CSE 167 (FA21)
Computer Graphics:
Digital Geometry Processing
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Surfaces

- A few weeks ago, we had a brief overview of surfaces
  - Triangle meshes
  - Modeling: generate surfaces by spline or subdivision

manifold (valid) mesh

non-manifold mesh

half-edge data structure
Geometry processing

- View discrete surface as a form of “signal data”
  - Traditional signal data: audio & images
  - Geometric signal
  - Upsampling / downsampling / filtering / aliasing
Geometry processing

- Scan
- Process
- Print
- Physics
- Animation / Render
Tasks of geometry processing

- reconstruction
- filtering
- remeshing
- shape analysis
- parameterization
- compression
Symposium on Geometry Processing

- SGP summer school
  http://school.geometryprocessing.org/

- Shape approximation
- Maps between surfaces
- Directional field
Today

- Let’s study one topic today: **Remeshing**
• Let’s study one topic today: **Remeshing**

- Reduce polygon
- Obtain better triangle mesh
- Subdivision
What is a “good” mesh?

- One idea: good approximation of the original shape!
- Keep only elements that contribute information about shape
- Add element where, e.g., curvature is large
What is a “good” mesh?

• One idea: good approximation of the original shape!
• Keep only elements that contribute information about shape
• Add element where, e.g., curvature is large
• “Good approximation” can be deceiving sometimes

vertices exactly on smooth cylinder

smooth cylinder

surface area doesn’t converge under refinement
What is a “good” mesh?

- Another rule of thumb: *triangle shape*

  ![Diagram showing good and bad triangle shapes]

  - “Good”
  - “Bad”

- For example, all angles close to 60 degrees
- A concrete characterization: Delaunay condition
Delaunay condition

- A triangle mesh is **Delaunay** if the circumcircle of each triangle does not contain any vertex of any adjacent triangle.
- Many desirable properties
  - Helps numerical accuracy / stability
  - Maximizes minimal angle
  - Smoothest linear interpolation
- Tradeoffs with efficient shape approximation
What is a “good” mesh?

• Another rule of thumb: *regular vertex degree* (valence)
  ▶ Regular: Vertex degree = 6
What is a “good” mesh?

- Another rule of thumb: regular vertex degree (valence)
  - Regular: Vertex degree = 6

- It may be impossible to have all vertex degree = 6

**Euler–Poincaré Theorem**

\[(\text{Vertices}) - \text{(Edges)} + \text{(Faces)} = 2 - 2 \text{ genus}\]

- If all vertices are regular, then genus must be 1
Remeshing

- General objectives of re-meshing
  - Shape approximation
  - As Delaunay (or equilateral-triangle) as possible
  - Vertex degree as regular as possible

- Mesh simplification

- Improve mesh quality
Remeshing

- Mesh simplification
- Improve mesh quality
Mesh simplification

- Popular scheme: Iteratively collapse edges
- Greedy algorithm
  - Assign each edge a cost
  - Collapse edge with least cost
  - Repeat until target number of elements is reached
- Particularly effective cost function: \textit{quadratic error metric}
Mesh simplification

• If we would collapse this edge, where should we set the new vertex?

• Minimize the distance-squared to neighboring triangle planes
Mesh simplification

average

median

error quadric
Mesh simplification

• What is the distance $d$ between a point $\mathbf{q} \in \mathbb{R}^3$ and a plane?

• Suppose the plane passes through $\mathbf{p} \in \mathbb{R}^3$ with unit normal $\mathbf{n} \in \mathbb{S}^2$

\[
d = \mathbf{n} \cdot (\mathbf{q} - \mathbf{p})
\]
\[
= \begin{bmatrix}
- & \mathbf{n}^T & - & (-\mathbf{n} \cdot \mathbf{p})
\end{bmatrix}
\begin{bmatrix}
\mathbf{q} \\
1
\end{bmatrix}
\]
\[
= \mathbf{a}_{4D}^T \mathbf{q}_{4D}
\]

• Every plane is now a 4D row vector $\mathbf{a}_{4D}^T$

\[
d^2 = \mathbf{q}_{4D}^T (\mathbf{a}_{4D} \mathbf{a}_{4D}^T) \mathbf{q}_{4D}
\]
Mesh simplification

- What is the distance $d$ between a point $q \in \mathbb{R}^3$ and a plane?
  $$d^2 = q_{4D}^\top (a_{4D} a_{4D}^\top) q_{4D}$$

- The total spring energy
  $$U(q) = \frac{1}{2} q_{4D}^\top \left( \sum_{j \in \text{neighboring triangles}} a_j a_j^\top \right) q_{4D}$$
  with $K \in \mathbb{R}^{4 \times 4}$
Mesh simplification

- The position $\mathbf{q} \in \mathbb{R}^3$ that minimizes the spring energy

$$U(\mathbf{q}) = \frac{1}{2} \begin{bmatrix} \mathbf{q}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{K}_{4 \times 4} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ 1 \end{bmatrix}$$

is the solution to $\mathbf{Aq} = -\mathbf{b}$
Mesh simplification

Algorithm (Assign energy per edge)

• For each edge
  ▶ Compute the spring matrix \( K \)
    \[
    U(q) = \frac{1}{2} q^{\top}_4 D \left( \sum_{j \in \text{neighboring triangles}} a_j a_j^{\top} \right) q_{4D} \]
    \[
    K = \begin{bmatrix} A_{3 \times 3} & b_{3 \times 1} \\ b_{1 \times 3}^{\top} & c \end{bmatrix}
    \]
  ▶ Compute the optimal new vertex position \( q = -A^{-1}b \)
  ▶ Assign the edge energy as \( U(q) \)
• EndFor
Mesh simplification

Algorithm (Edge Collapse)

‣ Assign each edge a cost
‣ Collapse edge with least cost
  ▶ Exclude the cases where the triangles fold over
‣ Repeat until target number of elements is reached
Remeshing

- Mesh simplification

- Improve mesh quality
Remeshing

- Mesh simplification

- Improve mesh quality
A triangle mesh is **Delaunay** if the circumcircle of each triangle does not contain any vertex of any adjacent triangle.

How do we make a mesh “more Delaunay”? 

- Delaunay
- Non-Delaunay
- If $\alpha + \beta > 180^\circ$, flip the edge.

(Delaunay condition is equivalent to $\alpha + \beta \leq 180^\circ$ for all edges)

- For a planar (2D) mesh, this eventually yields Delaunay mesh
- For surfaces in 3D, this is still good heuristics in practice
Another algorithm for planar Delaunay

- Notice that every planar section of the paraboloid \( z = x^2 + y^2 \) is always circle when viewed from top.
- Lift the vertices to the paraboloid.
- Delaunay condition in the plane is equivalent to convexity of the lifted triangular mesh.
Alternatively: Improve vertex degree

• How do we improve vertex degree?
• Same tool: Edge flip.
• If total deviation from 6 gets smaller, flip it!

```
|d_i - 6| + |d_j - 6| + |d_k - 6| + |d_l - 6|
```

• Works well in practice. No known guarantees.
Smoothing

- Delaunay doesn’t guarantee triangle angles are close to 60°
- Improve triangle shapes by centering vertices
  - On surface: move only in the *tangent* direction
    - How? Remove the normal component of the update vector
Isotropic Remeshing Algorithm

- Put all tricks together: make triangles uniform in shape & size
- Repeat the 4 steps
  - Split any edge over 4/3rds mean edge length
  - Collapse edge less than 4/5ths mean edge length
  - Flip edges to improve vertex degree
  - Center vertices tangentially
What we’ve learned today

- Surfaces are geometric signals
- Basic remeshing