CSE 167 (FA22)
Computer Graphics: Linear Algebra

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Linear algebra in graphics
Linear algebra in graphics

- Geometry of the 3D world.

**Often not taught in your linear algebra class:**

- It makes distinctions between **positions** and **directions**.
- In 3D, a **4th coordinate** is needed to unify the algebra for positions, vectors, and their perspective transformations.
Linear algebra in graphics

- Matrix algebra
- Vectors, change of basis, linear transformations
- 3D rotations
- Affine geometry
- Projective geometry
Matrix algebra

- Matrix algebra
- Vectors
- Change of basis
- Linear transformations
  - Linear transformations in graphics
- Inner product
- Special 2D linear transforms
• A **matrix** is a rectangular array of **objects** (usually numbers)

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\]

is called an m-by-n matrix.
• A **matrix** is a rectangular array of objects *(usually numbers)*

\[
A = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\]

is called an m-by-n matrix.

• In your OpenGL code (**GLSL** or **GLM**) \[A[j][i] = a_{ij}\]

  j-th column
Matrix addition

• If the objects in the matrices have the operator “+,” then we can add matrices of the same size (element-wise addition).

\[
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22} \\
a_{31} & a_{32}
\end{bmatrix}
+ 
\begin{bmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22} \\
b_{31} & b_{32}
\end{bmatrix}
= 
\begin{bmatrix}
a_{11} + b_{11} & a_{12} + b_{12} \\
a_{21} + b_{21} & a_{22} + b_{22} \\
a_{31} + b_{31} & a_{32} + b_{32}
\end{bmatrix}
\]
Matrix multiplication

- Suppose there is a multiplication between type1 and type2 that produces an object of type3, and there is an addition “+” for type3.

\[ a \cdot b = c \]

- Then we have a multiplication \( A B = C \) between type1 matrices and the type2 matrices that produce type3 matrices when the number of columns of \( A \) and the number of rows of \( B \) match.

\[
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22} \\
a_{31} & a_{32}
\end{bmatrix}
\begin{bmatrix}
b_{11} & b_{12} & b_{13} & b_{14} \\
b_{21} & b_{22} & b_{23} & b_{24}
\end{bmatrix}
= 
\begin{bmatrix}
c_{11} & c_{12} & c_{13} & c_{14} \\
c_{21} & c_{22} & c_{23} & c_{24} \\
c_{31} & c_{32} & c_{33} & c_{34}
\end{bmatrix}
\]
Matrix multiplication

- Then we have a multiplication $A B = C$ between type1 matrices and the type2 matrices that produce type3 matrices when the number of columns of $A$ and the number of rows of $B$ match.

$$
\begin{bmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22} \\
  a_{31} & a_{32}
\end{bmatrix}
\begin{bmatrix}
  b_{11} & b_{12} & b_{13} & b_{14} \\
  b_{21} & b_{22} & b_{23} & b_{24}
\end{bmatrix}
= 
\begin{bmatrix}
  c_{11} & c_{12} & c_{13} & c_{14} \\
  c_{21} & c_{22} & c_{23} & c_{24} \\
  c_{31} & c_{32} & c_{33} & c_{34}
\end{bmatrix}
$$

$$
c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}
$$
Example: Linear system of equations

- Matrix algebra allows us to write linear systems of equations concisely.

\[
\begin{cases}
x + 2y = 8 \\
x + y = 5
\end{cases}
\]

\[
\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}
\]
Associativity and commutativity

• A multiplication (within the same type) is called **associative** if

\[(ab)c = a(bc)\] for all \(a, b, c\) of the type.

• A multiplication is called **commutative** if

\[ab = ba\] for all \(a, b\) of the type.
Matrix algebra is associative

• If the matrix elements have associative multiplication, then the matrix multiplications are also associative.

\[(AB)C = A(BC)\]

• In general, matrix multiplications are not commutative (even if their elements have commutative multiplications)

\[AB \neq BA\]
Matrix transpose

- The **transpose** of a matrix is the matrix with the roles of column and row swapped.

\[
\begin{bmatrix}
 a_{11} & a_{12} & a_{13} \\
 a_{21} & a_{22} & a_{23}
\end{bmatrix}^T =
\begin{bmatrix}
 a_{11} & a_{21} \\
 a_{12} & a_{22} \\
 a_{13} & a_{23}
\end{bmatrix}
\]

\[
B = A^T, \quad b_{ij} = a_{ji}
\]

- Transposition reverses the order of multiplication:

\[
(AB \cdots C)^T = C^T \cdots B^T A^T
\]
Identity matrix

- The n-by-n identity matrix is given by

\[
I = \begin{bmatrix}
1 & & \\
& 1 & \\
& & \ddots \\
& & & 1
\end{bmatrix}
\]

where 1 is the element for which \( 1 \cdot a = a \cdot 1 = a \) for all \( a \) of the type.

- Property of identity matrix: \( IA = AI = A \) for any n-by-n matrix \( A \).
Matrix inversion

- Similar to taking reciprocal
  \[
  \frac{1}{a} \cdot a = a \cdot \frac{1}{a} = 1
  \]
- The inverse of a square matrix $A$ is the unique matrix $A^{-1}$ (if any exists) such that
  \[
  A^{-1}A = AA^{-1} = I
  \]
- Inversion reverses the order of multiplication
  \[
  (AB \cdots C)^{-1} = C^{-1} \cdots B^{-1}A^{-1}
  \]
Example: Linear system of equations

\[
\begin{cases}
    x + 2y = 8 \\
    x + y = 5
\end{cases}
\]

i.e.

\[
\begin{bmatrix}
    1 & 2 \\
    1 & 1
\end{bmatrix}
\begin{bmatrix}
    x \\
    y
\end{bmatrix}
=
\begin{bmatrix}
    8 \\
    5
\end{bmatrix}
\]

\[
\begin{bmatrix}
    1 & 2 \\
    1 & 1
\end{bmatrix}^{-1}
\begin{bmatrix}
    1 & 2 \\
    1 & 1
\end{bmatrix}
\begin{bmatrix}
    x \\
    y
\end{bmatrix}
=
\begin{bmatrix}
    1 & 2 \\
    1 & 1
\end{bmatrix}^{-1}
\begin{bmatrix}
    8 \\
    5
\end{bmatrix}
\]

\[
\begin{bmatrix}
    x \\
    y
\end{bmatrix}
=
\begin{bmatrix}
    1 & 2 \\
    1 & 1
\end{bmatrix}^{-1}
\begin{bmatrix}
    8 \\
    5
\end{bmatrix}
=
\begin{bmatrix}
    -1 & 2 \\
    1 & -1
\end{bmatrix}
\begin{bmatrix}
    8 \\
    5
\end{bmatrix}
=
\begin{bmatrix}
    2 \\
    3
\end{bmatrix}
\]

- There are efficient formulas for inverting small matrices (Exer2)
Summary (matrix algebra)

• Matrices are arrays of numbers \( A = [a_{ij}] \)

• Matrix multiplication: \( C = AB \) \( \iff \) \( c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \)

• Associative: \((AB)C = A(BC)\)

• Not necessarily commutative: \( AB \neq BA \)

• Transpose swaps row/col indices, and \((AB \cdots C)^T = C^T \cdots B^T A^T\)

• Inverse \( AA^{-1} = A^{-1} A = I \), and \((AB \cdots C)^{-1} = C^{-1} \cdots B^{-1} A^{-1}\)
  
  ▶ Only square matrices (\#rows=\#cols) can have inverse
  
  ▶ Some square matrix don’t have inverse. Invertible \( \iff \) nonzero determinant

\( A[j][i] \) in GLM/GLSL!
Vectors

- Matrix algebra
- Vectors
- Change of basis
- Linear transformations
- Linear transformations in graphics
- Inner product
- Special 2D linear transforms
Geometric and algebraic vectors

**Geometrically**
a vector is an arrow, i.e. it has a magnitude and a direction.

**Algebraically**
a vector is an array of numbers, i.e. an n-by-1 matrix.

\[
\begin{bmatrix}
  v_1 \\
  \vdots \\
  v_n
\end{bmatrix}
\]
Notations

Symbols accented with an arrow are arrows in space:

\[ \vec{v} = \]

Bold symbols are arrays of numbers:

\[ \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \]
Notations

Symbols accented with an arrow are arrows in space:

\[ \vec{v} = \vec{v}_1 \ldots \vec{v}_n \]

Bold symbols accented with an arrow are arrays of vectors.

\[ \vec{v} = \begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_n \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \]

Bold symbols are arrays of numbers:

\[ v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \]
Geometric and algebraic vectors

**Geometrically**

\[ \vec{v} = \cdot \cdot \cdot \]

**Algebraically**

\[ \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \]

*How are they related to each other?*

- Each of them forms a structured space called *vector space* sharing the same rules.
- The geometric vectors and algebraic vectors are mapped to each other using an array of vectors called *basis*. 

A (real) vector space (also called a real linear space) $V$ is a set of objects called **vectors** $\vec{v} \in V$ with the two operations

- **Addition** \((\vec{u} \in V, \vec{v} \in V) \mapsto \vec{u} + \vec{v} \in V\)

- **Scaling** \((a \in \mathbb{R}, \vec{v} \in V) \mapsto a\vec{v} \in V\)
Vector space

Definition

A (real) vector space (also called a real linear space) $V$ is a set of objects called vectors $\vec{v} \in V$ with the two operations

- **Addition** $(\vec{u}, \vec{v}) \mapsto \vec{u} + \vec{v} \in V$
  - Associative $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
  - Commutative $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
  - There exists $\vec{0} \in V$ so that $\vec{0} + \vec{v} = \vec{v}$ for all $\vec{v} \in V$
  - For each $\vec{v} \in V$ there is $(-\vec{v}) \in V$ so that $\vec{v} + (-\vec{v}) = \vec{0}$

- **Scaling** $(a, \vec{v}) \mapsto a\vec{v} \in V$
Vector space

Definition

A (real) vector space (also called a real linear space) \( V \) is a set of objects called vectors \( \vec{v} \in V \) with the two operations

- **Addition** \((\vec{u} \in V, \vec{v} \in V) \mapsto \vec{u} + \vec{v} \in V\)

- **Scaling** \((a \in \mathbb{R}, \vec{v} \in V) \mapsto a\vec{v} \in V\)
  - Distributive law
    \[
    (a + b)\vec{v} = a\vec{v} + b\vec{v}
    \]
    \[
    a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}
    \]
Vector space

Definition

A (real) vector space (also called a real linear space) $V$ is a set of objects called vectors $\vec{v} \in V$ with the two operations

- **Addition** \((\vec{u} \in V, \vec{v} \in V) \mapsto \vec{u} + \vec{v} \in V\)

- **Scaling** \((a \in \mathbb{R}, \vec{v} \in V) \mapsto a\vec{v} \in V\)

Main usage of the structure: linear combination

\[ a\vec{u} + b\vec{v} + \cdots + c\vec{w} \]
Definition

A (real) **vector space** (also called a real linear space) $V$ is a set of objects called **vectors** $\vec{v} \in V$ with the two operations

- **Addition** $(\vec{u} \in V, \vec{v} \in V) \mapsto \vec{u} + \vec{v} \in V$
- **Scaling** $(a \in \mathbb{R}, \vec{v} \in V) \mapsto a\vec{v} \in V$

Example  
Array of $n$ real numbers form a real vector space $V = \mathbb{R}^n = \left\{ \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \mid v_1, \ldots, v_n \in \mathbb{R} \right\}$ with element-wise addition and scaling.
Example  Arrows based at a point in a space (in which we can construct parallel lines) form a real vector space.

\[ V = \left\{ \vec{v} = \right\} \]

- Addition
- Scaling

The addition and scaling structures are inherited from parallelism
Basis

Definition

A list of vectors $\vec{e}_1, \ldots, \vec{e}_n \in V$ are said to form a basis for $V$ if for each $\vec{v} \in V$ there exists a unique array of numbers $\nu_1, \ldots, \nu_n \in \mathbb{R}$ such that

$$\vec{v} = \nu_1 \vec{e}_1 + \cdots + \nu_n \vec{e}_n$$

$\vec{v} = 2\vec{e}_1 + 0.7\vec{e}_2$
Definition

A list of vectors \( \vec{e}_1, \ldots, \vec{e}_n \in V \) are said to form a **basis** for \( V \) if for each \( \vec{v} \in V \) there exists a *unique* array of numbers \( \nu_1, \ldots, \nu_n \in \mathbb{R} \) such that

\[
\vec{v} = \nu_1 \vec{e}_1 + \cdots + \nu_n \vec{e}_n
\]
Basis

Definition
A list of vectors $\vec{e}_1, \ldots, \vec{e}_n \in V$ are said to form a **basis** for $V$ if for each $\vec{v} \in V$ there exists a **unique** array of numbers $\nu_1, \ldots, \nu_n \in \mathbb{R}$ such that

$$\vec{v} = \nu_1 \vec{e}_1 + \cdots + \nu_n \vec{e}_n$$

Property

None of $\vec{e}_k$ can be written as linear combination of other $\vec{e}_j$'s.

- The basis vectors must be **linearly independent**.
- The number of vectors in a basis is the **dimension** of the space.
Basis

Definition
A list of vectors $\vec{e}_1, \ldots, \vec{e}_n \in V$ are said to form a basis for $V$ if for each $\vec{v} \in V$ there exists a unique array of numbers $\nu_1, \ldots, \nu_n \in \mathbb{R}$ such that

$$\vec{v} = \nu_1 \vec{e}_1 + \cdots + \nu_n \vec{e}_n$$

$$= \begin{bmatrix} \vec{e}_1 & \cdots & \vec{e}_n \end{bmatrix} \begin{bmatrix} \nu_1 \\ \vdots \\ \nu_n \end{bmatrix} = \vec{e}^T \nu$$
A basis bridges the geometric and algebraic aspects of vectors.

\[
\vec{v} = [\vec{e}_1 \  \cdots \  \vec{e}_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \vec{e}^T \vec{v}
\]

geometric vector

algebraic array of numbers

basis
Change of basis

- Matrix algebra
- Vectors
- Change of basis
- Linear transformations
- Linear transformations in graphics
- Inner product
- Special 2D linear transforms
Change of basis

A vector in vector space

\[ \vec{v} = [\vec{e}_1 \ldots \vec{e}_n] \]

basis coefficients relative to the basis

\[ \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \vec{e}^T \vec{v} \]

another basis

\[ \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \vec{f}^T \vec{w} \]

coefficients have changed

• How are the coefficients changed?
Change of basis

\[ \vec{v} = \vec{e}^T \mathbf{v} = \vec{f}^T \mathbf{w} \]

- Example for n=2

Given a relationship between the bases

\[ \vec{e}_1 = \vec{f}_1 + \vec{f}_2 \]
\[ \vec{e}_2 = \vec{f}_1 - \vec{f}_2 \]

Write them in a matrix form with matrix acting from the right

\[
\begin{bmatrix}
\vec{e}_1 \\
\vec{e}_2
\end{bmatrix} = 
\begin{bmatrix}
\vec{f}_1 \\
\vec{f}_2
\end{bmatrix}
\]
Change of basis

\[ \vec{v} = \vec{e}^T \vec{v} = \vec{f}^T \vec{w} \]

- Example for n=2
  
  Given a relationship between the bases
  
  \[ \vec{e}_1 = \vec{f}_1 + \vec{f}_2 \]
  \[ \vec{e}_2 = \vec{f}_1 - \vec{f}_2 \]

  Write them in a matrix form with matrix acting from the right

  \[ \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} = \begin{bmatrix} \vec{f}_1 & \vec{f}_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \]

  Shorthand:

  \[ \vec{e}^T = \vec{f}^T \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \vec{e}^T \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \vec{f}^T \]
Change of basis

\[ \vec{v} = \vec{e}^T \vec{v} = \vec{f}^T \vec{w} \]

- Example for n=2

Given a relationship between the bases

\[ \vec{e}_1 = \vec{f}_1 + \vec{f}_2 \]
\[ \vec{e}_2 = \vec{f}_1 - \vec{f}_2 \]

\[ \vec{e}^T = \vec{f}^T \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \vec{e}^T \left[ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right]^{-1} = \vec{f}^T \]

Simple substitution:

\[ \vec{f}^T \vec{w} = \vec{e}^T \vec{v} = \vec{f}^T \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \vec{v} \]
Change of basis

\[ \vec{v} = \vec{e}^T \vec{v} = \vec{f}^T \vec{w} \]

- Example for \( n=2 \)

Given a relationship between the bases

\[ \vec{e}_1 = \vec{f}_1 + \vec{f}_2 \]
\[ \vec{e}_2 = \vec{f}_1 - \vec{f}_2 \]

*matrix acts on basis from the right*

\[ \vec{e}^T = \vec{f}^T \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \vec{e}^T \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \vec{f}^T \]

Simple substitution:

\[ \vec{f}^T \vec{w} = \vec{f}^T \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \vec{v} \quad \implies \quad \vec{w} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \vec{v} \]

*matrix acts on coefficients from the left*
Change of basis summary

• For every pair of bases \( \mathbf{e}, \mathbf{f} \), there is a matrix \( \mathbf{A} \) such that

\[
\mathbf{e}^T = \mathbf{f}^T \mathbf{A} \quad \text{equivalently} \quad \mathbf{e}^T \mathbf{A}^{-1} = \mathbf{f}^T
\]

• One fixed vector but written in two different bases:

\[
\mathbf{v} = \mathbf{e}^T \mathbf{v} = \mathbf{f}^T \mathbf{w}
\]

• By substitution, we get the transformation rule for the coefficients

\[
\mathbf{w} = \mathbf{A} \mathbf{v} \quad \mathbf{v} = \mathbf{A}^{-1} \mathbf{w}
\]
Linear Transformations

- Matrix algebra
- Vectors
- Change of basis
- Linear transformations
- Linear transformations in graphics
- Inner product
- Special 2D linear transforms
Deforming vectors

- What we just discussed (change of basis) is
  \[ \tilde{v} = \tilde{e}^T v = \tilde{f}^T w \]
  vector is *fixed*  
  basis is changing  
  coefficients change correspondingly

- Now we will **transform** vectors, i.e. the vector will change

- Scenario example: Rotate all vectors by 30 degrees and then stretch horizontally by a factor of 2

- Strategy: reuse some ideas of change of basis
A **linear transformation** is a function $A : V \rightarrow V$ so that

- it preserves the addition: $A(\vec{u} + \vec{v}) = A(\vec{u}) + A(\vec{v})$
- it preserves the scaling: $A(a \vec{v}) = a A(\vec{v})$

Consequences:

- it preserves the zero: $A(\vec{0}) = \vec{0}$
- it preserves straightness, parallelism, proportion
- We only need to know the transformation of a basis.
Linear transformation

- We only need to know the transformation of a basis.

\[
\begin{align*}
\vec{a}_1 &= A\vec{e}_1 \\
\vec{a}_2 &= A\vec{e}_2
\end{align*}
\]
Linear transformation

- We only need to know the transformation of a basis.

\[ \vec{a}_1 = A\vec{e}_1 \]
\[ \vec{a}_2 = A\vec{e}_2 \]
Relate the two bases

- We can express the new basis in terms of the old basis.

\[
\vec{a}_1 = A \vec{e}_1 = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} 1.2 \\ 0.3 \end{bmatrix} \vec{a}_1
\]

\[
\vec{a}_2 = A \vec{e}_2 = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} 0.1 \\ 1.1 \end{bmatrix} \vec{a}_2
\]

transformation operator

\[
\begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} 1.2 & 0.1 \\ 0.3 & 1.1 \end{bmatrix}
\]

A transformation matrix
Relate the two bases

- We can express the new basis in terms of the old basis.

\[
\begin{bmatrix}
\vec{a}_1 \\
\vec{a}_2
\end{bmatrix} = \begin{bmatrix}
\vec{e}_1 \\
\vec{e}_2
\end{bmatrix} \begin{bmatrix}
1.2 & 0.1 \\
0.3 & 1.1
\end{bmatrix}
\]

\[
A \vec{v} = \begin{bmatrix}
\vec{e}_1 \\
\vec{e}_2
\end{bmatrix} \begin{bmatrix}
1.2 & 0.1 \\
0.3 & 1.1
\end{bmatrix} \begin{bmatrix}
2.3 \\
1.5
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\vec{e}_1 \\
\vec{e}_2
\end{bmatrix} \begin{bmatrix}
2.91 \\
2.34
\end{bmatrix}
\]
Linear transformation

- We can express the new basis in terms of the old basis.

\[
\begin{bmatrix}
\vec{a}_1 & \vec{a}_2
\end{bmatrix} =
\begin{bmatrix}
\vec{e}_1 & \vec{e}_2
\end{bmatrix}
\begin{bmatrix}
1.2 & 0.1 \\
0.3 & 1.1
\end{bmatrix}
\]

\[
A\vec{v} =
\begin{bmatrix}
\vec{e}_1 & \vec{e}_2
\end{bmatrix}
\begin{bmatrix}
1.2 & 0.1 \\
0.3 & 1.1
\end{bmatrix}
\begin{bmatrix}
2.3 \\
1.5
\end{bmatrix}
\]

\[
= 
\begin{bmatrix}
\vec{e}_1 & \vec{e}_2
\end{bmatrix}
\begin{bmatrix}
2.91 \\
2.34
\end{bmatrix}
\]

coefficients for the transformed vector under the original basis
Linear transformation: summary

\[
A\tilde{e}^\mathsf{T} = \begin{bmatrix} A\tilde{e}_1 & \cdots & A\tilde{e}_n \end{bmatrix} = \begin{bmatrix} \tilde{a}_1 & \cdots & \tilde{a}_n \end{bmatrix} = \tilde{a}^\mathsf{T}
\]

Express each transformed basis vector under the original basis:

\[
\tilde{a}_j = \begin{bmatrix} \tilde{e}_1 & \cdots & \tilde{e}_n \end{bmatrix} \begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \tilde{e}_1 & \cdots & \tilde{e}_n \end{bmatrix}
\]
Linear transformation: summary

\[ \mathbf{A} \mathbf{\tilde{e}}^{\mathsf{T}} = \begin{bmatrix} \mathbf{A} \mathbf{\tilde{e}}_1 & \cdots & \mathbf{A} \mathbf{\tilde{e}}_n \end{bmatrix} = \begin{bmatrix} \mathbf{\tilde{a}}_1 & \cdots & \mathbf{\tilde{a}}_n \end{bmatrix} = \mathbf{\tilde{a}}^{\mathsf{T}} \]

Express each transformed basis vector under the original basis:

\[
\begin{bmatrix} \mathbf{\tilde{a}}_1 & \cdots & \mathbf{\tilde{a}}_n \end{bmatrix} = \begin{bmatrix} \mathbf{\tilde{e}}_1 & \cdots & \mathbf{\tilde{e}}_n \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}
\]

- \( \mathbf{A} \) is the **matrix representation** associated to the linear transformation \( \mathbf{A} \) under the basis \( \mathbf{\tilde{e}} \).
Linear transformation: summary

- $A$ is the **matrix representation** associated to the linear transformation $A$ under the basis $\tilde{e}$.

$$A\tilde{e}^T = \tilde{e}^T A$$

- Linear transformation under a basis is a matrix multiplication:

$$A\tilde{v} = A\tilde{e}^T v = \tilde{e}^T A v$$

- A matrix records the “ratio” between two bases:

$$\tilde{a}^T = \tilde{e}^T A$$
Transformation v.s. change of basis

\[ \tilde{a}^T = \tilde{e}^T A \]

- Scenario 1: Change of basis
  - vector is invariant
    \[ \tilde{w} = \tilde{a}^T v \]
    \[ = \tilde{e}^T Av \]
  - coefficients change co-variantly

- Scenario 2: Vector is transformed
  - \[ \tilde{v} = \tilde{e}^T v \]
  - coefficients are invariant
  - \[ A\tilde{v} = \tilde{a}^T v = \tilde{e}^T Av \]
  - vector is transformed
Transformation v.s. change of basis

\[ \tilde{a}^T = \tilde{e}^T A \]

- **Scenario 1:** Change of basis
  
  vector is invariant  \[ \tilde{w} = \tilde{a}^T v \]
  
  \[ = \tilde{e}^T A v \]

- **Scenario 2:** Vector is transformed
  
  \[ \tilde{v} = \tilde{e}^T v \]
  
  \[ A\tilde{v} = \tilde{a}^T v = \tilde{e}^T A v \]

  vector is transformed  \[ same \ basis \]

coefficients change co-variantly

coefficients are transformed by matrix
Linear Transformations in Computer Graphics

- Matrix algebra
- Vectors
- Change of basis
- Linear transformations
- Linear transformations in graphics
- Inner product
- Special 2D linear transforms
• In a scene, geometric objects have vertex positions thought of as geometric vectors (displacements from the origin).

$$\vec{v} \in V$$

• When we store the vertex positions, we represent the vectors as algebraic vectors, i.e. tuples of real numbers.

$$\mathbf{v} \in \mathbb{R}^n$$

• The conversion between the above two types of vectors require a basis.

$$\vec{v} = \mathbf{b}^T \mathbf{v}$$
There are usually several sets of basis in one scene.
• In this example, there are 3 sets of basis $\mathbf{e}, \mathbf{a}, \mathbf{b}$.

• In order not to confuse ourselves for “which basis should we use to interpret an algebraic vector?” we think of 3 separate spaces of algebraic vectors:

\[
\mathbb{R}^n_\mathbf{e} \quad \mathbb{R}^n_\mathbf{a} \quad \mathbb{R}^n_\mathbf{b}
\]

\[\mathbf{v} \in \mathbb{R}^n_\mathbf{b}\] means that this tuple of numbers is to be realized as a geometric vector using the $\mathbf{b}$ basis: $\mathbf{b}^T \mathbf{v}$
• We describe these bases “relatively.” The information that we will store is the matrix between the bases.

\[
\begin{align*}
\bar{a}^T &= \bar{e}^T A \\
\bar{b}^T &= \bar{a}^T B
\end{align*}
\]

• The above relationship translates to the following diagram.
Linear transformations in graphics

\[
\begin{align*}
\vec{a}^\top &= \vec{e}^\top A \\
\vec{b}^\top &= \vec{a}^\top B \\
ABv &\in \mathbb{R}^n_{\vec{e}} \\
Bv &\in \mathbb{R}^n_{\vec{a}}
\end{align*}
\]
Linear transformations in graphics

Example

\[
\tilde{\mathbf{a}}^T = \tilde{\mathbf{e}}^T \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}
\]

\[
\tilde{\mathbf{b}}^T = \tilde{\mathbf{a}}^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}
\]

\[
\begin{array}{ccc}
\mathbb{R}^2_{\tilde{\mathbf{b}}} & \xrightarrow{\tilde{\mathbf{a}}} & \mathbb{R}^2_{\tilde{\mathbf{a}}} & \xrightarrow{\tilde{\mathbf{e}}} & \mathbb{R}^2_{\tilde{\mathbf{e}}} \\
\begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} & \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{array}
\]
Hierarchical modeling

- A scene graph

When we change the matrix on one edge, all objects beneath it move together as a group.
Euclidean Vector Space

- Matrix algebra
- Vectors
- Change of basis
- Linear transformations
- Linear transformations in graphics
- Inner product
- Special 2D linear transforms
Length and angle

• A vector space $V$ is a **Euclidean vector space** if we can measure lengths of vectors and angles between vectors.

• Mathematically, a Euclidean vector space is a vector space that has an **inner product**:

$$ (\mathbf{u} \in V, \mathbf{v} \in V) \mapsto \mathbf{u} \cdot \mathbf{v} \in \mathbb{R} $$

obeying the following assertions:

▶ **Symmetric:** $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

▶ **Bilinear:**

$$ \mathbf{u} \cdot (c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2) = c_1 (\mathbf{u} \cdot \mathbf{v}_1) + c_2 (\mathbf{u} \cdot \mathbf{v}_2) $$

$$ (c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2) \cdot \mathbf{v} = c_1 (\mathbf{u}_1 \cdot \mathbf{v}) + c_2 (\mathbf{u}_2 \cdot \mathbf{v}) $$

▶ **Positive-definite:** $\mathbf{u} \cdot \mathbf{u} > 0$ for all $\mathbf{u} \neq \mathbf{0}$
Length and angle

Inner product is product of the magnitudes of the vectors after one is orthogonally projected to the other

\[ \vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta \]
A basis $\tilde{e}$ is **orthonormal** if the basis vectors are orthogonal to each other, and each of them has length 1.

If $\tilde{e}$ is orthonormal,

$$\tilde{u} = \tilde{e}^T u$$
$$\tilde{v} = \tilde{e}^T v$$

then $\tilde{u} \cdot \tilde{v} = u^T v$
Special linear transformations (2D)

- Matrix algebra
- Vectors
- Change of basis
- Linear transformations
- Linear transformations in graphics
- Inner product
- Special 2D linear transforms
We’ll look at transformation matrices

• We assume that there is a fixed *orthonormal basis* $\mathbf{e}$

• Each linear transformation $\mathbf{v} \mapsto A(\mathbf{v})$

$$
\mathbf{e}^T \mathbf{v} \mapsto A(\mathbf{e}^T \mathbf{v}) = \mathbf{e}^T A \mathbf{v}
$$

$$
\mathbf{v} \mapsto A \mathbf{v}
$$

is written in terms of a transformation matrix (in *uppercase boldface*)

• Keep in mind that the matrix might look different if you change the basis.
Uniform (isotropic) scaling

\[
\begin{bmatrix}
s \\
\end{bmatrix}
\begin{bmatrix}
s \\
\end{bmatrix}
\cdot \text{Constant times identity}
\]
Anisotropic scaling

\[
\begin{bmatrix}
s_1 \\
s_2
\end{bmatrix}
\]

- Diagonal entries are not constant
- Stretch along basis vectors
Shearing

\[
\begin{bmatrix}
1 & s \\
0 & 1 \\
\end{bmatrix}
\]
• Reflection in the plane normal to \( \mathbf{a}, (|\mathbf{a}| = 1) \):

\[
R_{\mathbf{a}} = I - 2\mathbf{a}\mathbf{a}^T
\]
• Reflection in the plane normal to $\mathbf{a}$, ($|\mathbf{a}| = 1$):

$$\mathbf{R}_a = \mathbf{I} - 2\mathbf{a}\mathbf{a}^\top$$

• In GLM and GLSL

$$\mathbf{ab}^\top \quad \text{outerProduct}(\mathbf{a}, \mathbf{b})$$

$$\mathbf{a}^\top \mathbf{b} \quad \text{dot}(\mathbf{a}, \mathbf{b})$$
Rotation

- Rotation by angle $\theta$

$$R^{\theta} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$
Orthogonal matrix

• If a transformation maps orthonormal basis to orthogonal basis then we call the transformation orthogonal or unitary

• Reflections and rotations are orthogonal

• Orthogonal transform if and only if

$$R^T = R^{-1}$$
Suppose the underlying basis is orthonormal.

- Isotropic scaling \[ \begin{bmatrix} s \\ s \end{bmatrix} \]
- Anisotropic scaling \[ \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \]
- Shearing \[ \begin{bmatrix} 1 \\ s \\ 1 \end{bmatrix} \]
- Reflection in the plane normal to \( a \), (\(|a| = 1\)): \( R_a = I - 2aa^\top \)
- Rotation \( R^\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \) \( (R_a)^{-1} = (R_a)^\top \) \( (R^\theta)^{-1} = (R^\theta)^\top \)
Next topic: 3D rotations