CSE 167 (FA21)
Computer Graphics: Computer Animation
Albert Chern
• **Animation** is a sequence of **frames** containing images that differ from frame to frame slightly,

• so as the animation is played in dozens of frames per second, the viewer gets the illusion of watching moving objects.

▷ Optical illusion: Beta movement
Animation: historical

A pot in Shahr-e Sukhteh, Iran, 3200 BCE
Animation: Phenakistoscope

Phenakistoscope (19th century)
Animation: flipbook

Linnett’s Flipbook animation (1868)
First hand-drawn featured film
Disney’s Snow White and the Seven Dwarfs (1939)
Animation: key-framing

First outline the key frames
Then fill in the intermediate frames
Animation: key-framing

Keyframing techniques: convey motions in keyframes

- anticipation
- squash and stretch
Animation: key-framing

More animation techniques

Secondary motion: Additional motion occurring as a consequence
Animation: key-framing

More animation techniques

Follow through till the resolution / termination of action
Disney’s 12 principles of animation

https://en.wikipedia.org/wiki/Twelve_basic_principles_of_animation
Animation: live-action reference

Live-action reference for accurate timing

Disney’s Alice in Wonderland (1951)
Animation: stop-motion

Pingu
Animation: computer animation

Star Wars (1977)
Animation: computer animation

Carl Sagan’s Cosmos (1980)
Animation: computer animation

First 3D computer animation
Toy Story (1995)
Computer animation techniques

Traditional Animation
- Key-framing
- Live action reference
- Stop motion

Computational physics
- Physics modeling
- Differential equation
- Numerical methods

Computer Animation
- Key-frame animation
- Motion capture
- Rigging, inverse kinematics
- Physics simulation
- Art-directed physics
Rigging and Key-frame animation

Animate through the control skeleton

Spline interpolation between keyframes
Stop motion + computer graphics

Dinosaur input device (Jurassic Park 1993)
Live actor reference => motion capture

Jar Jar Binks in *Star Wars I: the Phantom Menace* 1999

Gollum in *The Lord of the Rings: the Two Towers* 2002

Motion capture technique becomes an industry standard
Simulated animation

Flock of birds

Flock simulation
Crowd simulation

Crowd simulation (World War Z 2013)
Physics simulation

Rigid body motions
Physics simulation

Fluid simulation (Good Dinosaur 2015)
Computer animation techniques

**Traditional Animation**
- Key-framing
- Live action reference
- Stop motion

**Computational physics**
- Physics modeling
- Differential equation
- Numerical methods

**Computer Animation**
- Key-frame animation
- Motion capture
- Rigging, inverse kinematics
- Physics simulation
- Art-directed physics
Physics based animation
Physics based motion

Rough idea:

• Position of each object is governed by Newton’s law of motion
• Rate of change of position is called velocity
• Rate of change of velocity is called acceleration
• Model “force” as a function of position and velocity
• Newton’s law of motion: Mass x acceleration = force
Example

- Animate an object attached to a spring
- Identify the moving position: \( x \)
- Associated velocity \( v \)
Example

- Force

\[ f(x, v) = -k(x - x_0) - \mu v \]

rest position
Example

- Force

\[ f(x, v) = -k(x - x_0) - \mu v \]

*stiffness of spring*
Example

- Force

\[ f(x, v) = -k(x - x_0) - \mu v \]

*friction*
Example

- Force

\[ f(x, v) = -k(x - x_0) - \mu v \]

- Equations of motion

\[
\begin{align*}
\frac{dx}{dt} &= v \\
\frac{dv}{dt} &= \frac{1}{m} f(x, v)
\end{align*}
\]

relationship between position and velocity

relationship between acceleration and force

mass
Example

• Force

\[ f(x, v) = -k(x - x_0) - \mu v \]

• Equations of motion

\[
\begin{align*}
\frac{dx}{dt} &= v \\
\frac{dv}{dt} &= \frac{1}{m} f(x, v) \\
&= -\frac{k}{m} (x - x_0) - \frac{\mu}{m} v
\end{align*}
\]
Example

- We use the overhead dots to indicate time derivatives

\[
\frac{dx(t)}{dt} = \dot{x}(t) \quad \frac{d^2x(t)}{dt^2} = \ddot{x}(t)
\]
Example

• Equations of motion

\[ \dot{x} = v \]

\[ \dot{v} = -\frac{k}{m} (x - x_0) - \frac{\mu}{m} v \]

• Substitute \( v \):

\[ \ddot{x} = -\frac{k}{m} (x - x_0) - \frac{\mu}{m} \dot{x} \]

- Equation involving \( x \) and its derivatives
- This is called an ordinary differential equation (ODE)
In general

To simulate a physical dynamical system

• Derive the equation of motion
  ▶ Basically, \( F = ma \)
  ▶ Tricks from classical mechanics

• Solve the differential equation (ODE)
  ▶ Hard to solve it by hand most of the time
  ▶ Numerical method is needed
In general

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Equations of motion

- List all the variables that are sufficient to determine the motion
  \[ x : \text{position of the object} \]
  \[ v : \text{velocity of the object} \]

- These variables describe the state. Let them be functions of time.
  \[
  x(t) \quad v(t)
  \]

- Describe the force as a function of state \( f(x, v) \)
  (this can be difficult!)

- Newton’s Law of motion
  \[
  \dot{x} = v \\
  m \ddot{v} = f(x, v)
  \]
Multiple objects

- N-body problem

\[
\begin{align*}
\dot{x}_1 &= v_1 \\
&\vdots \\
\dot{x}_N &= v_N \\
\dot{v}_1 &= f_1(x_1, \ldots, x_N, v_1, \ldots, v_N) \\
&\vdots \\
\dot{v}_N &= f_N(x_1, \ldots, x_N, v_1, \ldots, v_N)
\end{align*}
\]
Multiple objects

- N-body problem

\[
\begin{align*}
\dot{x}_1 &= v_1 \\
& \quad \vdots \\
\dot{x}_N &= v_N \\
\dot{v}_1 &= f_1(x_1, \ldots, x_N, v_1, \ldots, v_N) \\
& \quad \vdots \\
\dot{v}_N &= f_N(x_1, \ldots, x_N, v_1, \ldots, v_N)
\end{align*}
\]
“F = ma” -based equation

- If we know the forces given a given state of positions and velocities
- Then we obtain the equation of motion

- Not the end of the story…
  - Even for simple system, finding all forces is challenging
Physics simulation

To simulate a physical dynamical system

• Derive the equation of motion
  ▶ Basically, $F = ma$
  ▶ Tricks from classical mechanics

• Solve the differential equation (ODE)
  ▶ Hard to solve it by hand most of the time
  ▶ Numerical method is needed
• Given any differential equation, for example,

\[ \dddot{x} + \ddot{x} \dot{x} + \sin(\dot{x}) = 1 \]

• Convert it into a 1st order system of ODEs (involving at most first first derivative)
  ▶ Give each derivative a separate name (except for the highest order derivative) \[ v = \dot{x} \quad a = \ddot{x} \]

\[ \begin{aligned}
\dot{x} &= v \\
\dot{v} &= a \\
\dot{a} &= 1 - av - \sin(a)
\end{aligned} \]
• Given any differential equation, for example,

\[ \ddot{x} + \dot{x} \dot{x} + \sin(\dot{x}) = 1 \]

Then

\[
\begin{cases}
\dot{x} = v \\
\dot{v} = a \\
\dot{a} = 1 - av - \sin(a)
\end{cases}
\]

Let \( y = \begin{bmatrix} x \\ v \\ a \end{bmatrix} \) ODE becomes

\[ \dot{y} = f(y) \]
Numerical ODE

- **Generic ODE** \(\dot{y} = f(y)\)

- Discretize time into time-frames \(y^{(n)} = y(n\Delta t)\)

- **Euler method**
  \[
  \frac{y^{(n+1)} - y^{(n)}}{\Delta t} \approx f(y^{(n)})
  \]

  \[
  y^{(n+1)} \approx y^{(n)} + \Delta t \cdot f(y^{(n)})
  \]

  (not accurate and usually unstable)
Numerical ODE

- **Euler method**
  \[ y^{(n+1)} \approx y^{(n)} + \Delta t \cdot f(y^{(n)}) \]

- **Runge–Kutta method (RK4)** (very powerful! Accurate and stable)
  \[
  \begin{align*}
  k_1 &= f(y^{(n)}) \\
  k_2 &= f(y^{(n)} + \frac{\Delta t}{2} k_1) \\
  k_3 &= f(y^{(n)} + \frac{\Delta t}{2} k_2) \\
  k_4 &= f(y^{(n)} + \Delta t k_3) \\
  y^{(n+1)} &= y^{(n)} + \frac{\Delta t}{6} (k_1 + 2k_2 + 2k_3 + k_4)
  \end{align*}
  \]
• In most cases the RK4 method works very well

• Sometimes the underlying physical system has additional structures (energy conservation, momentum conservation)

• Special algorithm (non-RK4) aims at preserving energy or momentum
  ▶ Variational integrator
  ▶ Symplectic integrator
  ▶ Lie group integrator
Physics simulation

To simulate a physical dynamical system

• Derive the equation of motion
  ▶ Basically, $F = ma$
  ▶ Tricks from classical mechanics

• Solve the differential equation (ODE)
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  ▶ Numerical method is needed

Let’s return to deriving the differential equation
• How do you write down the equation of motion for the following pendulum?

• For such a simple physical system, finding all the forces and torques correctly is very challenging.
Physics simulation

To simulate a physical dynamical system

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Classical mechanics beyond Newton

- 17th century: Newton’s $F = ma$
- Early 18th century
  - Conservation of energy
  - Work, potential energy

Isaac Newton 1643–1727

Émilie du Châtelet 1706–1749

Daniel Bernoulli 1709–1791

John Smeaton 1724–1792
Classical mechanics beyond Newton

- Every static configuration is equipped with a potential energy
  - The amount of work the system can possibly do after connected to any mechanical device
    - Denoted by $U(x)$
    - Example: gravitational potential $U = mgh$

- Motion itself has kinetic energy
  - Denoted by $T(x, v)$
  - For example, $T = \frac{1}{2} m |v|^2$

- $F = ma$ is transfer between different form of energies.
Classical mechanics beyond Newton

- Example: gravitational potential $U = mgh = mgx_3$
- For example, $T = \frac{1}{2}m|v|^2 = \frac{1}{2}m(v_1^2 + v_2^2 + v_3^2)$

- $F = ma$ is transfer between different form of energies.
  - Mass and acceleration is encoded in kinetic energy
    \[ \frac{\partial T}{\partial v_j} = mv_j \quad d \left( \frac{\partial T}{\partial v_j} \right) = m\dot{v}_j \]
  - Force is encoded in potential energy
    \[ f_j = -\frac{\partial U}{\partial x_j} \quad f = (0, 0, -mg) \]

- Lagrange equation
  \[ \frac{d}{dt} \left( \frac{\partial T}{\partial v_j} \right) + \frac{\partial U}{\partial x_j} = 0 \]
  \[ F = ma \text{ rewritten in terms of just kinetic and potential energies.} \]
Classical mechanics beyond Newton

- Lagrange equation

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial v_j} \right) + \frac{\partial U}{\partial x_j} = 0
\]

- Hamilton’s principle:
  Physical paths minimizes “length” (action) measured in terms of kinetic energy and potential energy.

- Lagrangian mechanics is still the basis for modern physics (general relativity, quantum field theory,...)
Lagrangian Mechanics

- Lagrange equation
  \[ \frac{d}{dt} \left( \frac{\partial T}{\partial v_j} \right) + \frac{\partial U}{\partial x_j} = 0 \]

- Instead of trying to account for all forces
- We only need to model two scalar functions
  - Kinetic energy \( T(x, v) \)
  - Potential energy \( U(x) \)

- The position \( x \) can be general parameters of the mechanical system (e.g. joint angles of a robotic arm).
Lagrangian Mechanics

\[
\frac{d}{dt} \left( \frac{\partial}{\partial v} T(x, v) \right) + \frac{\partial}{\partial x} U(x) = 0
\]

- Example: compound pendulum
  - State variables: \( \theta, \dot{\theta} \)
  - Kinetic energy \( T = \int_0^\ell \frac{1}{2} \frac{m}{\ell} (r \dot{\theta})^2 \, dr = \frac{1}{6} \frac{m}{\ell} \ell^3 \dot{\theta}^2 = \frac{1}{6} m \ell^2 \dot{\theta}^2 \)
  - Potential energy \( U = \int_0^\ell -\frac{m}{\ell} g r \cos \theta \, dr = -\frac{1}{2} mg \ell \cos \theta \)
  - Derivatives: \( \frac{\partial T}{\partial \dot{\theta}} = \frac{1}{3} m \ell^2 \dot{\theta}, \quad \frac{\partial U}{\partial \theta} = \frac{1}{2} mg \ell \sin \theta \)
  - Equation of motion \( \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}} \right) + \frac{\partial U}{\partial \theta} = 0 \)

\[ \implies \ddot{\theta} = -\frac{3}{2} \frac{g}{\ell} \sin \theta \]
Lagrangian Mechanics

\[
\frac{d}{dt} \left( \frac{\partial}{\partial \dot{v}} T(x, v) \right) + \frac{\partial}{\partial x} U(x) = 0
\]

• Example: double compound pendulum
  ▶ State variables: \( \theta, \phi, \dot{\theta}, \dot{\phi} \)
  ▶ Kinetic energy \( T = \int_0^\ell \frac{1}{2} \frac{m}{\ell} (r \dot{\theta})^2 \)
    \[+ \int_0^\ell \frac{1}{2} \frac{m}{\ell} \left( (\ell \dot{\theta} + r \cos(\phi - \theta) \dot{\phi})^2 + (r \sin(\phi - \theta) \dot{\phi})^2 \right) dr \]
  ▶ Potential energy \( U = -mg \frac{\ell}{2} \cos \theta - mg \left( \ell \cos \theta + \frac{\ell}{2} \cos \phi \right) \)
  ▶ Equation of motion
    \[
    \begin{cases}
      \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}} \right) + \frac{\partial U}{\partial \theta} = 0 \\
      \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\phi}} \right) + \frac{\partial U}{\partial \phi} = 0
    \end{cases}
    \]
Rigid body simulation
Rigid body motion
Rigid body motion

• The motion’s state variable is just the translation and rotation (and their time-derivatives)
  ▶ Rotation matrix $R$ and the translation vector $c$
  ▶ Denote the derivative of the translation
    $\dot{c} = v$
  ▶ The rotation matrix satisfies $R^T R = R R^T = \text{id}$
  ▶ This implies that $\dot{R}$ always take the form
    $\dot{R} = WR = RA$
    for some skew-symmetric matrices $W, A$.
  ▶ Skew-symmetric matrices are cross product with vectors
    $\dot{R} = [\omega \times] R = R[\Omega \times]$
Rigid body motion

- The motion’s state variable is just the translation and rotation (and their time-derivatives) \( \mathbf{c}, R, \mathbf{v}, \omega \)
  - Derive the kinetic energy as a quadratic function of \( \mathbf{v}, \omega \)
  - Use the Lagrange’s principle to derive the equations of motion.
    \[
    \begin{aligned}
    \dot{\mathbf{c}} &= \mathbf{v} \\
    \dot{R} &= [\mathbf{\omega} \times ]R \\
    \frac{d}{dt} \left( \frac{\partial T}{\partial \mathbf{v}} \right) &= 0 \\
    \frac{d}{dt} \left( \frac{\partial T}{\partial \mathbf{\omega}} \right) &= 0
    \end{aligned}
    \]
  - If we take \( \mathbf{c} \) to be the center of mass (CM), then the 3rd and 4th equations decouple.
  - We can focus on the rotation part.
Rigid body motion

- **State variables**: rotation $\mathbf{R}$, angular velocity vector $\mathbf{\omega}$
  - Angular velocities are defined so that $\dot{\mathbf{R}} = [\mathbf{\omega} \times] \mathbf{R} = \mathbf{R} [\mathbf{\Omega} \times]$
  - World and body coordinate: $\mathbf{\omega} \in \mathbb{R}_\text{World}^3$, $\mathbf{\Omega} \in \mathbb{R}_\text{Body}^3$, $\mathbf{\omega} = \mathbf{R} \mathbf{\Omega}$

- **Kinetic energy is a quadratic function**
  - $T = \frac{1}{2} \mathbf{\omega}^T \mathbf{I}_\text{Body} \mathbf{\Omega} = \frac{1}{2} \mathbf{\omega}^T \mathbf{I}_\text{World} \mathbf{\omega}$
  - The 3x3 symmetric positive definite matrix $\mathbf{I}$ is called the moment of inertia.
  - For rigid body, $\mathbf{I}_\text{body}$ is time-independent;
  - $\mathbf{I}_\text{World}(t) = \mathbf{R}(t) \mathbf{I}_\text{Body} \mathbf{R}(t)^T$ co-rotates.

- **Derivative** $\frac{\partial T}{\partial \mathbf{\omega}} = \mathbf{I}_\text{World} \mathbf{\omega}$
Rigid body motion

- Derivative \( \frac{\partial T}{\partial \omega} = I_{\text{World}} \omega \)

- Equation of motion \( \frac{d}{dt} \left( \frac{\partial T}{\partial \omega} \right) = 0 \)

\[
\Rightarrow \frac{d}{dt} \left( I_{\text{World}}(t) \omega(t) \right) = 0
\]

\[
\Rightarrow \frac{d}{dt} \left( R(t)I_{\text{Body}}R(t)^T \omega(t) \right) = 0
\]

\[
\Rightarrow \dot{R}I_{\text{Body}}R^T \omega + I_{\text{Body}}\dot{R}^T \omega + R I_{\text{Body}}R^T \dot{\omega} = 0
\]

\[
\Rightarrow \omega \times [R I_{\text{Body}}R^T \omega + I_{\text{Body}}R^T[\omega \times]R^T \omega + R I_{\text{Body}}R^T \dot{\omega}] = 0
\]

\[
\Rightarrow \omega \times (I_{\text{World}} \dot{\omega}) + I_{\text{World}} \ddot{\omega} = 0
\]

Euler equation of rigid body motion
Tennis Racket Theorem

Dzhanibekov effect
Tennis Racket Theorem
Tennis Racket Theorem