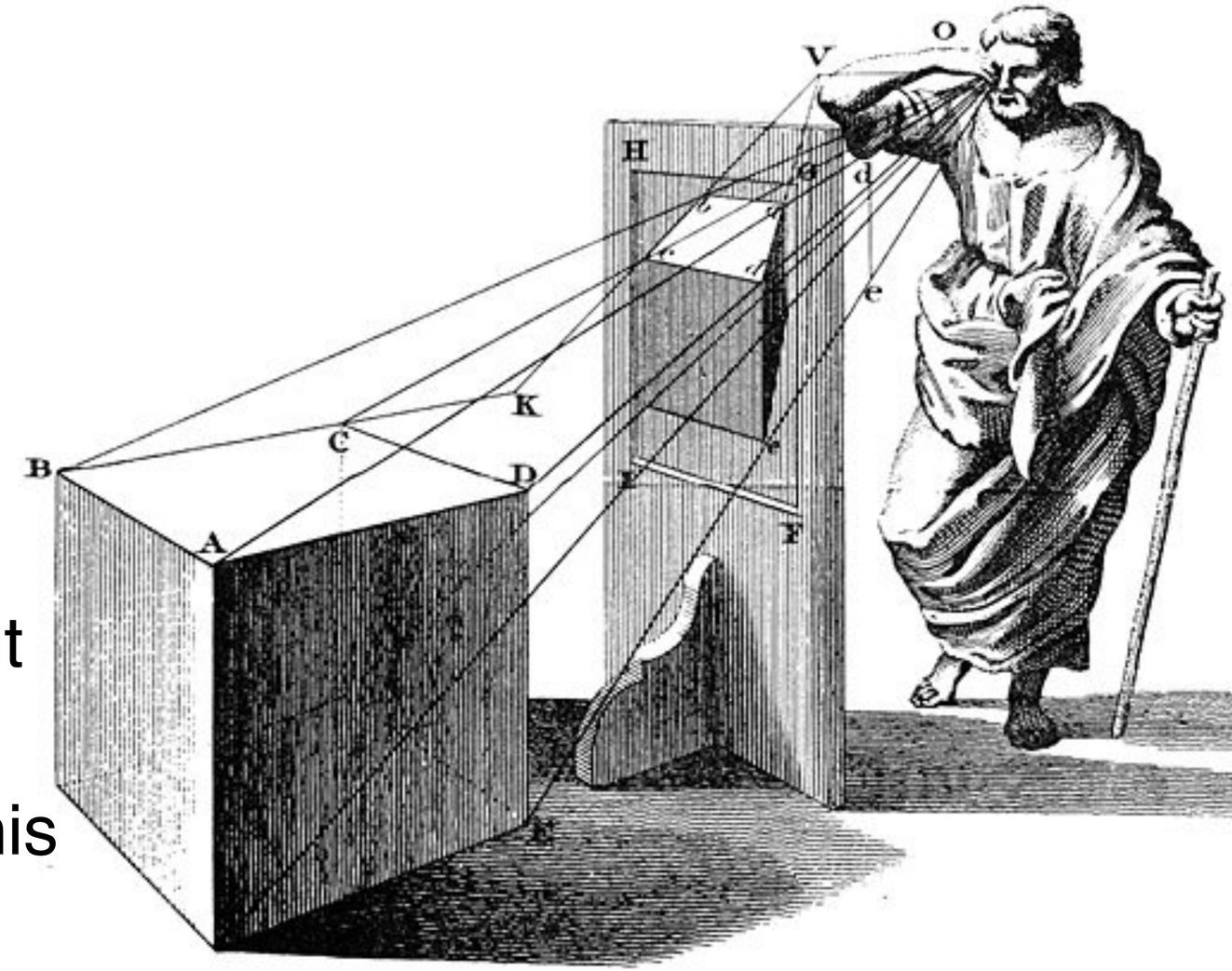
- Objects on canvas represents certain objects in 3D space

Objects on Canvas	Objects in 3D space
A point	A line through origin (eye)
A line	A plane through origin
A conic section	A quadratic cone with apex at origin



Linear perspective

Objects on Canvas	Objects in 3D space
A point	A line through origin (eye)
A line	A plane through origin
A conic section	A quadratic cone with apex at origin



Perspective drawing

- A 3D point P is mapped to a point on canvas, given by $\overleftrightarrow{OP}$.
- Collinearity is preserved under this projection from 3D to Canvas.
- Incidence relation is preserved.

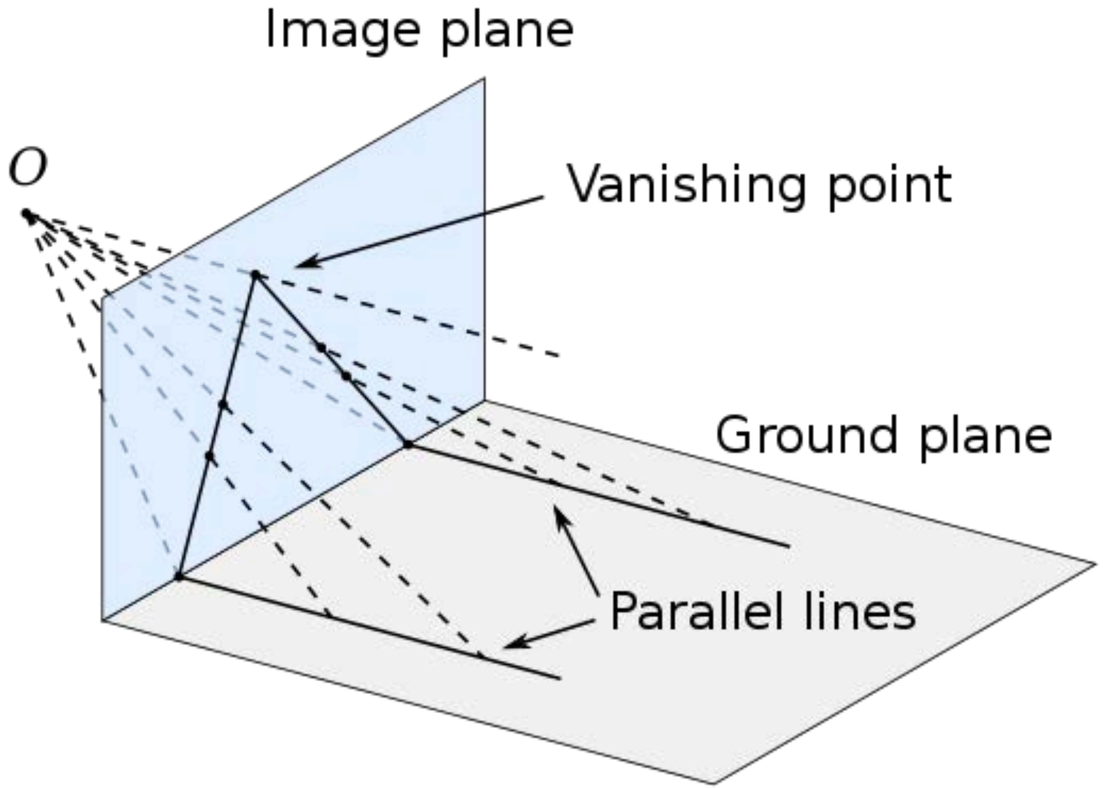
Horizons and vanishing points

Definition

A 3D plane gives rise to a **horizon line** on the canvas, defined as the unique parallel plane thru eye.

Definition

A 3D line gives rise to a **vanishing point** on the canvas, defined as the unique parallel line thru eye.



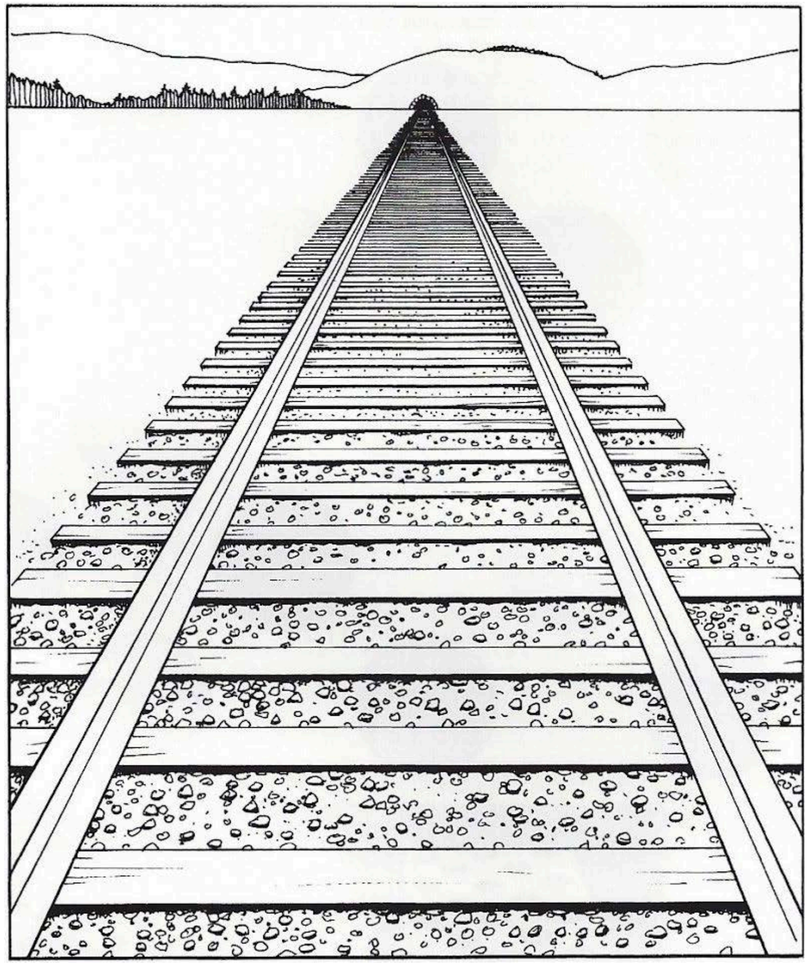
Horizons and vanishing points

Theorem

Parallel planes in 3D share a common horizon on canvas. Parallel lines in 3D share a common vanishing point on canvas.

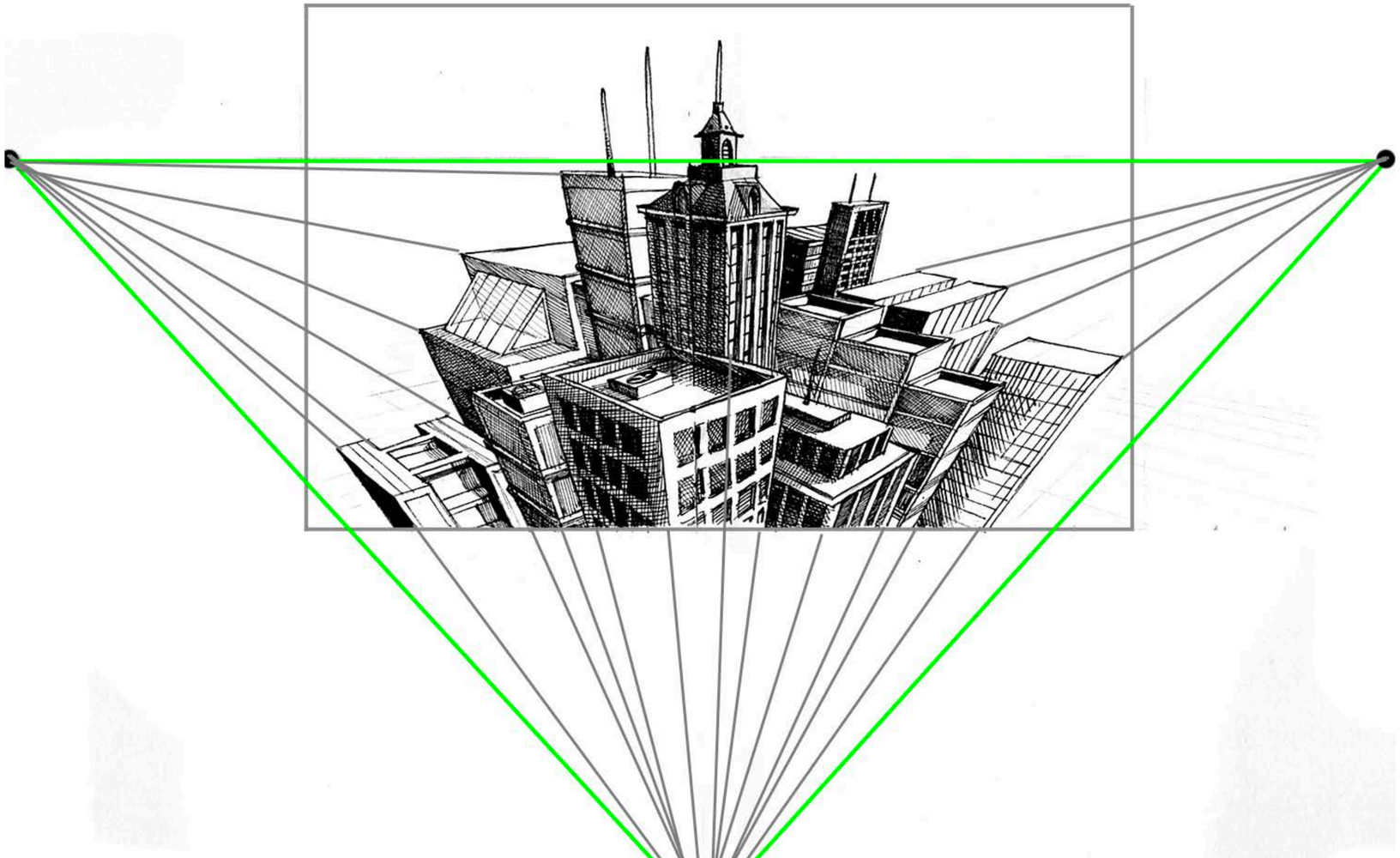
Theorem

If in 3D a line is parallel to or lies in a plane, then on canvas the corresponding vanishing point lies on the horizon for the plane.

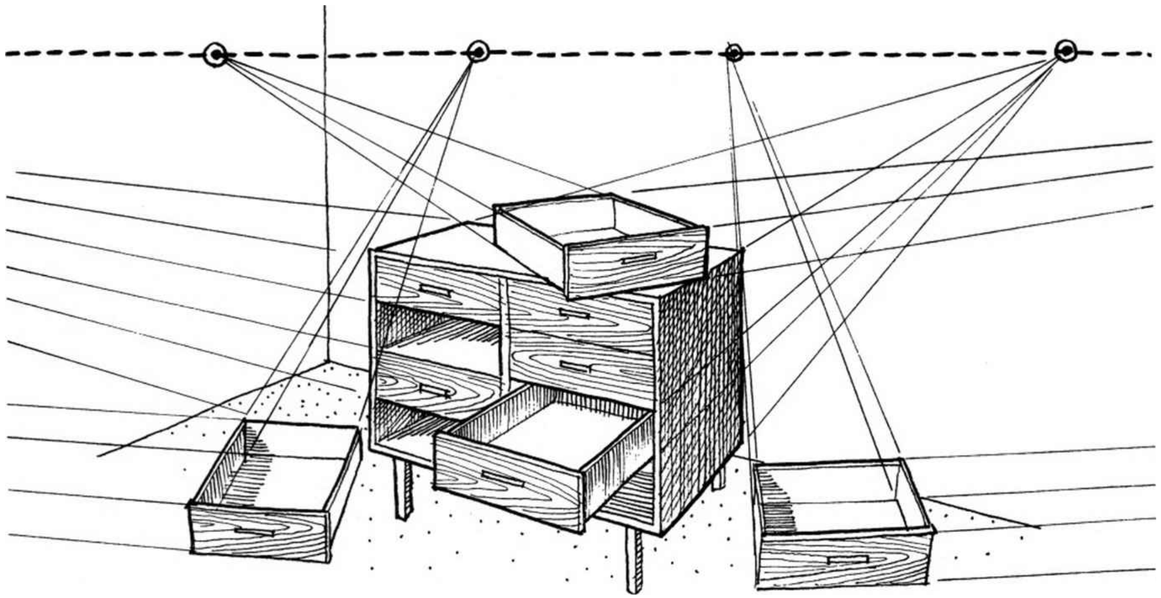


Vanishing point of parallel lines

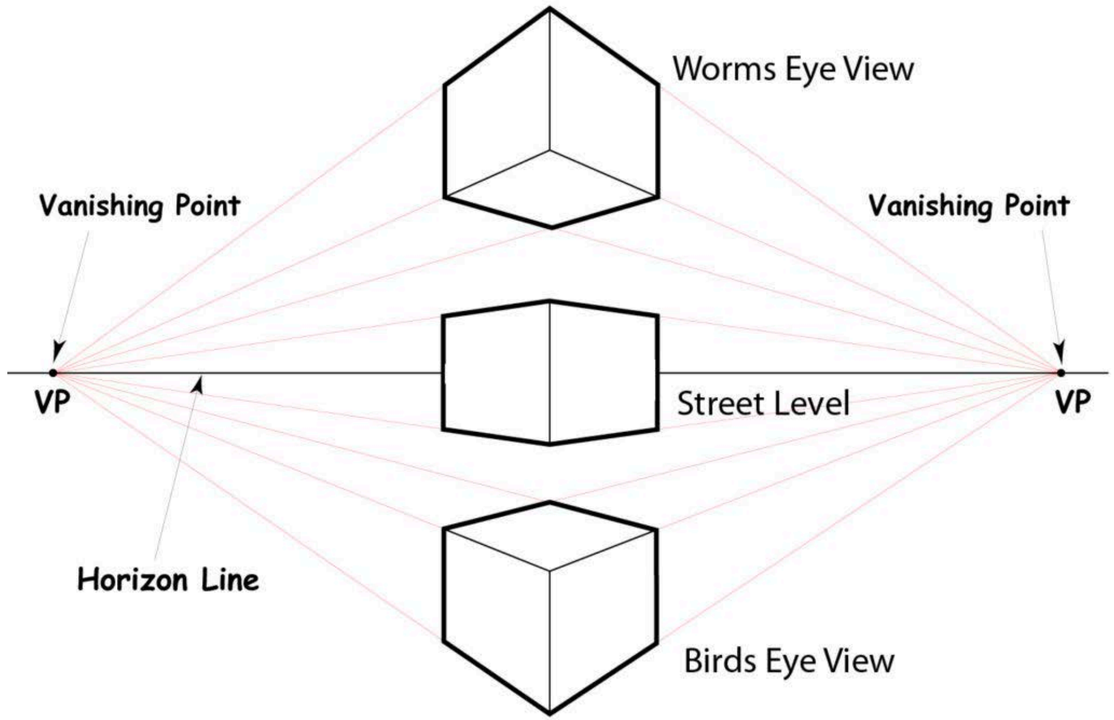
- Most scenery of cities or human made objects have 3 families of parallel lines, leading to 3-(vanishing)-point perspective



Vanishing points and horizon

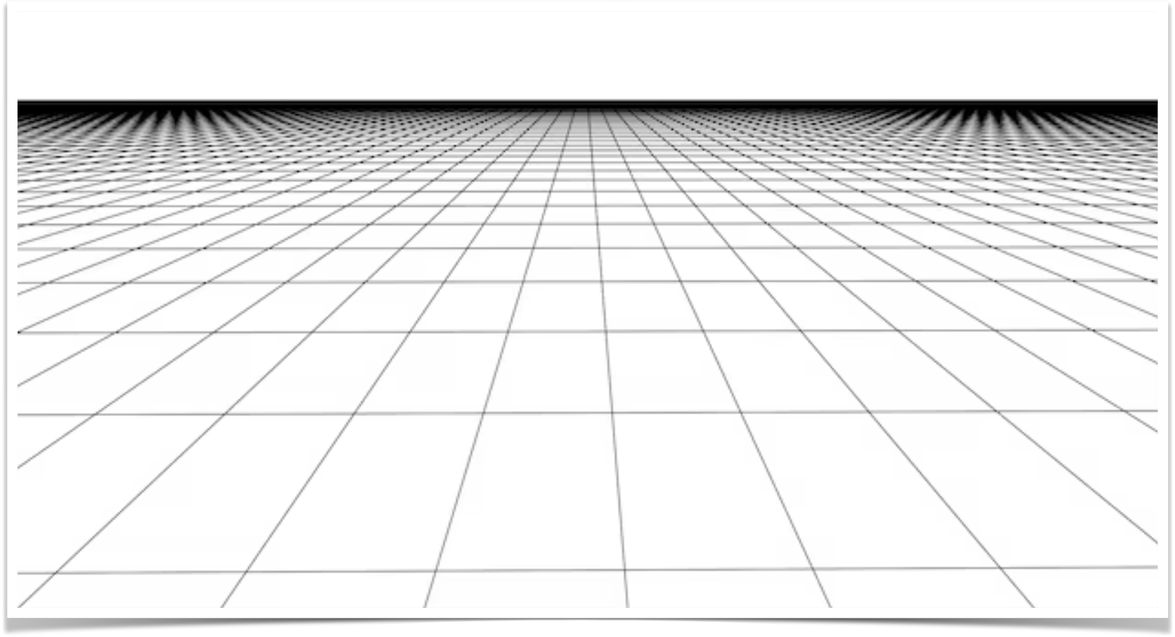


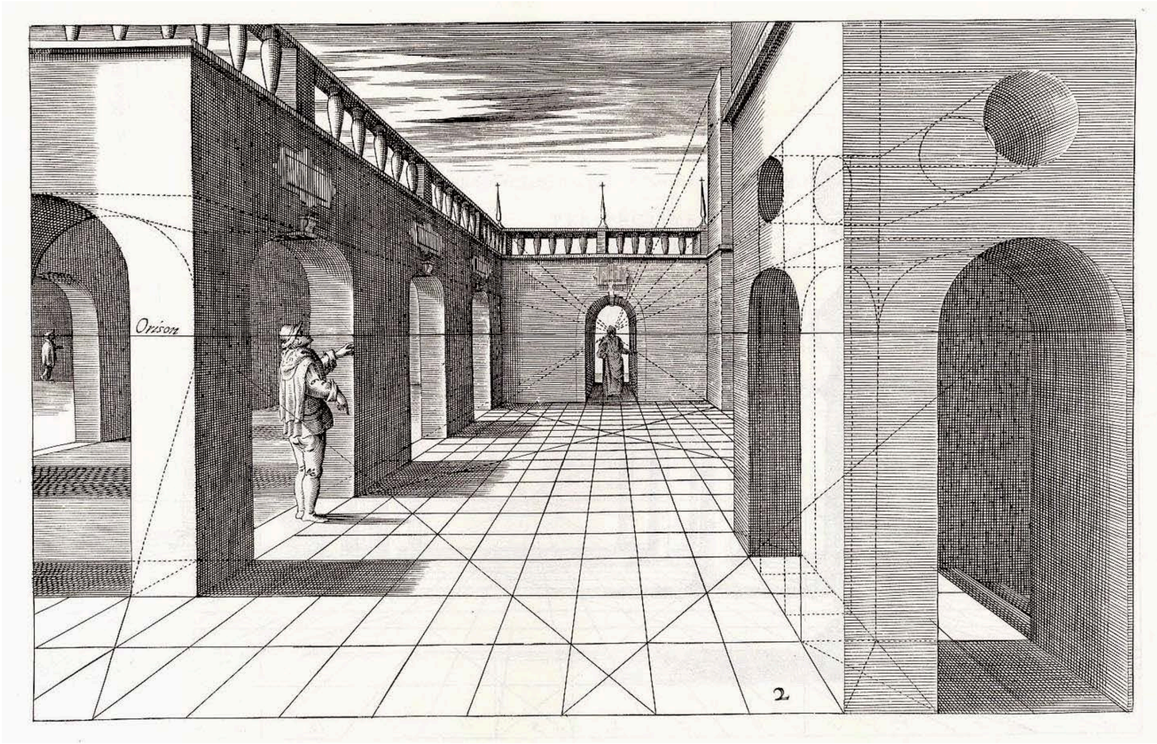
IIIlustration: Joshua Nava Art



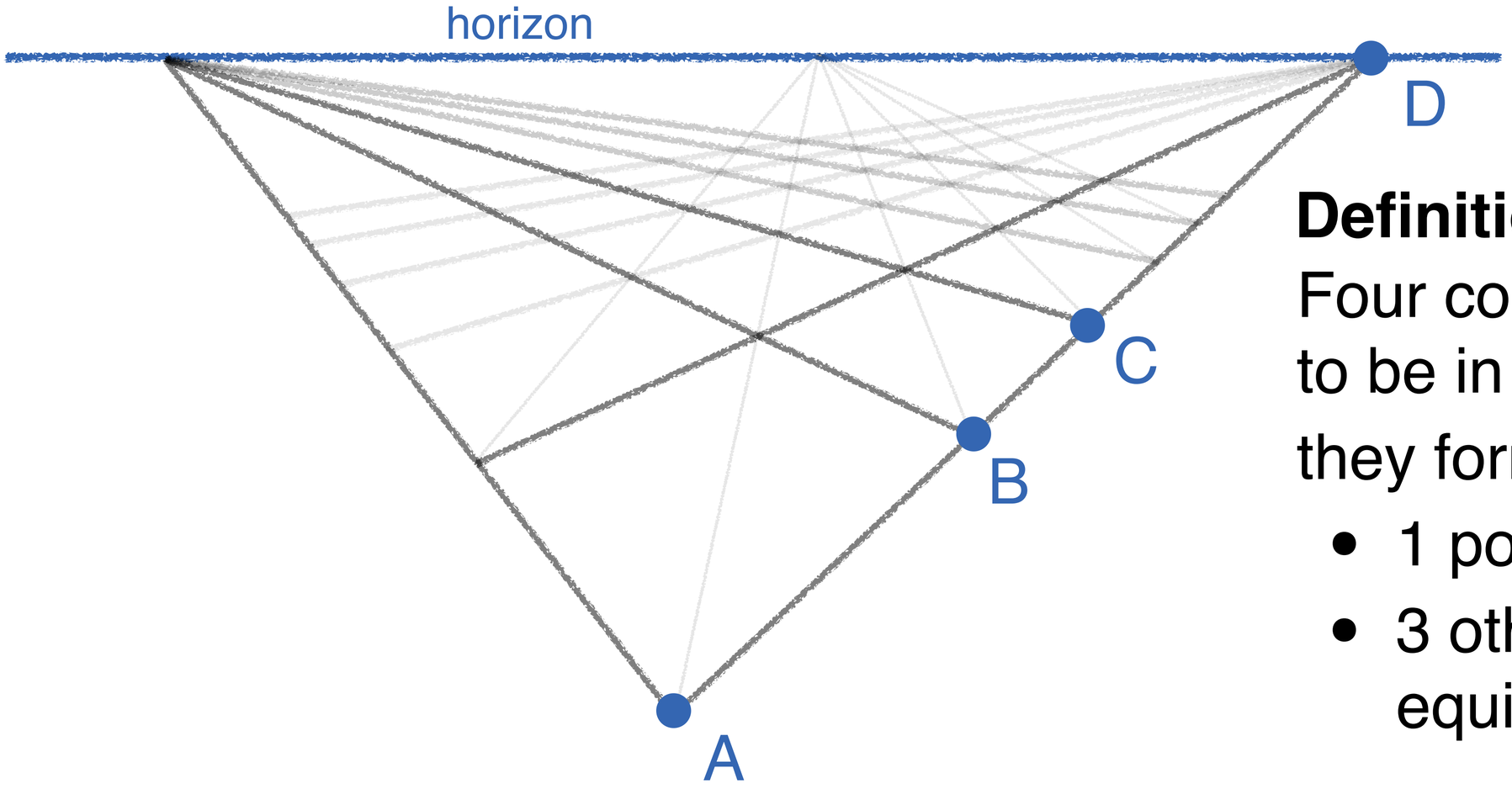
Perspective drawing

Task: Given one quad on canvas, tile the plane.





Harmonic range



Definition

Four collinear points are said to be in a ***harmonic range*** if they form a picture that

- 1 point is at infinity, and
- 3 other points are equidistant

Given 3 collinear points, we can construct "uniquely" the 4th pt for the harmonic range

4 points on a line

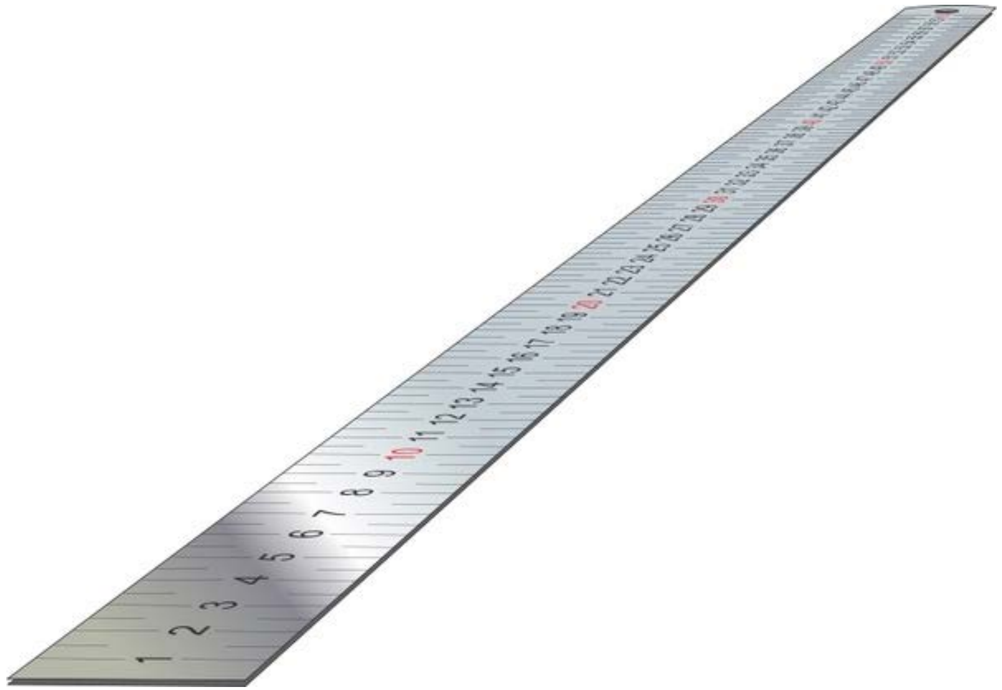
Corollary

Given 3 collinear points A,B,V, we can build C so that they can be interpreted as ABC are equi-distant and V is the vanishing pt.

Corollary

*4 collinear points ABCV have a projectively invariant **cross-ratio**, given by the ratio of AC and AB when V is the vanishing point.*

Definition *4 collinear points are harmonic if the cross ratio is -1,2,or 1/2.*



Cross ratio

- Art of projective geometry
  - Perspective drawings
  - Cross ratio

Cross ratio (anharmonic ratio)



- Given 4 points A,B,C,D along real line, the **cross ratio** is defined by

$$\mathrm{Cr}(A,B,C,D) = \frac{(B-A)(D-C)}{(C-B)(A-D)} \quad \text{or} \quad \pm\frac{\overline{AB}\cdot\overline{CD}}{\overline{BC}\cdot\overline{AD}}$$

- Other convention:

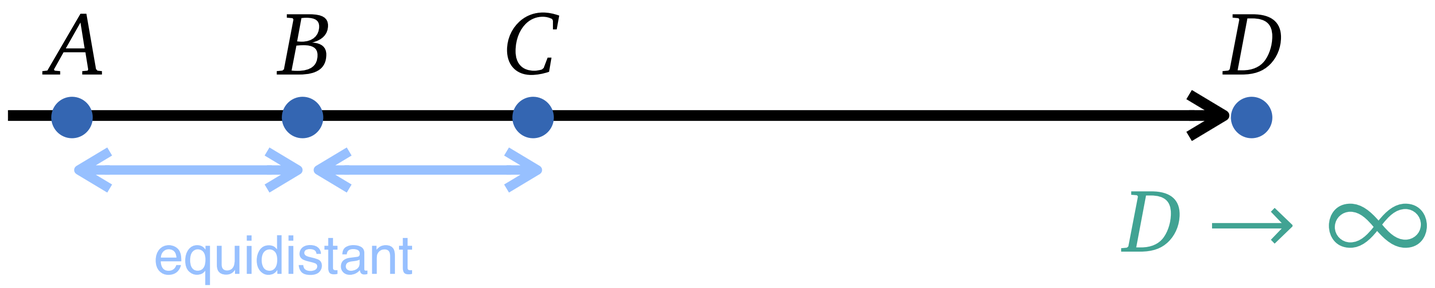
$$\mathrm{Cr}_{\text{Wikipedia}}(A,B,C,D) = \frac{(C-A)(D-B)}{(B-C)(A-D)}$$

Visit 4 pts cyclicly, and alternate between multiplying and dividing lengths

- Cross ratio is projectively invariant. (Very powerful measurement!)
- ABCD are harmonic when Cr(A,B,C,D)= –1

Example: cross ratio of harmonic range

- ABCD are harmonic when Cr(A,B,C,D)= –1



$$\mathrm{Cr}(A,B,C,D) = \frac{(B-A)(D-C)}{(C-B)(A-D)}$$

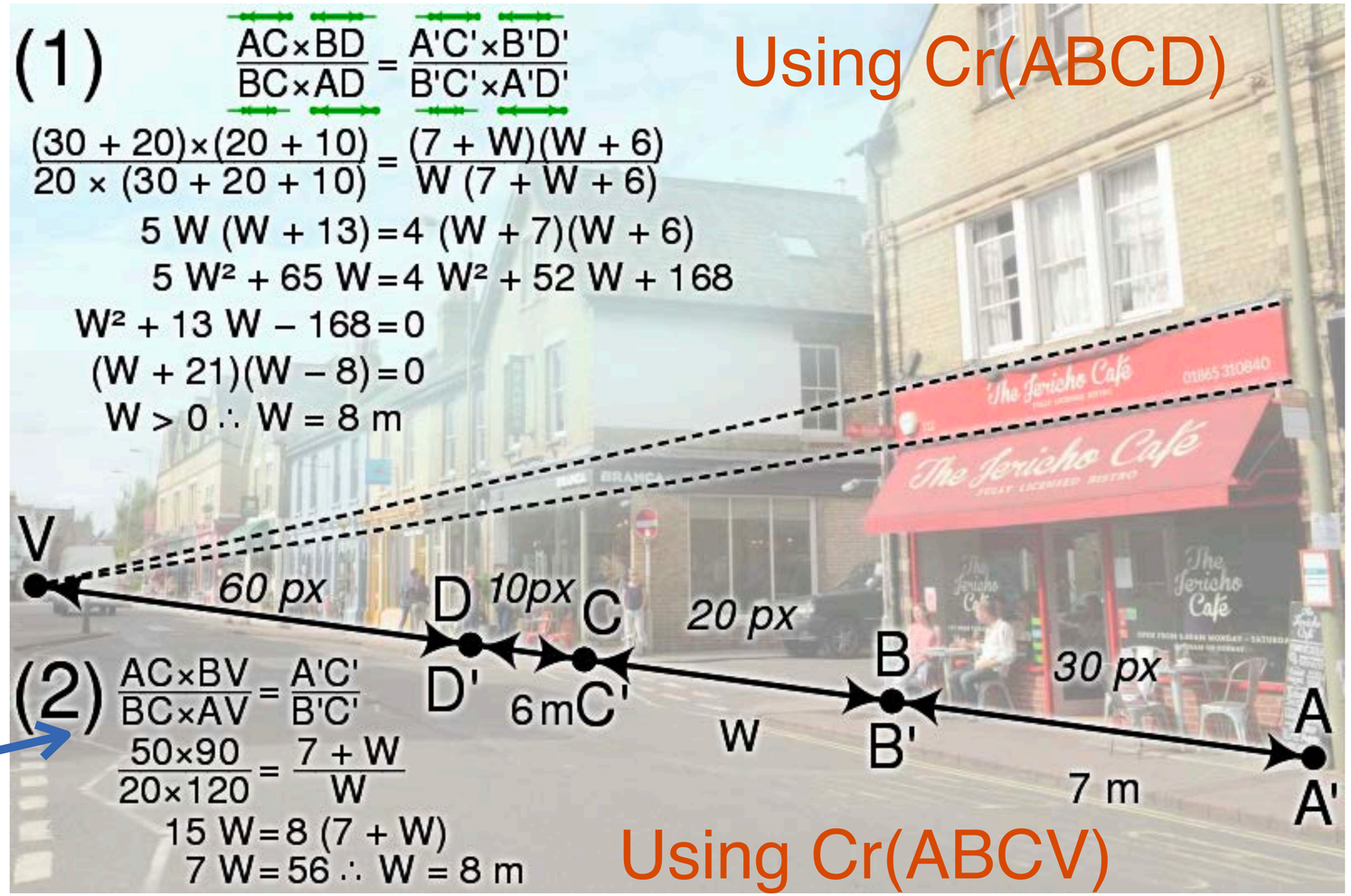
$$\frac{D-C}{A-D} = \frac{1-\frac{C}{D}}{\frac{A}{D}-1} \xrightarrow{D\to\infty} \frac{1-0}{0-1} = -1$$

$$= \frac{B-A}{C-B}\cdot(-1) = -1$$

=1 (equidistance)

Application of cross ratio

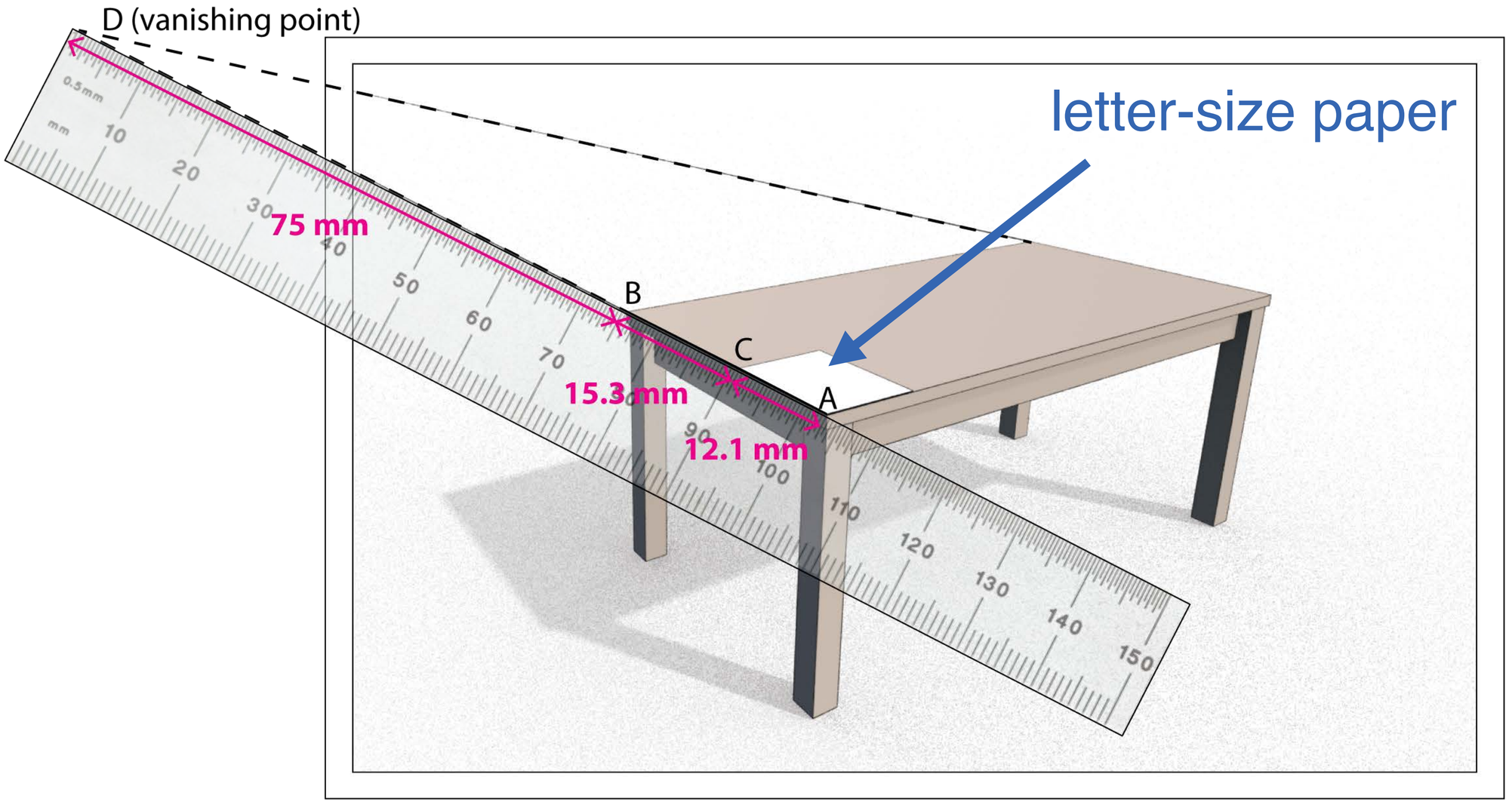
- Let A'B'C'D'V' denote pts in real life, and ABCDV corresponding pts in photograph
- Knowing A'B' = 7m, and all distances between ABCV, we can compute W = distance between B'C'.



When V' is at infinity: $\frac{\overline{B'V'}}{\overline{A'V'}}=1$

Wikipedia: Cross-ratio

Application of cross ratio



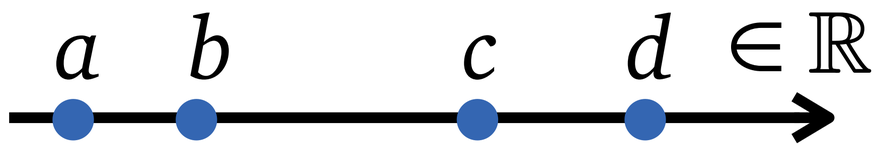
What's the width of the table in real life?

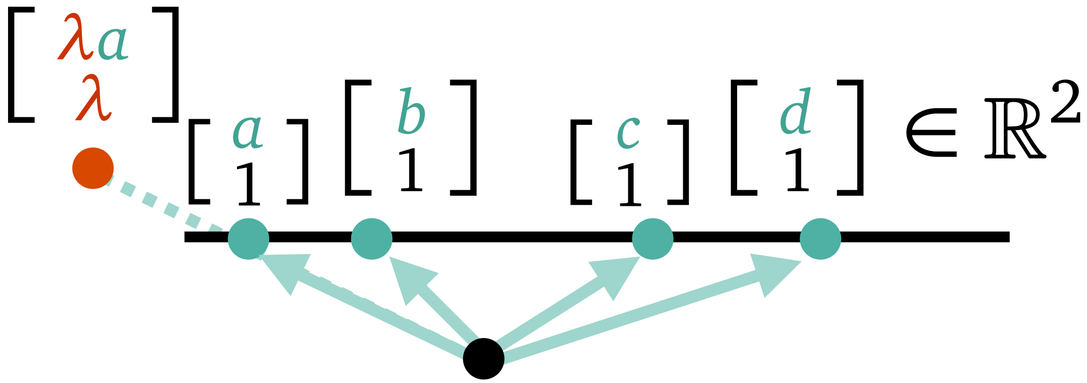
Appendix: Proof of projective invariance of cross ratio

Proof of proj. invariance of cross ratio

- On 1D, the **cross ratio** of 4 points

$$\mathrm{Cr}(a,b,c,d) = \frac{(b-a)(d-c)}{(c-b)(a-d)}$$





- Lift it to homogeneous coordinate

$$\mathrm{Cr}(a,b,c,d) = \frac{\begin{vmatrix} b & a \\ 1 & 1 \end{vmatrix}\begin{vmatrix} d & c \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} c & b \\ 1 & 1 \end{vmatrix}\begin{vmatrix} a & d \\ 1 & 1 \end{vmatrix}}$$

(2x2 determinants)

- This expression is consistent even if we scale individual point

$$\frac{\begin{vmatrix} b & \lambda a \\ 1 & \lambda \end{vmatrix}\begin{vmatrix} d & c \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} c & b \\ 1 & 1 \end{vmatrix}\begin{vmatrix} \lambda a & d \\ \lambda & 1 \end{vmatrix}} = \frac{\lambda\begin{vmatrix} b & a \\ 1 & 1 \end{vmatrix}\begin{vmatrix} d & c \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} c & b \\ 1 & 1 \end{vmatrix}\lambda\begin{vmatrix} a & d \\ 1 & 1 \end{vmatrix}} = \frac{\begin{vmatrix} b & a \\ 1 & 1 \end{vmatrix}\begin{vmatrix} d & c \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} c & b \\ 1 & 1 \end{vmatrix}\begin{vmatrix} a & d \\ 1 & 1 \end{vmatrix}}$$

Proof of proj. invariance of cross ratio

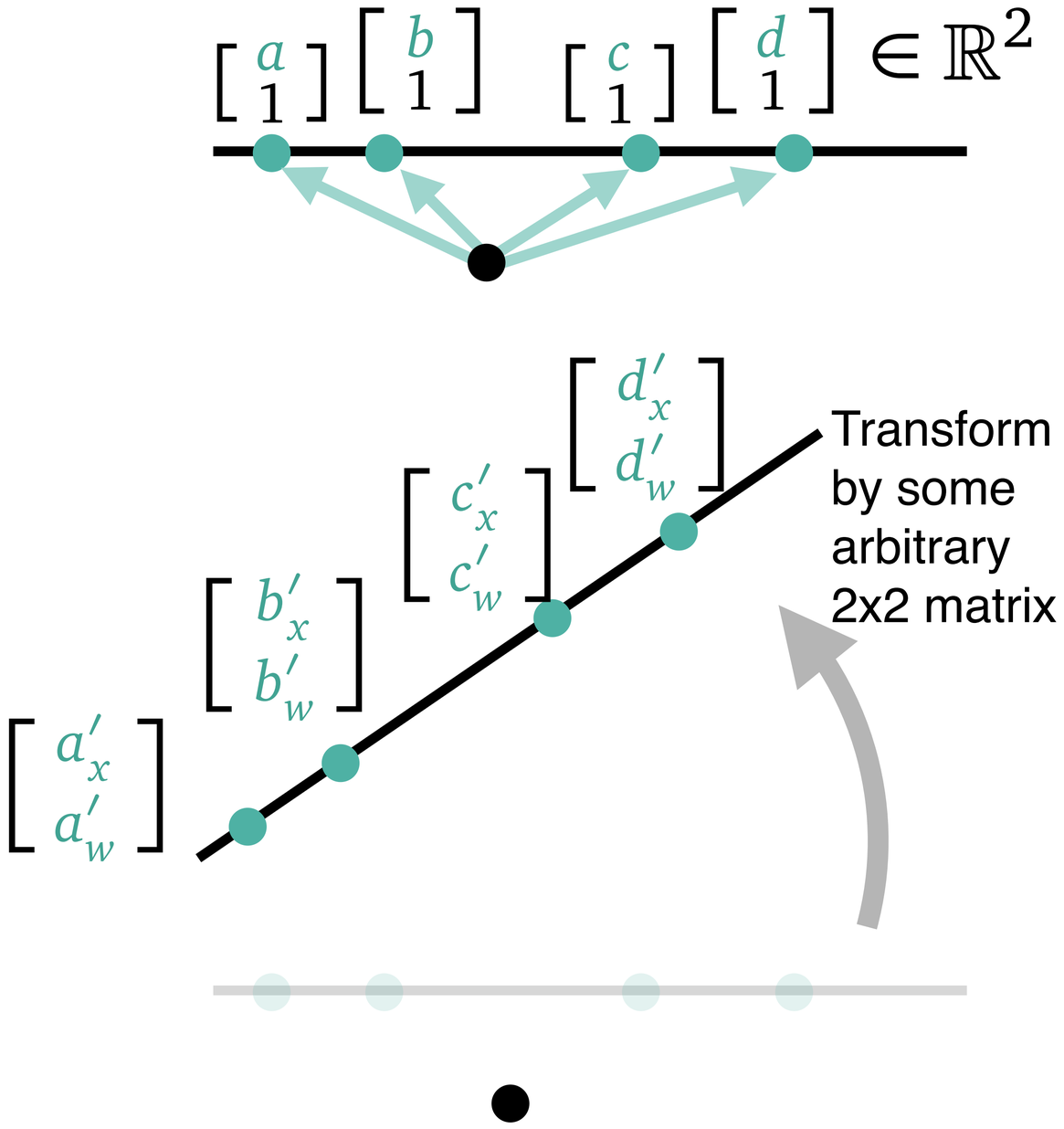
- On 1D, the **cross ratio** of 4 points

$$\mathrm{Cr}(a,b,c,d) = \frac{\begin{vmatrix} b & a \\ 1 & 1 \end{vmatrix}\begin{vmatrix} d & c \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} c & b \\ 1 & 1 \end{vmatrix}\begin{vmatrix} a & d \\ 1 & 1 \end{vmatrix}}$$

- Consider arbitrary transformation

$$\begin{bmatrix} a'_x & b'_x & c'_x & d'_x \\ a'_w & b'_w & c'_w & d'_w \end{bmatrix} = \underbrace{\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}}_{\mathbf{M}} \begin{bmatrix} a & b & c & d \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a'_x & b'_x & c'_x & d'_x \\ a'_w & b'_w & c'_w & d'_w \end{bmatrix} = [\mathbf{M}] \begin{bmatrix} a & b & c & d \\ 1 & 1 & 1 & 1 \end{bmatrix}$$



Proof of proj. invariance of cross ratio

- On 1D, the **cross ratio** of 4 points

$$\mathrm{Cr}(a,b,c,d) = \frac{\begin{vmatrix} b & a \\ 1 & 1 \end{vmatrix} \begin{vmatrix} d & c \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} c & b \\ 1 & 1 \end{vmatrix} \begin{vmatrix} a & d \\ 1 & 1 \end{vmatrix}}$$

- Consider arbitrary transformation

$$\begin{bmatrix} a'_x & b'_x & c'_x & d'_x \\ a'_w & b'_w & c'_w & d'_w \end{bmatrix} = [\mathbf{M}] \begin{bmatrix} a & b & c & d \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

- We can slice/swizzle the channels to get, for example,

$$\begin{bmatrix} b'_x & a'_x \\ b'_w & a'_w \end{bmatrix} = [\mathbf{M}] \begin{bmatrix} b & a \\ 1 & 1 \end{bmatrix} \quad \Longrightarrow \quad \begin{vmatrix} b'_x & a'_x \\ b'_w & a'_w \end{vmatrix} = \underbrace{\left|\mathbf{M}\right|}_{=:m} \begin{vmatrix} b & a \\ 1 & 1 \end{vmatrix}$$

- Fact: determinant distributes over matrix multiplication

Proof of proj. invariance of cross ratio

$$\mathrm{Cr}(a,b,c,d) = \frac{\begin{vmatrix} b & a \\ 1 & 1 \end{vmatrix}\begin{vmatrix} d & c \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} c & b \\ 1 & 1 \end{vmatrix}\begin{vmatrix} a & d \\ 1 & 1 \end{vmatrix}}$$

- Consider arbitrary transformation

$$\begin{bmatrix} a'_x & b'_x & c'_x & d'_x \\ a'_w & b'_w & c'_w & d'_w \end{bmatrix} = [\mathbf{M}] \begin{bmatrix} a & b & c & d \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \Longrightarrow \quad \begin{vmatrix} b'_x & a'_x \\ b'_w & a'_w \end{vmatrix} = \underbrace{|\mathbf{M}|}_{=:m} \begin{vmatrix} b & a \\ 1 & 1 \end{vmatrix} \text{ etc}$$

- The new cross ratio is

$$\mathrm{Cr}' = \frac{\begin{vmatrix} b'_x & a'_x \\ b'_w & a'_w \end{vmatrix}\begin{vmatrix} d'_x & c'_x \\ d'_w & c'_w \end{vmatrix}}{\begin{vmatrix} c'_x & b'_x \\ c'_w & b'_w \end{vmatrix}\begin{vmatrix} a'_x & d'_x \\ a'_w & d'_w \end{vmatrix}} = \frac{m\begin{vmatrix} b & a \\ 1 & 1 \end{vmatrix} m\begin{vmatrix} d & c \\ 1 & 1 \end{vmatrix}}{m\begin{vmatrix} c & b \\ 1 & 1 \end{vmatrix} m\begin{vmatrix} a & d \\ 1 & 1 \end{vmatrix}} = \frac{\begin{vmatrix} b & a \\ 1 & 1 \end{vmatrix}\begin{vmatrix} d & c \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} c & b \\ 1 & 1 \end{vmatrix}\begin{vmatrix} a & d \\ 1 & 1 \end{vmatrix}} = \mathrm{Cr}$$

QED