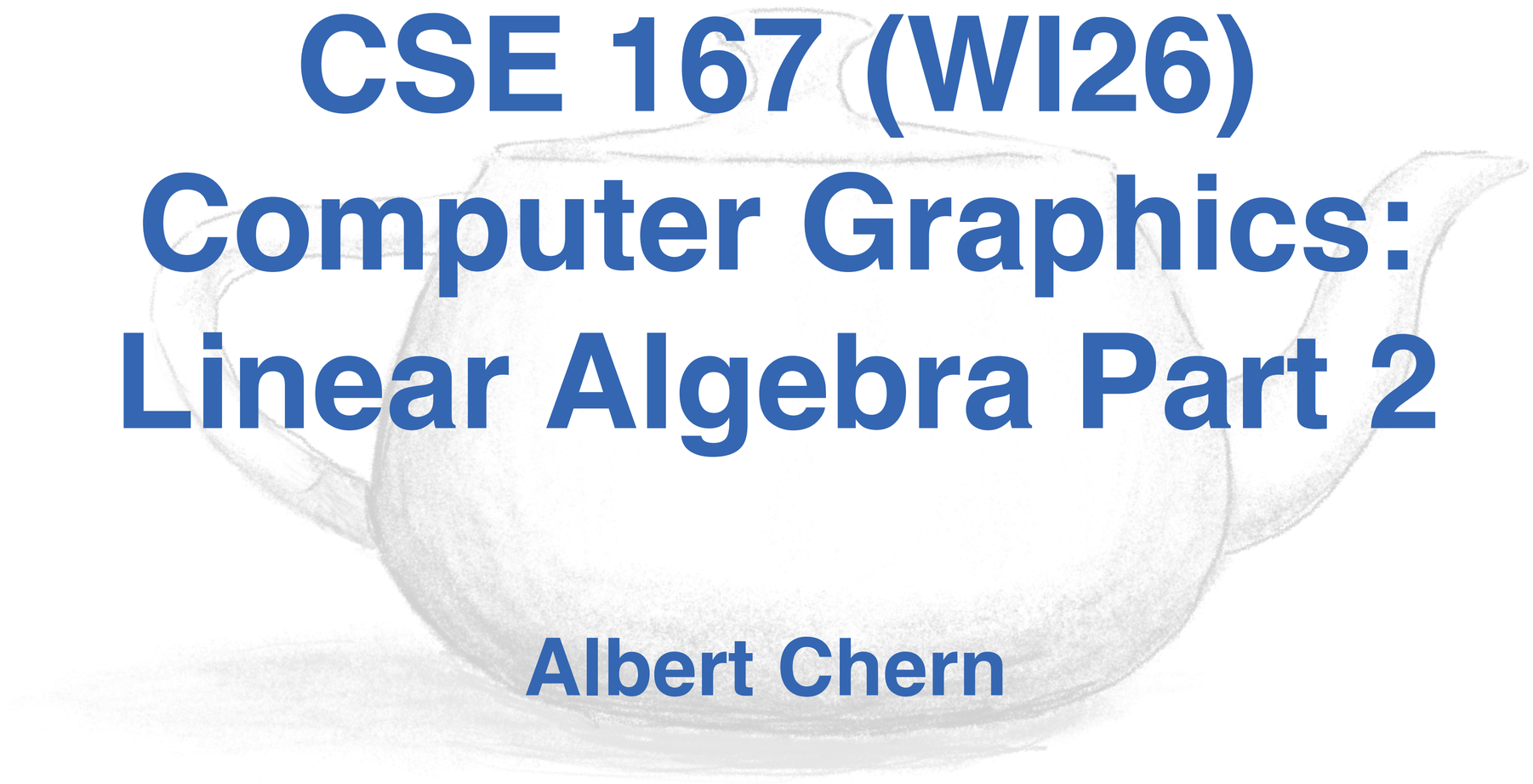
CSE 167 (WI26)
Computer Graphics:
Linear Algebra Part 2



Albert Chern

UC San Diego
JACOBS SCHOOL OF ENGINEERING

Number with unit; Vector with basis

- Number with unit; vector with basis
- Mathematics behind vectors & basis
- Change of basis
- Linear transformations in graphics

More “precise” linear algebra

- In Part 1, we discussed that each vector $\mathbf{v} = \begin{bmatrix} v_1 \\ | \\ v_n \end{bmatrix}$ is
  - an array of numbers;
  - an arrow pointing from origin to a point in *n*-D space.
- The 1st description is a set of numerical values, while the 2nd description is a geometrical/physical object.
- It is not always clear how to convert the two descriptions.
- Let’s study a similar question: how do we describe a physical measurement (e.g. length, weight,…)

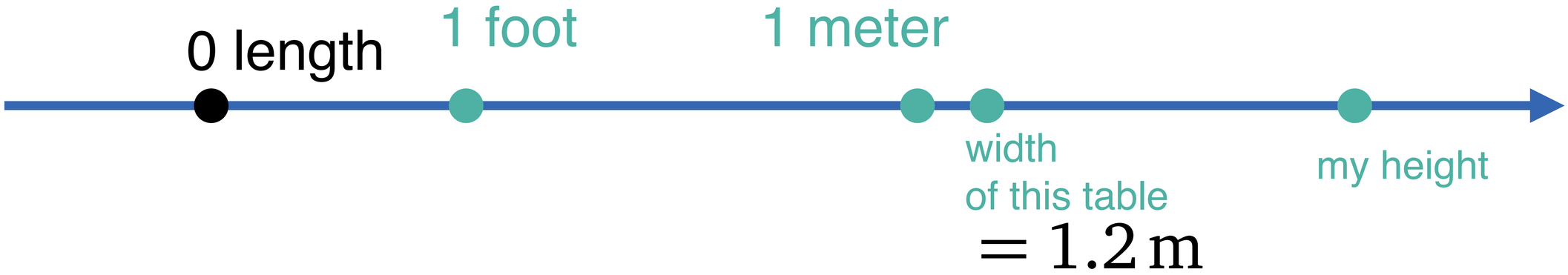
Physical measurements

- How do we describe a length measurement?
  - Examples: 2 meters, 5 inches, 7 miles, …
  - A **measurement** is a number together with a unit.
- We can add length measurements
  - Concatenate the two lengths and measure the total length.
  - Numerically, may need unit conversion:

$$\begin{aligned} 2\,\text{mile} + 3\text{km} &= 2 \cdot 1.6\,\text{km} + 3\text{km} \\ &= (2 \cdot 1.6 + 3)\,\text{km} = 6.2\,\text{km} \end{aligned}$$

Physical measurements

- We can add length measurements
  - Even if the unit hasn't matched yet (just require unit conversion)
- We cannot add length with mass (different physical measurements)
- Same type of physical measurement (e.g. length) form a 1D space:
  - Every point in this space represents a length measurement.
  - A **unit** is any *non-zero* point used as a reference to turn any length to a number.



Vectors

- Vectors are arrows that can add and scale.
- Same type of vectors form a space, called a **vector space**.
- Every point in a vector space represents a vector.
- A **basis** is like a **unit** that is used as a reference to convert vectors to an array of numbers.
- When conveying measurement, we cannot just say a number without also saying the unit.
  Similarly, when conveying a geometric arrow, we cannot just give an array of numbers without mentioning the basis.
- Between two units, we have a number to convert the two: $\frac{\text{mile}}{\text{km}} = 1.6$
- Between two basis, we have a matrix to convert the two.

Explicitly writing basis

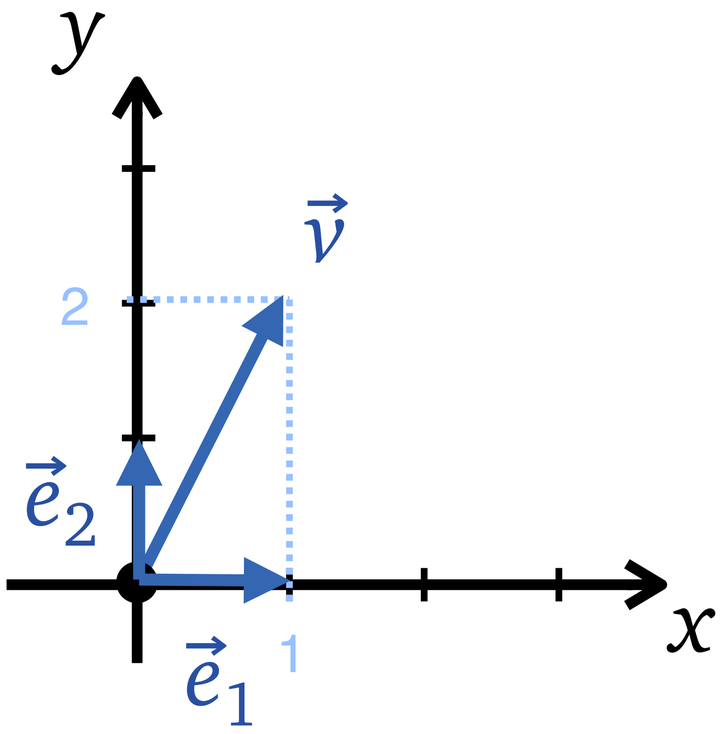
- In part 1, we represent $\vec{v}$ by an array of numbers $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
- To be more precise, there are **basis vectors** $\vec{e}_1$, $\vec{e}_2$ of the coordinate system, and

$$\vec{v} = 1\,\vec{e}_1 + 2\,\vec{e}_2$$

- Treat basis like "physical unit" accompanying array of numbers

$$\vec{v} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

unit *array of numbers*



- The basis is on the left, and the column of numbers is on the right.

Mathematics behind vectors and basis

- Number with unit; vector with basis
- Mathematics behind vectors & basis
- Change of basis
- Linear transformations in graphics

Vector space

Definition

A **(real) vector space** V is a set (whose elements are called **vectors** $\vec{v} \in V$) with two operations:

- **Addition** $(\vec{u} \in V, \vec{v} \in V) \mapsto \vec{u} + \vec{v} \in V$
- **Scaling** $(a \in \mathbb{R}, \vec{v} \in V) \mapsto a\vec{v} \in V$

Vector space

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  - Associative $(\vec{u} + \vec{v}) + \vec{w} = u + (\vec{v} + \vec{w})$
  - Commutative $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
  - There exists $\vec{0} \in V$ so that $\vec{0} + \vec{v} = \vec{v}$ for all $\vec{v} \in V$
  - For each $\vec{v} \in V$ there is $(-\vec{v}) \in V$ so that $\vec{v} + (-\vec{v}) = \vec{0}$
- **Scaling** $(a \in \mathbb{R}, \vec{v} \in V) \mapsto a\vec{v} \in V$

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- **Scaling** $(a \in \mathbb{R}, \vec{v} \in V) \mapsto a\vec{v} \in V$
  - Distributive law

$$(a + b)\vec{v} = a\vec{v} + b\vec{v}$$

$$a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$$

Vector space

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Main usage of the structure: **linear combination**

$$a\vec{u} + b\vec{v} + \cdots + c\vec{w}$$

Vector space

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Example Array of n real numbers form a real vector space

$$V = \mathbb{R}^n = \left\{ \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \middle| \, v_1, \ldots, v_n \in \mathbb{R} \right\}$$

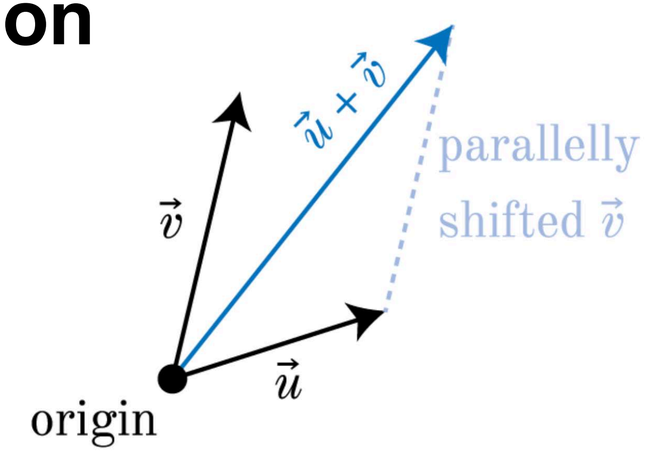
with element-wise addition and scaling.

Vector space

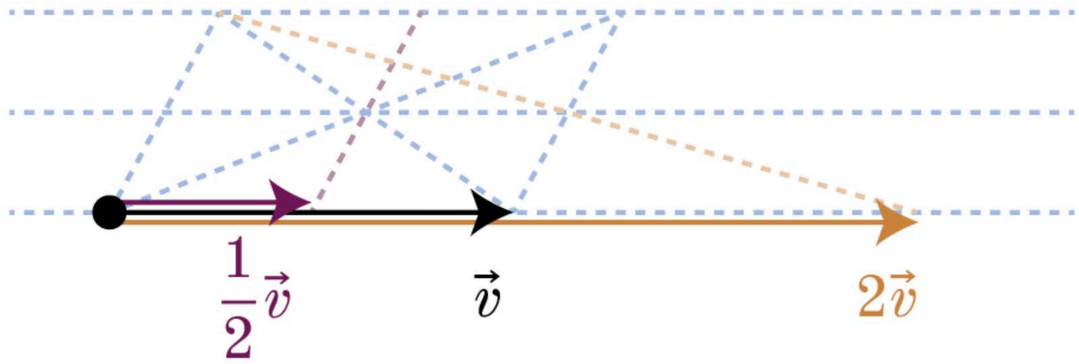
Example Arrows based at a point in a space (in which we can construct parallel lines) form a real vector space.

$$V = \left\{ \vec{v} = \right\}$$

- **Addition**



- **Scaling**



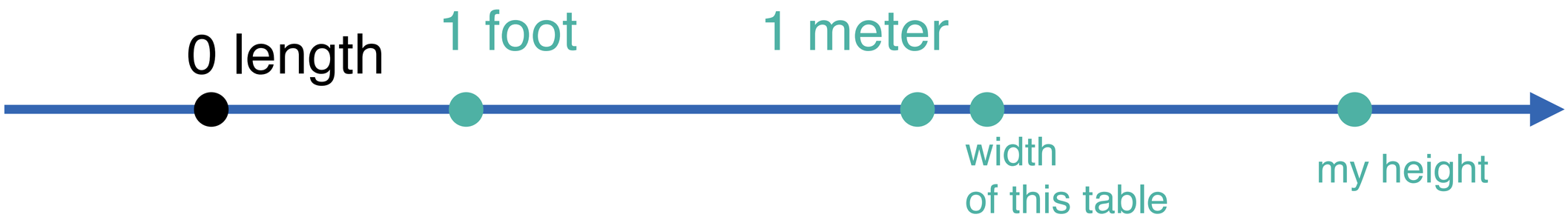
The addition and scaling structures are inherited from parallelism

More examples of vector spaces

Example The set of all cubic polynomials is a vector space.

$$P = \{at^3 + bt^2 + ct + d \,|\, a, b, c, d \in \mathbb{R}\}$$

Example The collection of all length measurements is a vector space.



Example The set of all complex numbers is a real vector space

$$\mathbb{C} = \{a + b\,\mathrm{i} \,|\, a, b \in \mathbb{R}\}$$

Linear independence

- Let V be any vector space.

Definition

A set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in V$ are **linearly independent**

if none of them can be written as a linear combination of the other ones in the list.

Linear independence

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- Example $V = \mathbb{R}^3$

$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are not linearly independent

since $\mathbf{v}_3 = \mathbf{v}_1 - \mathbf{v}_2$

These 3 vectors lie in the same 2D plane.

Basis

- Let V be any vector space.

Definition

A **basis** for V is a "maximal" linearly independent set of vectors

$$\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n \in V$$

That is, they are linearly independent, and any vector $\vec{v} \in V$ would make the list $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n, \vec{v}$ not linearly independent.

- **Consequence**

For each $\vec{v} \in V$ there exists a *unique* array of numbers $v_1, \ldots, v_n \in \mathbb{R}$ such that

$$\vec{v} = v_1\vec{e}_1 + \cdots + v_n\vec{e}_n$$

Basis

- Let V be any vector space.

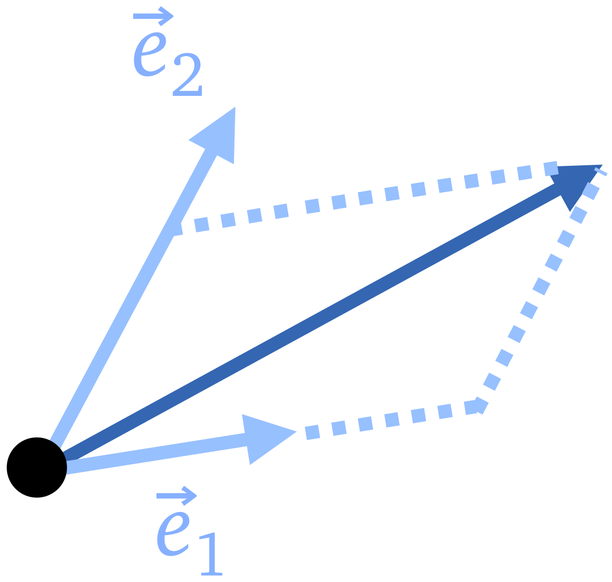
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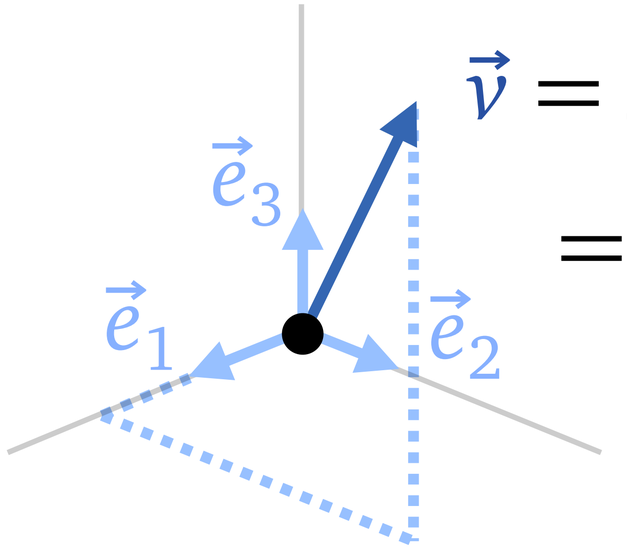
That is, they are linearly independent, and any vector $\vec{v} \in V$ would make the list $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n, \vec{v}$ not linearly independent.

Definition The size n of any basis for V is called the **dimension** of the vector space V.

Basis



$$\vec{v} = 2\vec{e}_1 + 0.7\vec{e}_2 = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} 2 \\ 0.7 \end{bmatrix}$$



$$\vec{v} = 2\vec{e}_1 + 3\vec{e}_2 + 4\vec{e}_3 = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

Basis

A basis bridges the geometric (vector space) and algebraic (array of numbers) aspects of vectors

geometric vector

algebraic array of numbers

$$\vec{v} = \begin{bmatrix} \vec{e}_1 & \cdots & \vec{e}_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \vec{\mathbf{e}}^{\mathsf{T}} \mathbf{v}$$

basis

Change of basis

- Number with unit; vector with basis
- Mathematics behind vectors & basis
- Change of basis
- Linear transformations in graphics

Change of basis

$$\vec{v} = \begin{bmatrix} \vec{e}_1 & \cdots & \vec{e}_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \vec{\mathbf{e}}^{\mathsf{T}}\mathbf{v}$$

a vector in vector space

basis

coefficients relative to the basis

$$= \begin{bmatrix} \vec{f}_1 & \cdots & \vec{f}_n \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \vec{\mathbf{f}}^{\mathsf{T}}\mathbf{w}$$

another basis

coefficients have changed

- How are the coefficients changed?

Change of basis

$$\vec{v} = \vec{\mathbf{e}}^{\mathsf{T}}\mathbf{v} = \vec{\mathbf{f}}^{\mathsf{T}}\mathbf{w}$$

- Example for n=2

  Given a relationship between the bases

$$\vec{e}_1 = \vec{f}_1 + \vec{f}_2$$
$$\vec{e}_2 = \vec{f}_1 - \vec{f}_2$$

  Write them in a matrix form *with matrix acting from the right*

$$\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} = \begin{bmatrix}\vec{f}_1 & \vec{f}_2\end{bmatrix}\begin{bmatrix} & \end{bmatrix}$$

Change of basis

$$\vec{v} = \vec{\mathbf{e}}^{\mathsf{T}}\mathbf{v} = \vec{\mathbf{f}}^{\mathsf{T}}\mathbf{w}$$

- Example for n=2

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$$\vec{e}_1 = \vec{f}_1 + \vec{f}_2$$
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  Write them in a matrix form *with matrix acting from the right*

$$\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} = \begin{bmatrix}\vec{f}_1 & \vec{f}_2\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

  Shorthand:

$$\vec{\mathbf{e}}^{\mathsf{T}} = \vec{\mathbf{f}}^{\mathsf{T}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \quad \text{and} \quad \vec{\mathbf{e}}^{\mathsf{T}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}^{-1} = \vec{\mathbf{f}}^{\mathsf{T}}$$

Change of basis

$$\vec{v} = \vec{\mathbf{e}}^{\mathsf{T}}\mathbf{v} = \vec{\mathbf{f}}^{\mathsf{T}}\mathbf{w}$$

- Example for n=2

  Given a relationship between the bases

$$\vec{e}_1 = \vec{f}_1 + \vec{f}_2$$
$$\vec{e}_2 = \vec{f}_1 - \vec{f}_2$$

$$\vec{\mathbf{e}}^{\mathsf{T}} = \vec{\mathbf{f}}^{\mathsf{T}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \vec{\mathbf{e}}^{\mathsf{T}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \vec{\mathbf{f}}^{\mathsf{T}}$$

  Simple substitution:

$$\vec{\mathbf{f}}^{\mathsf{T}}\mathbf{w} = \vec{\mathbf{e}}^{\mathsf{T}}\mathbf{v} = \vec{\mathbf{f}}^{\mathsf{T}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\mathbf{v}$$

Change of basis

$$\vec{v} = \vec{\mathbf{e}}^{\mathsf{T}}\mathbf{v} = \vec{\mathbf{f}}^{\mathsf{T}}\mathbf{w}$$

- Example for n=2

  Given a relationship between the bases

$$\vec{e}_1 = \vec{f}_1 + \vec{f}_2$$
$$\vec{e}_2 = \vec{f}_1 - \vec{f}_2$$

*matrix acts on basis from the **right***

$$\vec{\mathbf{e}}^{\mathsf{T}} = \vec{\mathbf{f}}^{\mathsf{T}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \vec{\mathbf{e}}^{\mathsf{T}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \vec{\mathbf{f}}^{\mathsf{T}}$$

Simple substitution:

*matrix acts on coefficients from the **left***

$$\vec{\mathbf{f}}^{\mathsf{T}}\mathbf{w} = \vec{\mathbf{f}}^{\mathsf{T}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\mathbf{v} \quad \Longrightarrow \quad \boxed{\mathbf{w} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\mathbf{v}}$$

Change of basis summary

- For every pair of bases $\vec{\mathbf{e}}$, $\vec{\mathbf{f}}$, there is a matrix $\mathbf{A}$ such that

$$\vec{\mathbf{e}}^{\mathsf{T}} = \vec{\mathbf{f}}^{\mathsf{T}}\mathbf{A} \qquad \text{equivalently} \qquad \vec{\mathbf{e}}^{\mathsf{T}}\mathbf{A}^{-1} = \vec{\mathbf{f}}^{\mathsf{T}}$$

- One fixed vector but written in two different bases:

$$\vec{v} = \vec{\mathbf{e}}^{\mathsf{T}}\mathbf{v} = \vec{\mathbf{f}}^{\mathsf{T}}\mathbf{w}$$

- By substitution, we get the transformation rule for the coefficients

$$\mathbf{w} = \mathbf{A}\mathbf{v} \qquad \mathbf{v} = \mathbf{A}^{-1}\mathbf{w}$$

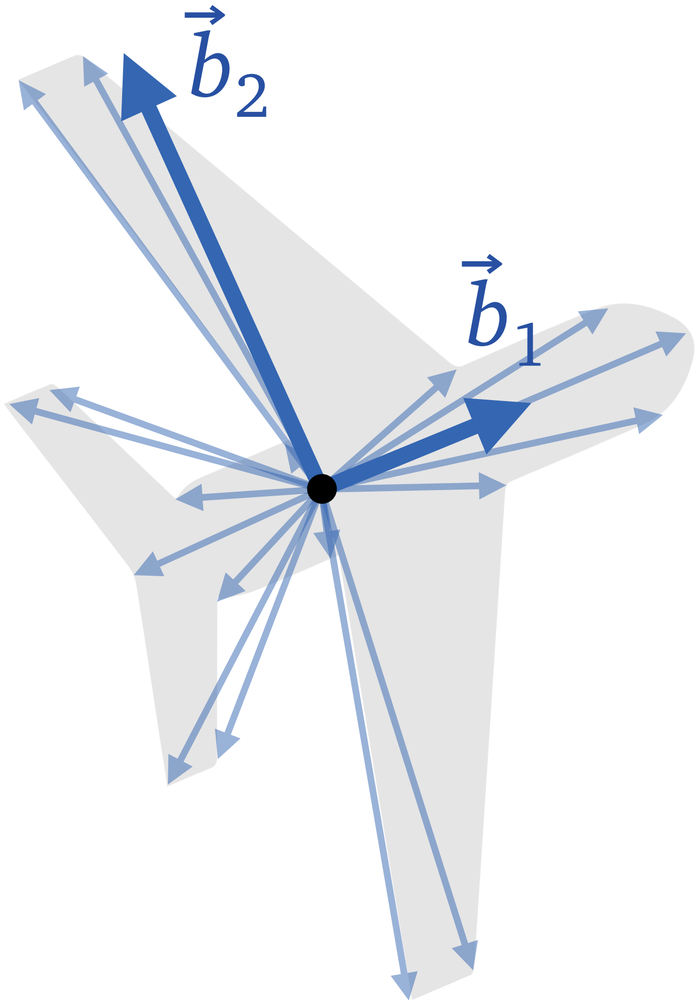
Linear Transformations in Computer Graphics

- Number with unit; vector with basis
- Mathematics behind vectors & basis
- Change of basis
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Linear transformations in graphics

- In a scene, geometric objects have vertex positions thought of as **geometric vectors** (displacements from the origin).

$$\vec{v} \in V$$



- When we store the vertex positions, we represent the vectors as **algebraic vectors**, i.e. tuples of real numbers.

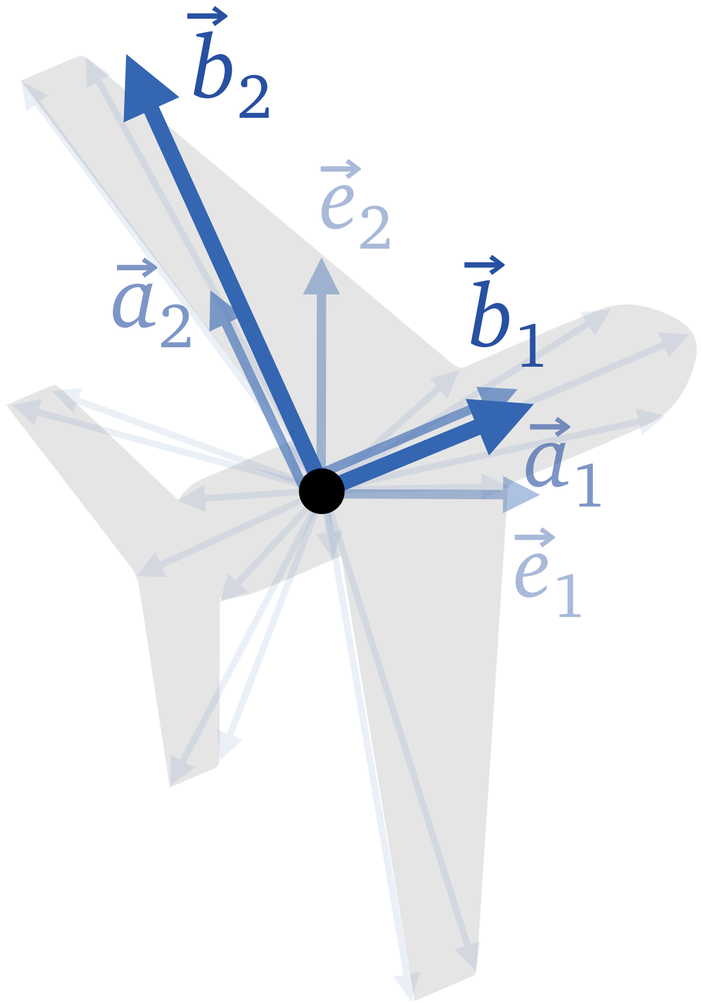
$$\mathbf{v} \in \mathbb{R}^n$$

- The conversion between the above two types of vectors require a basis.

$$\vec{v} = \vec{\mathbf{b}}^{\mathsf{T}}\,\mathbf{v}$$

Linear transformations in graphics

There are usually several sets of basis in one scene.



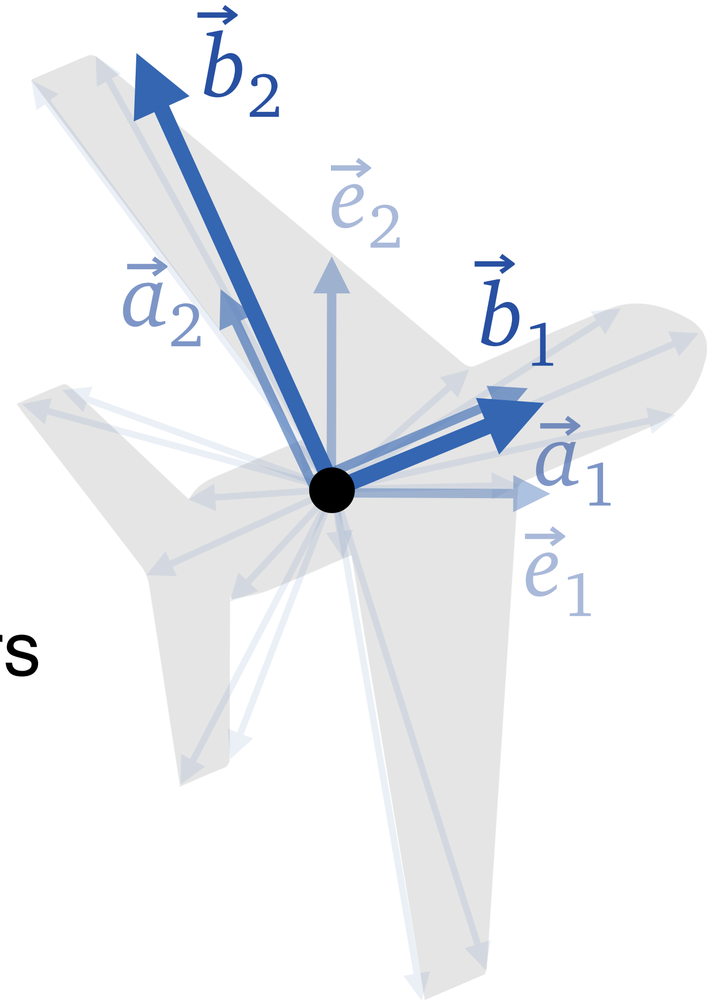
Linear transformations in graphics

- In this example, there are 3 sets of basis $\vec{\mathbf{e}}$, $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$.

- In order not to confuse ourselves for "which basis should we use to interpret an algebraic vector?" we think of 3 separate spaces of algebraic vectors:

$$\mathbb{R}^n_{\vec{\mathbf{e}}} \qquad \mathbb{R}^n_{\vec{\mathbf{a}}} \qquad \mathbb{R}^n_{\vec{\mathbf{b}}}$$

$\mathbf{v} \in \mathbb{R}^n_{\vec{\mathbf{b}}}$ means that this tuple of numbers is to be realized as a geometric vector using the $\vec{\mathbf{b}}$ basis: $\vec{\mathbf{b}}^{\mathsf{T}}\mathbf{v}$

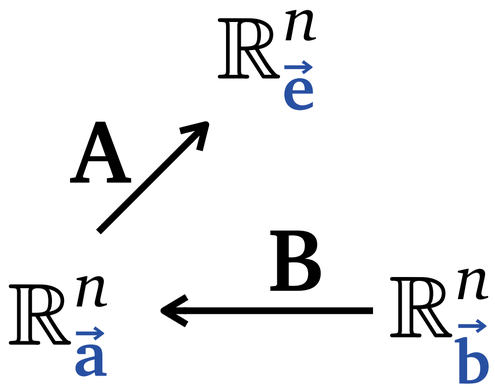


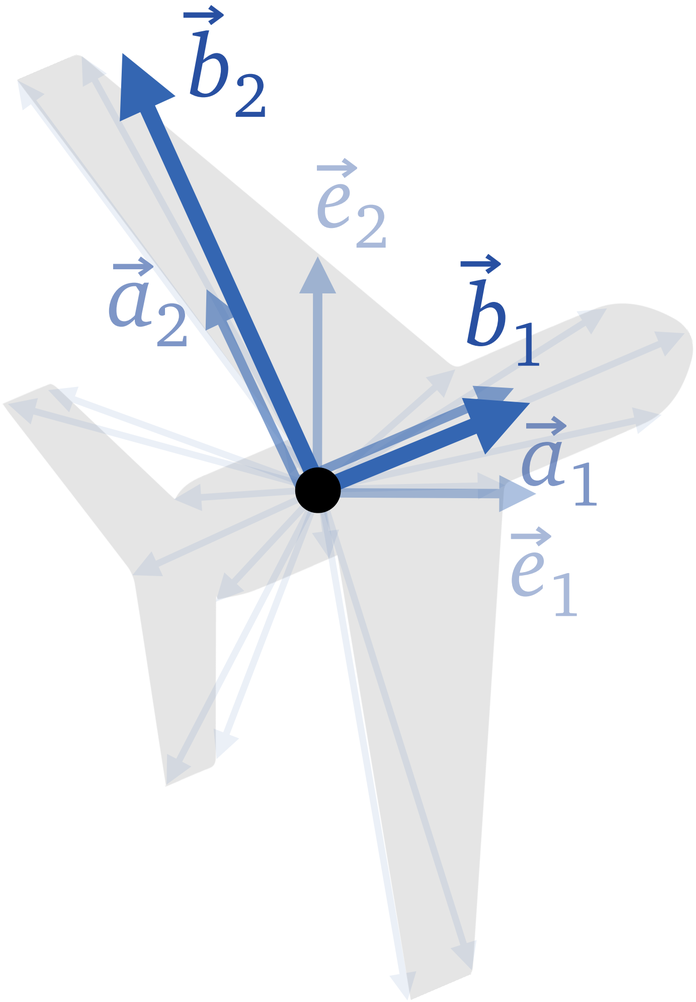
Linear transformations in graphics

- We describe these bases “relatively.” The information that we will store is the **matrix** between the bases.

$$\vec{\mathbf{a}}^{\mathsf{T}} = \vec{\mathbf{e}}^{\mathsf{T}}\mathbf{A}$$
$$\vec{\mathbf{b}}^{\mathsf{T}} = \vec{\mathbf{a}}^{\mathsf{T}}\mathbf{B}$$

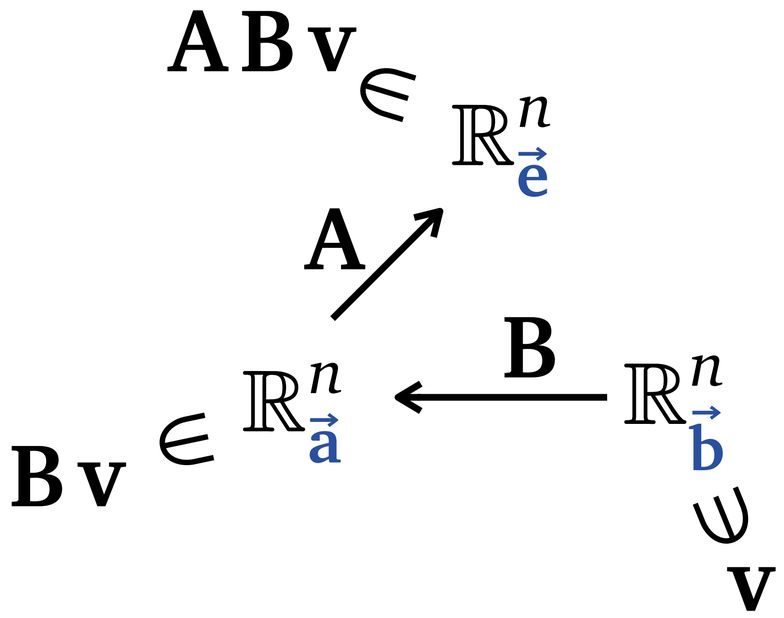
- The above relationship translates to the following diagram.

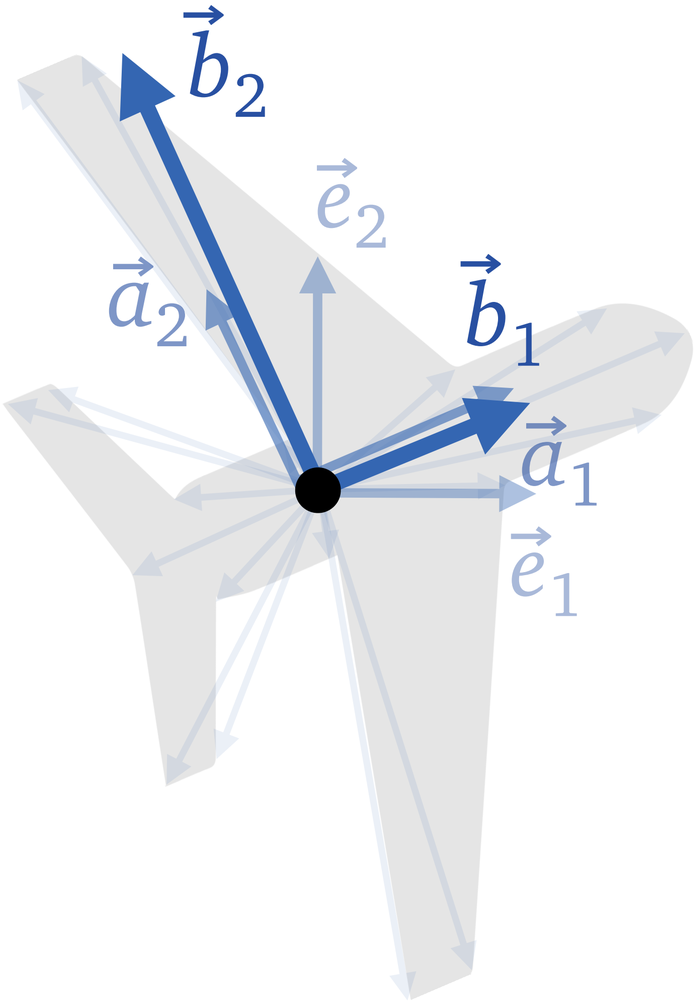




Linear transformations in graphics

$$\vec{\mathbf{a}}^{\mathsf{T}} = \vec{\mathbf{e}}^{\mathsf{T}}\mathbf{A}$$
$$\vec{\mathbf{b}}^{\mathsf{T}} = \vec{\mathbf{a}}^{\mathsf{T}}\mathbf{B}$$





Linear transformations in graphics

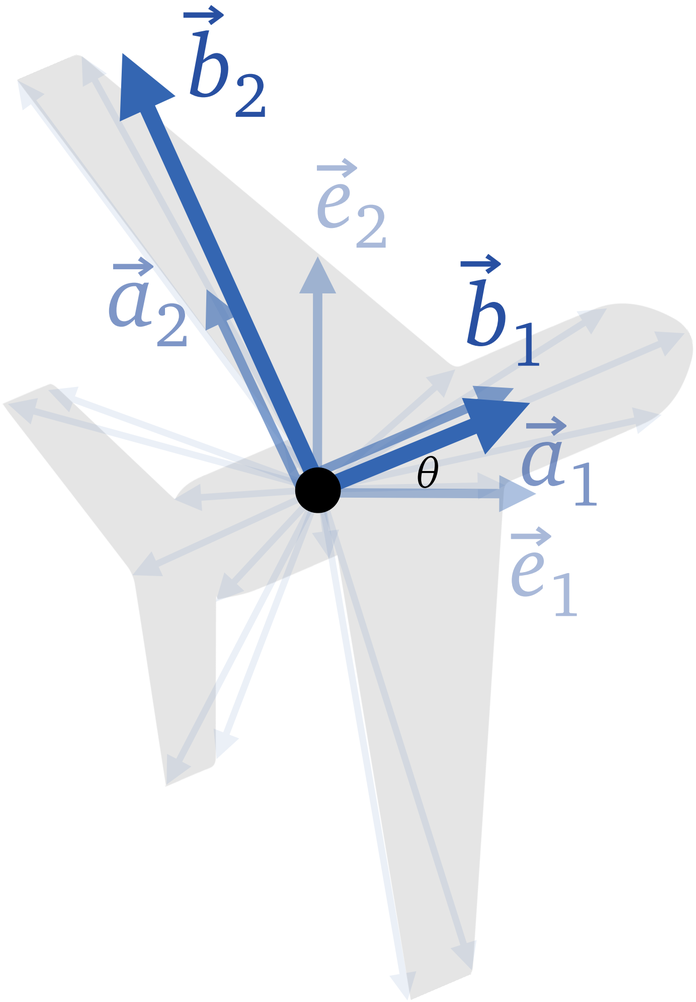
Example

$$\vec{\mathbf{a}}^{\mathsf{T}} = \vec{\mathbf{e}}^{\mathsf{T}} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\vec{\mathbf{b}}^{\mathsf{T}} = \vec{\mathbf{a}}^{\mathsf{T}} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

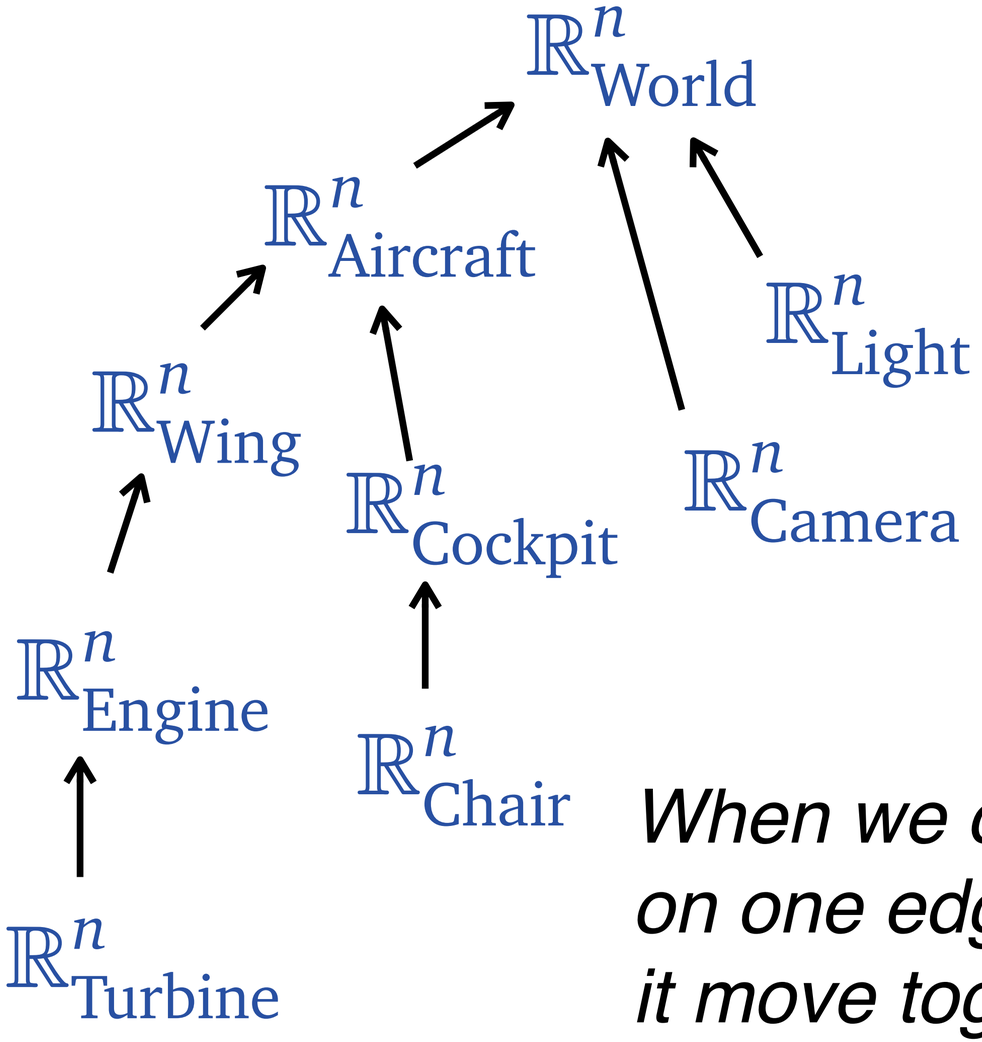
$$\mathbb{R}^2_{\vec{\mathbf{b}}} \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}} \mathbb{R}^2_{\vec{\mathbf{a}}} \xrightarrow{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}} \mathbb{R}^2_{\vec{\mathbf{e}}}$$

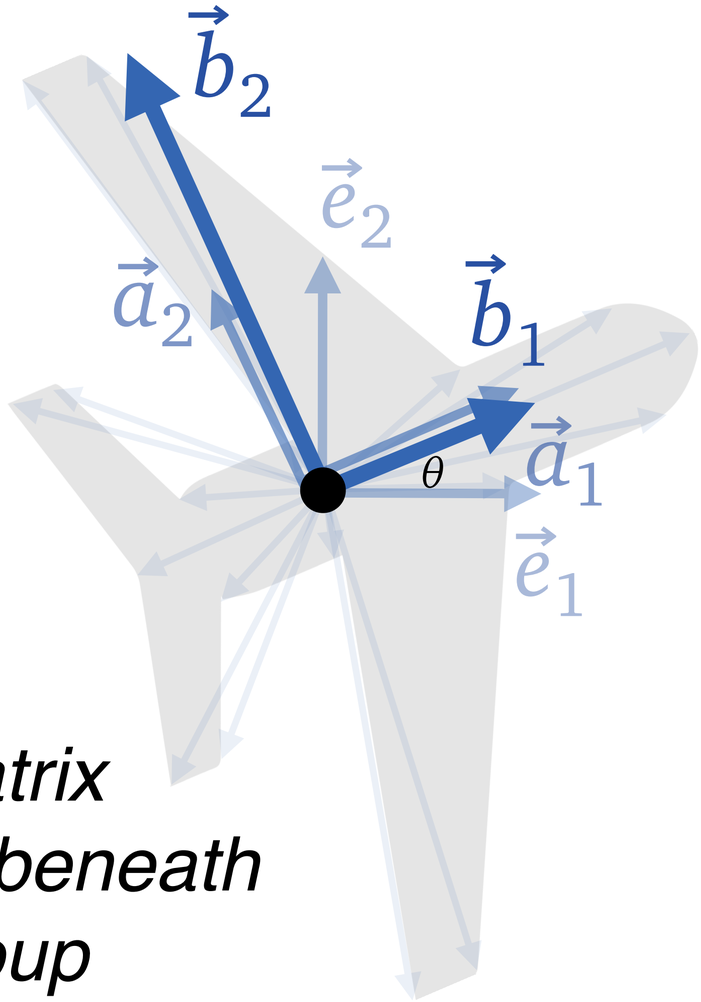
$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



Hierarchical modeling

- A scene graph





When we change the matrix on one edge, all objects beneath it move together as a group

Next topic: 3D rotations