Physics based rendering

- Derive the color in pixels based on physical principles
- Light transport simulation
- A direct discretization of the simulation is referred to as *radiosity.*
- Another way to evaluate the light transport equation becomes the ray tracing with global illumination
What is light?

- Wave optics
  - Lights are electromagnetic waves
  - Polarized waves
  - Diffraction
What is light?

- Ignore polarization
- Wavelength $\ll$ object to interact

- Ray optics (geometric optics)
- Lights are rays
  - Each ray has a distribution over all wavelengths
  - Or just simplify it as RGB channels
- Computer graphics rendering is mostly based on ray optics
Overview for today

- Space of rays
- Measures on space of rays
- Light intensity
- Equation satisfied by light intensities
A ray is described by a point \( p \in \mathbb{R}^3 \) and a direction \( d \in S^2 \).

\[ S^2 = \{ v \in \mathbb{R}^3 \mid |d| = 1 \} \]

The space of all rays

\[ \mathcal{R} = \mathbb{R}^3 \times S^2 \]

The ray space \( \mathcal{R} \) is a 5 dimensional space.
Surfaces

- A scene is made of bunch of surfaces
  \[ M \subset \mathbb{R}^3 \]
  2D surface

- Each surface point \( x \in M \) has a normal \( n_x \in S^2 \)

- The set of all out-pointing rays based at \( x \) is a hemisphere

\[ H_x = \{ v \in S^2 \mid n_x \cdot v > 0 \} \]
Rays based on surface

- Consider the surface ray space
  \[ S = \{ (x, v) \in \mathcal{R} \mid x \in M, v \in H_x \} \]

- Surface rays $S$ is a 4D subset of the 5D $\mathcal{R}$

- The **Cast** function
  \[
  \text{Cast}: \mathcal{R} \to S \\
  \text{Cast}(p, d) = (p + td, -d)
  \]

  *first intersection with $M*
Measures on 4D surface ray space

• In ray optics, a **light field** is “density” assigned over the 4D space of surface rays.

• Idea of density is similar to how we would describe mass density in 3D:
  - Mass: Measure of physical content
  - Volume: Measure of geometric content
  - Density = Mass/Volume over infinitesimal region
Density

• Idea of density is similar to how we would describe mass density in 3D
  ▶ Mass: Measure of physical content
  ▶ Volume: Measure of geometric content
  ▶ Density = Mass/Volume over infinitesimal region

\[
\text{Density}(p) = \lim_{\text{region} \to p} \frac{\text{Mass(\text{region})}}{\text{Volume(\text{region})}}
\]

\[
\text{Density} = \frac{d\text{Mass}}{d\text{Volume}}
\]

Radon–Nikodym derivative (1910s)
Measures on 4D surface ray space

- In ray optics, a **light field** is “density” assigned over the 4D space of surface rays.

- We need two 4D measures:
  - Measure of physical content
  - Measure of geometric content
  - Light intensity = Mass/Volume over infinitesimal region
Throughput: a 4D volume

- Consider a bundle of rays occupying
  - an area $\Delta A$ around point $x \in M$
  - a range of directions $\Delta \sigma$ around $v$
- This forms a 4D region in our 4D surface ray space $S$
Throughput: a 4D volume

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- This forms a 4D region in our 4D surface ray space $S$
- The total **throughput** is

$$(n \cdot v)\Delta A \Delta \sigma$$
Throughput: a 4D volume

- The total **throughput** is

\[(n \cdot v) \Delta A \Delta \sigma\]

- Define the *infinitesimal throughput element* at \((x, v) \in S\) as

\[(n_x \cdot v) dA d\sigma\]

\[= dA d\sigma^\perp\]

where \(d\sigma^\perp = (n_x \cdot v) d\sigma\)
Throughput: a 4D volume

- What are the following integrals?

\[
\int \int_{H_x} d\sigma = 2\pi \\
\int \int_{H_x} d\sigma^\perp = \pi
\]
Throughput: a 4D volume

- Given a 4D region $\mathcal{D} \subset S$
  the total throughput is written as

\[
\text{Throughput}(\mathcal{D}) = \int \int \int \int_\mathcal{D} dA d\sigma^\perp
\]

**SI-units:** $m^2 \text{sr}$

sr: steradian
which is the SI-unit for solid angle
Power: the physical measurement

- In a lit environment, given a 4D region $\mathcal{D} \subset S$

  we measure **power of light**

  entering a region of surface over a range of direction described by $\mathcal{D} \subset S$

  \[
  \text{Power}(\mathcal{D})
  \]

  **SI-units:** $W = J s^{-1}$  
  Watts = Joules per second

  or, after weighted by human eye sensitivity, **lumen**
Intensity: Density of light

• “Density of light” is given by **power per unit throughput**
• For each surface ray \((x, v) \in S\)

\[
L(x, v) = \lim_{D \to (x,v)} \frac{\text{Power}(D)}{\text{Throughput}(D)}
\]

**SI-unit:** \(W \text{ m}^{-2} \text{sr}^{-1}\) or **candela**

• Intensity \(L\) is also called **radiance**
• It is a function defined on the 4D space \(S\)
• You can overload \(L\) into a function on the whole 5D space \(\mathbb{R}\)
• You can overload $L$ into a function on the whole 5D space $\mathbb{R}$

$$L(p, -d) = L(\text{Cast}(p, d))$$

• In photography, this 5D function is called
  ▶ Light field
  ▶ Plenoptic function
Radiance, irradiance, power

- **Given radiance / luminance**
  \[ L : S \rightarrow \mathbb{R} \, \text{W} \, \text{m}^{-2} \, \text{sr}^{-1} \]

- **Irradiance / illuminance** is a function on the surface (integrate all directions)
  \[ E : M \rightarrow \mathbb{R} \, \text{W} \, \text{m}^{-2} \text{ or lux} \]
  \[ E(x) = \int \int_{H_x} L(x, v) \, d\sigma^\perp \]

- **Power / luminous flux** over a surface
  \[ \text{Power}(A) = \int \int_{A} E(x) \, dA = \int \int_{A} \int \int_{H_x} L(x, v) \, d\sigma^\perp \, dA \]
Radiance, irradiance, power

http://6degreesoffreedom.co/luminance-vs-illuminance/
Rendering equation
Suppose some surfaces in the scene are emitting light

Suppose material property (how light reflect) is also given

This will determine the 4D function $L$

The equation satisfied by $L$ is called the light transport equation, or the rendering equation
Rendering equation

• Given emission \( L_e : S \rightarrow \mathbb{R} \, \text{w} m^{-2} \, \text{sr}^{-1} \) ignore the unit from now on

• Unknowns: \( L_{\text{in}} : S \rightarrow \mathbb{R} \)
  \( L_{\text{out}} : S \rightarrow \mathbb{R} \)

• Rendering equation

\[
\begin{align*}
L_{\text{out}}(x, v_{\text{out}}) &= L_e(x, v_{\text{out}}) + \int \int_{H_x} f(x, v_{\text{out}}, v_{\text{in}}) L_{\text{in}}(x, v_{\text{in}}) \, d\sigma_{v_{\text{in}}} \\
L_{\text{in}}(x, v_{\text{in}}) &= L_{\text{out}}(\text{Cast}(x, v_{\text{in}}))
\end{align*}
\]
• Rendering equation

\[
\begin{align*}
L_{\text{out}}(x, v_{\text{out}}) &= L_e(x, v_{\text{out}}) + \int \int_{H_x} f(x, v_{\text{out}}, v_{\text{in}}) L_{\text{in}}(x, v_{\text{in}}) d\sigma_{v_{\text{in}}} \\
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\end{align*}
\]

• BRDF satisfies:
  ▶ Non-negative \( f(x, v_{\text{out}}, v_{\text{in}}) \geq 0 \)
  ▶ No production of energy \( \int \int_{H_x} f(x, v_{\text{out}}, v_{\text{in}}) d\sigma_{v_{\text{in}}} \leq 1 \)
  ▶ Symmetry \( f(x, v_{\text{out}}, v_{\text{in}}) = f(x, v_{\text{in}}, v_{\text{out}}) \)
• Rendering equation

\[
\begin{aligned}
L_{\text{out}}(x, v_{\text{out}}) &= L_{\text{e}}(x, v_{\text{out}}) + \int \int_{H_x} f(x, v_{\text{out}}, v_{\text{in}}) L_{\text{in}}(x, v_{\text{in}}) \, d\sigma_{v_{\text{in}}} \\
L_{\text{in}}(x, v_{\text{in}}) &= L_{\text{out}}(\text{Cast}(x, v_{\text{in}}))
\end{aligned}
\]

• Example: Pure diffuse material

\[
f(x, v_{\text{out}}, v_{\text{in}}) = \frac{C_{\text{diffuse}}}{\pi}
\]
Rendering equation

\[ L_{\text{out}}(x, v_{\text{out}}) = L_e(x, v_{\text{out}}) + \int \int_{H_x} f(x, v_{\text{out}}, v_{\text{in}}) L_{\text{in}}(x, v_{\text{in}}) d\sigma_{v_{\text{in}}} \]

\[ L_{\text{in}}(x, v_{\text{in}}) = L_{\text{out}}(\text{Cast}(x, v_{\text{in}})) \]

Simplifying notation:

\[ \begin{cases} 
L_{\text{out}} = L_e + \mathcal{R}(L_{\text{in}}) \\
L_{\text{in}} = \mathcal{G}(L_{\text{out}}) 
\end{cases} \]


Rendering equation

- Rendering equation

\[
\begin{aligned}
L_{\text{out}} &= L_e + \mathcal{R}(L_{\text{in}}) \\
L_{\text{in}} &= \mathcal{G}(L_{\text{out}})
\end{aligned}
\]

\[\implies L_{\text{out}} = L_e + \mathcal{R}(\mathcal{G}(L_{\text{out}}))\]

\[\implies L_{\text{out}} = L_e + \mathcal{T}(L_{\text{out}})\]

\[\implies (1 - \mathcal{T})L_{\text{out}} = L_e\]

\[\mathcal{T} = \mathcal{R} \circ \mathcal{G}\]

transport operator
• Rendering equation

\[(1 - \mathcal{T})L_{\text{out}} = L_e\]

• Computational methods
  ▶ Finite element radiosity
  ▶ Ray tracing
Finite element radiosity

- Rendering equation
  \[(1 - \mathcal{T})L_{\text{out}} = L_e\]

- Radiosity finite element
  - Discretize the 4D space $\mathcal{S}$ of surface rays (into N elements)
  - Then $L_e, L_{\text{out}}$ are vectors of dimension N
  - $(1 - \mathcal{T})$ is a matrix (compute each matrix element)
  - Solve the matrix system
Finite element radiosity

- Rendering equation

\[(1 - T) L_{\text{out}} = L_e\]

- Radiosity finite element
Ray tracing

- Rendering equation

\[(1 - \mathcal{T})L_{\text{out}} = L_e\]

- Ray tracing

\[L_{\text{out}} = (1 - \mathcal{T})^{-1}L_e\]

\[= (1 + \mathcal{T} + \mathcal{T}^2 + \mathcal{T}^3 + \cdots) L_e\]

- This justifies the ray tracing algorithm
- The integration in reflection is usually done with stochastic methods
- Russian Roulette method is usually used for the infinite sum
Next week

- Physics based animation