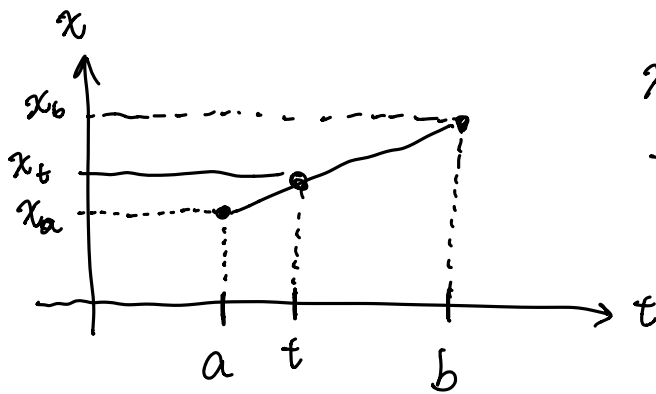


Recall linear interpolation



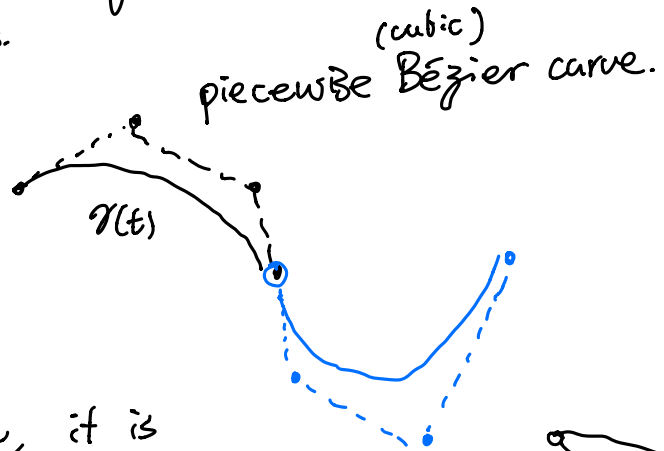
$$x_t = \overset{\lambda_1}{\frac{t-b}{a-b}} x_a + \overset{\lambda_2}{\frac{t-a}{b-a}} x_b$$

$$\begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} t \\ 1 \end{pmatrix}$$

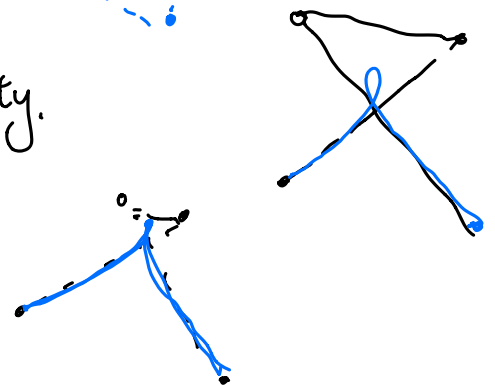
$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \frac{1}{a-b} \begin{pmatrix} 1 & -b \\ -1 & a \end{pmatrix} \begin{pmatrix} t \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{t-b}{a-b} \\ \frac{-t+a}{a-b} \end{pmatrix}$$

Bézier curve

- Bézier curve of degree n requires $(n+1)$ control pts.
- A typical Bézier curve design is composed of many Bézier curve segments.



- For piecewise Bézier curve, it is easy to obtain $C' (\underline{G'})$ continuity.
 $|\gamma'(t)| \neq 0$



- Difficult to obtain C^2 continuity (need to solve the placement of control pts)

Given $(n+1)$ -pts (here $n=3$) C_0, C_1, C_2, C_3 , consider a symmetric

$(0,0,1)C_1$

$$F(0,0,t) = (1-t)C_0 + tC_1$$

$F(0,0,t)$
 $F(0,t,0)$

$F(0,t,1)$

$(0,1,1)$
 C_2

$$= (1-t)F(0,0,t) + tF(0,t,1)$$

$F(t,t,t)$

$(t,t,1)$

$(t,1,1)$

$(1,1,1)$
 C_3

n -affine function

$F(t_1, t_2, t_3)$ n -Blossom
cubic blossom

and assign

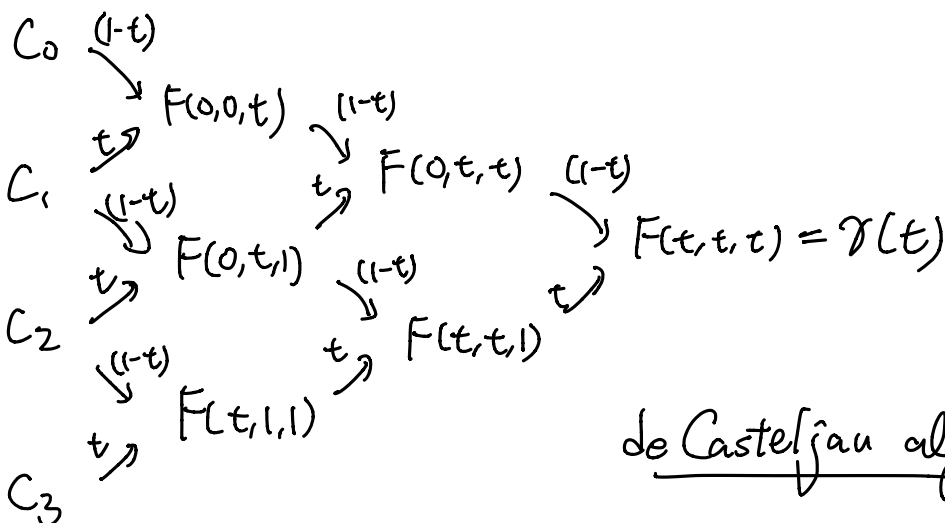
$$F(0,0,0) = C_0$$

$$F(0,0,1) = C_1$$

$$F(0,1,1) = C_2$$

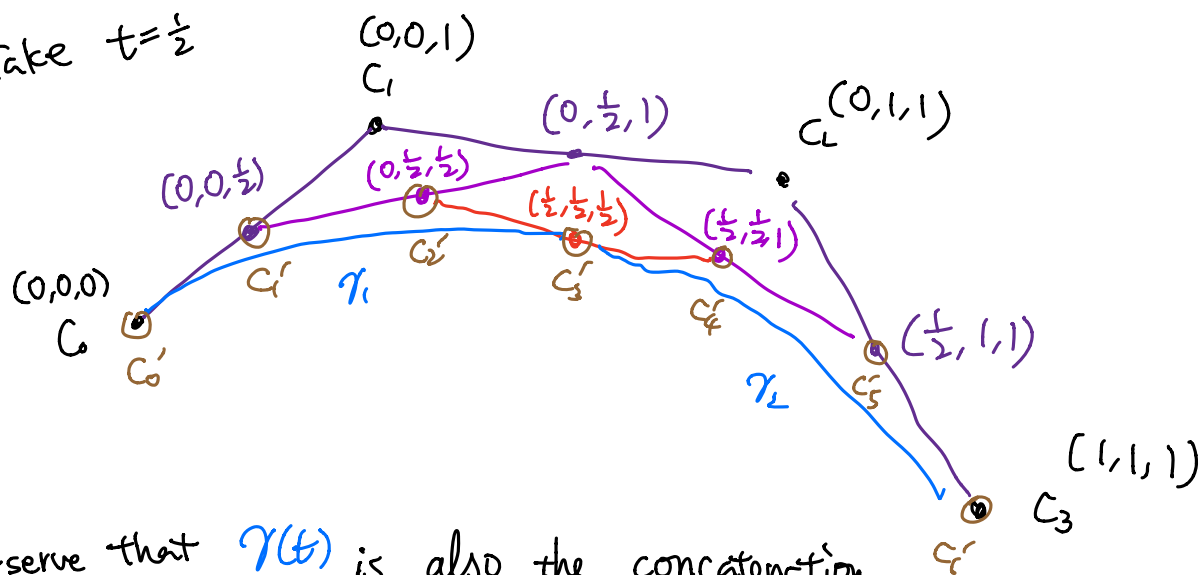
$$F(1,1,1) = C_3$$

Evaluate $\gamma(t) = F(t,t,t)$ $0 \leq t \leq 1$



de Casteljau algorithm

Take $t = \frac{1}{2}$



Observe that $\gamma(t)$ is also the concatenation

of two smaller Bézier curves $\gamma_1(t)$ $0 \leq t \leq \frac{1}{2}$ from C'_0, C'_1, C'_2, C'_3

$\gamma_2(t)$ $\frac{1}{2} \leq t \leq 1$ from C'_3, C'_4, C'_5, C'_6

$$C'_0 = C_0$$

$$C'_1 = \frac{1}{2}C_0 + \frac{1}{2}C_1$$

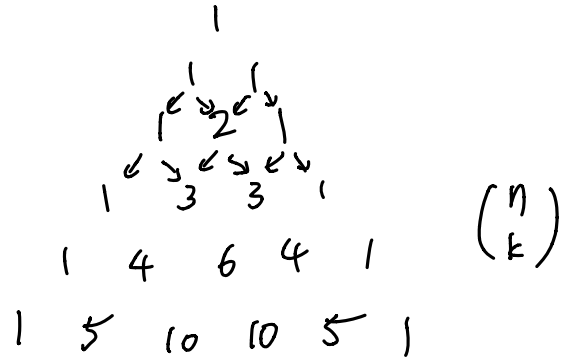
$$C'_2 = \frac{1}{2}\left(\frac{1}{2}C_0 + \frac{1}{2}C_1\right) + \frac{1}{2}\left(\frac{1}{2}C_1 + \frac{1}{2}C_2\right) = \frac{1}{4}C_0 + \frac{1}{2}C_1 + \frac{1}{4}C_2$$

$$= \frac{1}{4}(C_0 + 2C_1 + C_2)$$

$$C'_3 = \frac{1}{8}(C_0 + 3C_1 + 3C_2 + C_3)$$

$$r_1: \begin{bmatrix} C'_0 \\ C'_1 \\ C'_2 \\ C'_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

$$r_2: \begin{bmatrix} C'_3 \\ C'_4 \\ C'_5 \\ C'_6 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \\ & \frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\ & & \frac{1}{2} & \frac{1}{2} \\ & & & 1 \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{bmatrix}$$



Spline

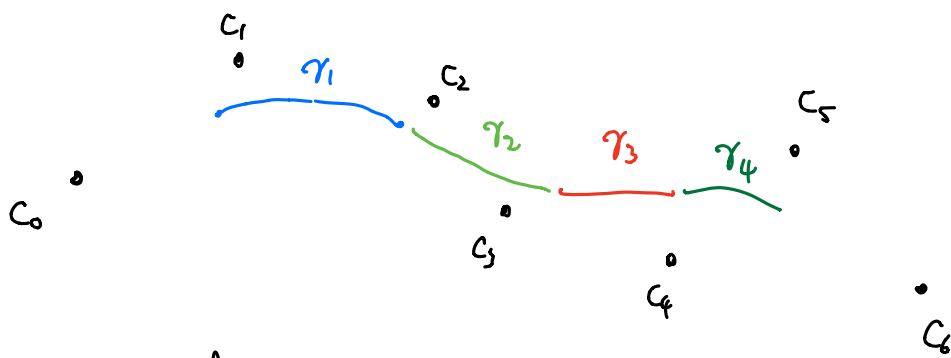
directly evaluate
 $r(t)$ for each t .

Subdivision

Instead of drawing $r(t)$,
we just draw the control polygon.
A subdivision is a refinement of the
control polygon.

Thm Under recursive subdivision, these
refined control polygons converge to the
spline $r(t)$.

B-spline (de Casteljau style)



A ^{polynomial} degree $(m-1)$ or order m B-spline is a curve $\gamma(t)$

$$0 \leq t \leq 6$$

$$\gamma(t) = \begin{cases} \gamma_0(t) & 0 \leq t < 1 \\ \gamma_1(t) & 1 \leq t < 2 \\ \vdots & \\ \gamma_5(t) & 5 \leq t < 6 \end{cases}$$

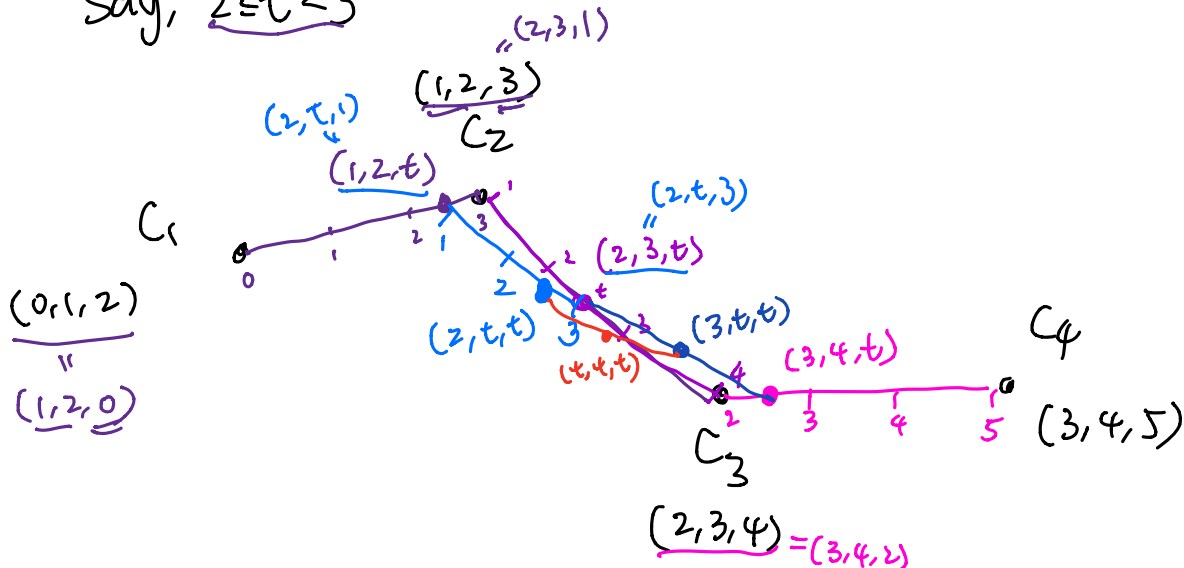
and $\gamma_i(t)$ is controlled by the neighboring m control pts.

For example, in a cubic B-spline (order-4 B-spline)

$\gamma_i(t)$ ($i \leq t < i+1$) is controlled by

$C_{i-1}, C_i, C_{i+1}, C_{i+2}$.

Say, $2 \leq t < 3$



Set $F(0,1,2) = C_1$
 $F(1,2,3) = C_2$
 $F(2,3,4) = C_3$
 $F(3,4,5) = C_4$

$\rightarrow F(1,2,t) = \frac{t-3}{0-3} C_1 + \frac{t-0}{3-0} C_2$
 $\rightarrow$
 $\rightarrow$
 $\rightarrow$

$$\gamma_2(t) = F(t, t, t)$$

- B-spline of order m is a piecewise $(m-1)$ polynomial with C^{m-2} continuity, with local control.

- What we have seen is the uniform B-spline where the partition in t is just $0, 1, 2, \dots$

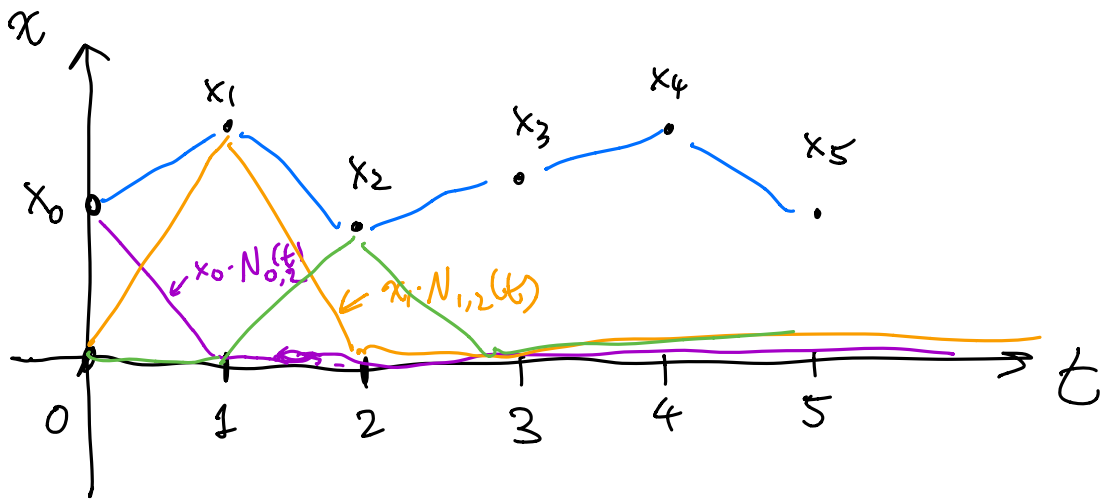
A non-uniform B-spline uses some partition such as

$$0 \leq t < 3 \quad 3 \leq t < 4 \quad 4 \leq t < 7, \dots$$

in which case set $F(0, 3, 4) = C_0$
 $F(3, 4, 7) = C_1$
 $\vdots$
 etc

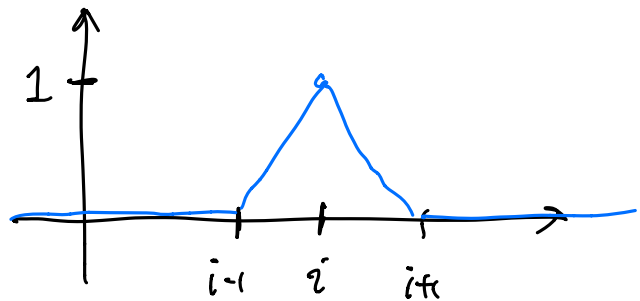
B-spline (as basis-spline)

Order 2 B-spline (piecewise linear spline)



$$\gamma(t) = \begin{cases} \gamma_0(t) = (1-t)x_0 + tx_1 & 0 \leq t < 1 \\ \gamma_1(t) = \frac{t-2}{1-2}x_1 + \frac{t-1}{2-1}x_2 & 1 \leq t < 2 \\ \vdots \end{cases}$$

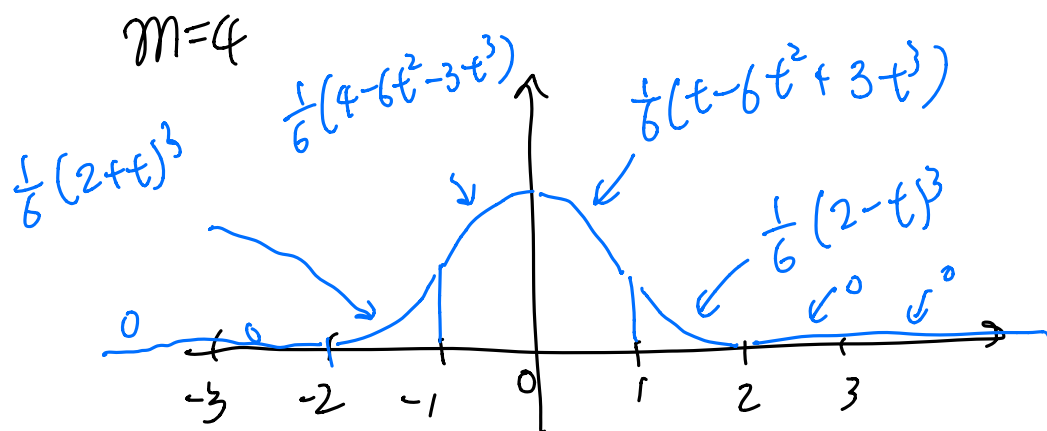
$$= \sum_{i=0}^4 \underbrace{N_{i,2}(t)}_{\text{basis}} x_i \quad \text{where } N_{i,2}(t) \text{ looks like}$$



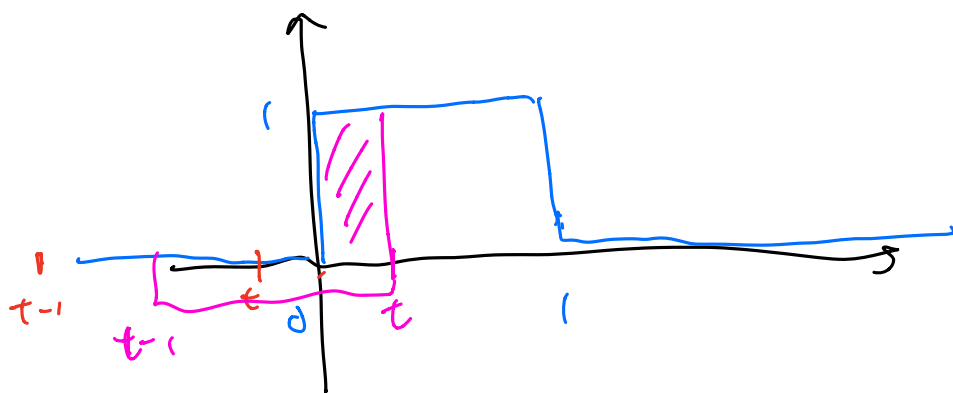
For order m B-spline

$$\gamma(t) = \sum_{i=0}^n N_{i,m}(t) x_i \quad \text{where}$$

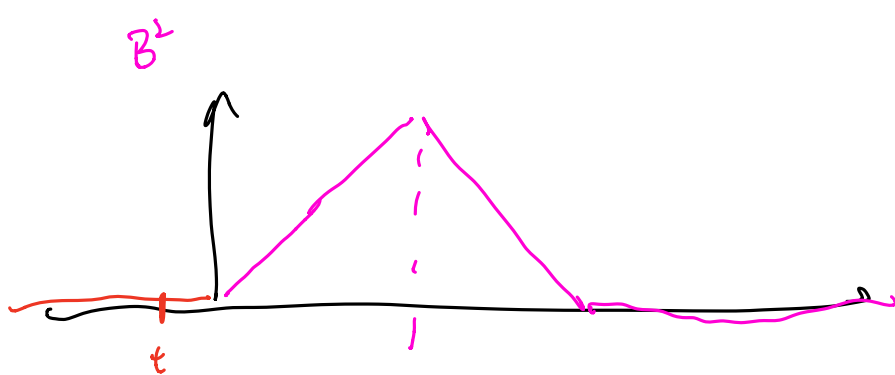
- $N_{i,m}(t) = N_{0,m}(t-i)$
- $N_{0,m}(t)$ is piecewise degree $(n-1)$ polynomial in each $[j, j+1)$.
- $N_{0,m} \in C^{m-2}$
- $N_{0,m}(t) = 0$ for t outside a window of width m .



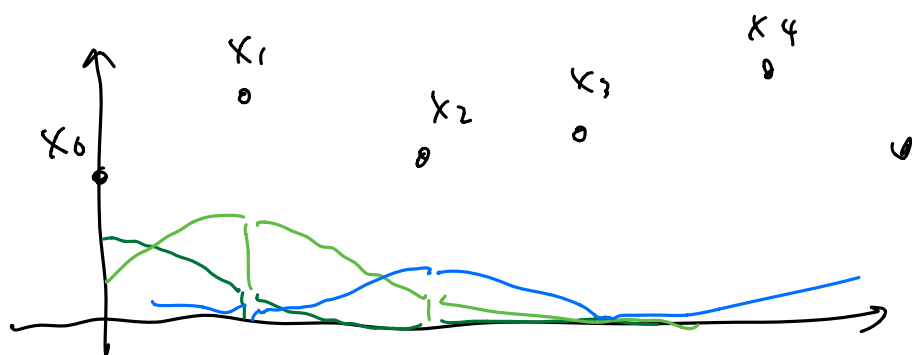
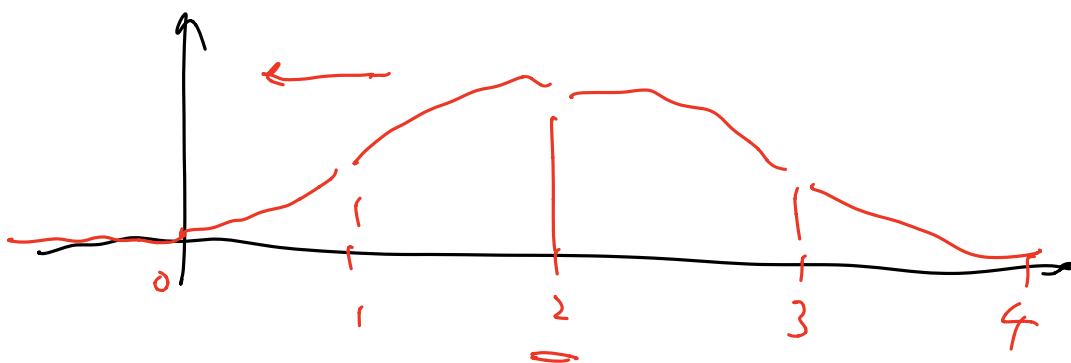
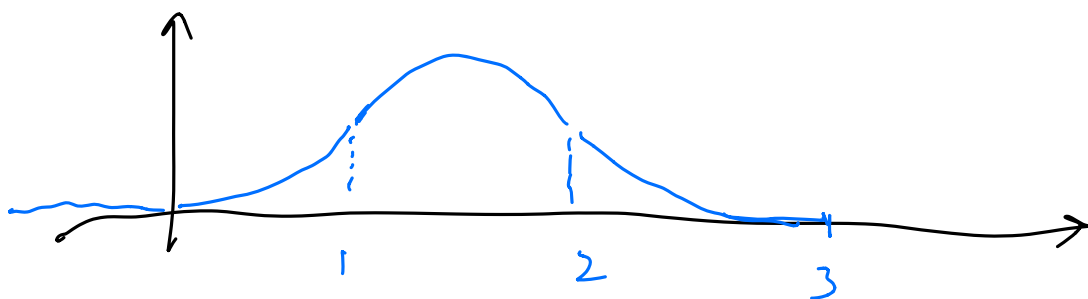
Start with $B'(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$



$$B^2(t) = \int_{\tau=0}^1 \overset{u}{B'(t-\tau)} d\tau = \int_{t-1}^t B'(u) du$$



$$B^3(t) = \int_{t-1}^t B^2(u) du$$



$$2 \leq t < 3$$

$$\gamma_2(t) = \underbrace{\text{cubic polynomial}}_{x_1} + \underbrace{\quad}_{x_2} + \underbrace{\quad}_{x_3} + \underbrace{\quad}_{x_4}$$

$$\text{let } \tau = t^{-2} \quad \text{if } \tau < 1$$

$$= \begin{bmatrix} 1 & \tau & \tau^2 & \tau^3 \end{bmatrix} \begin{bmatrix} \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix} \begin{Bmatrix} C_{i4} \\ C_i \\ C_{i41} \\ C_{i42} \end{Bmatrix} \quad i=2$$