

Splines

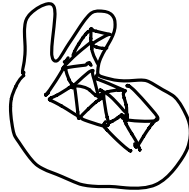
Motivation

We have been using
low level representation of geometry

- lines



- triangles



Pros:

- Easy to store
- Easy to render

Cons:

- Not easy to edit

- Not easy to make a smooth curve or surface

- Not good for design.

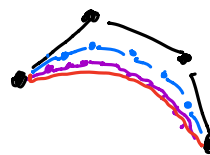


Need:

high level representation of geometry.

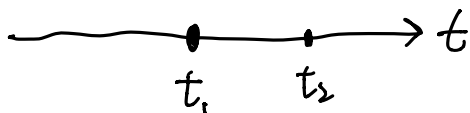
(generate lines & triangles easily)

- Hermite spline
 - Bézier curve
 - B-spline
 - Subdivision
- } splines

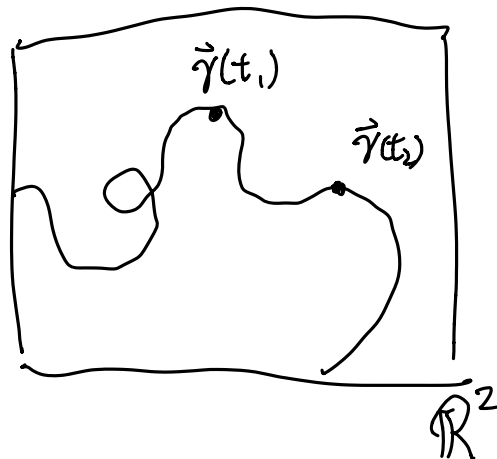


Parametric geometry

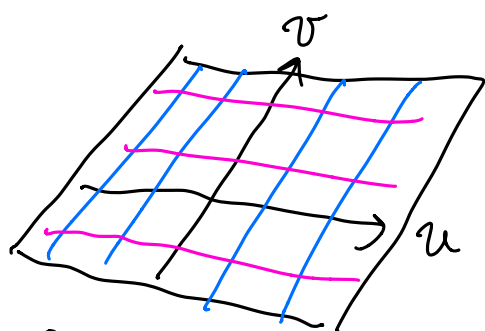
$$\vec{\gamma}: t \mapsto \vec{\gamma}(t) \in \mathbb{R}^2$$



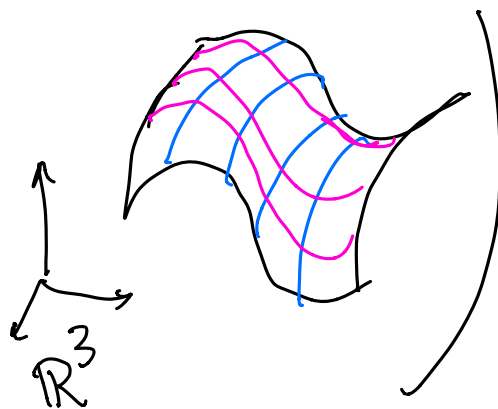
easy to evaluate



surface



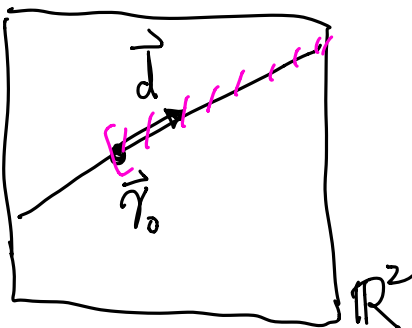
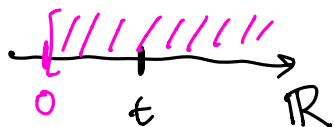
$$\vec{f}(u,v)$$



example

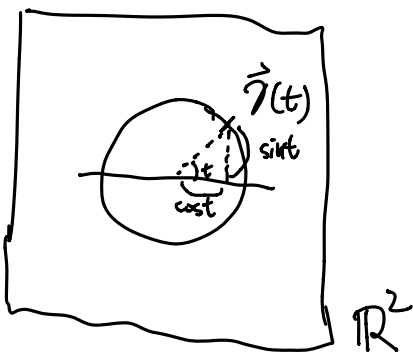
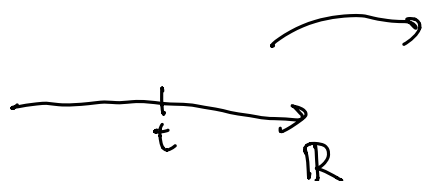
• Straight line

$$\vec{\gamma}(t) = \vec{\gamma}_0 + t\vec{d}$$



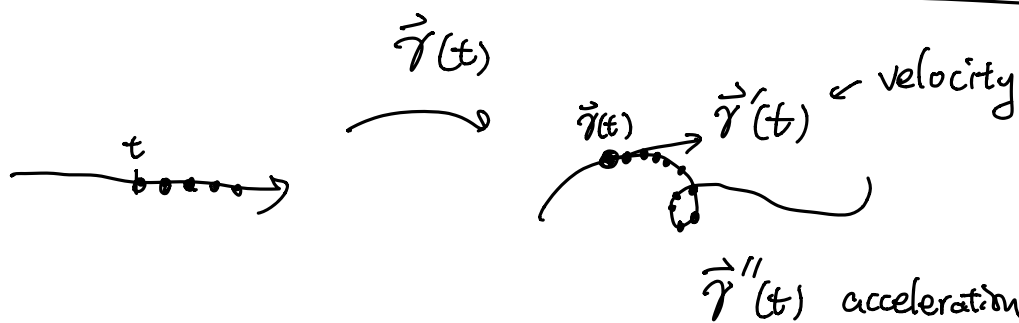
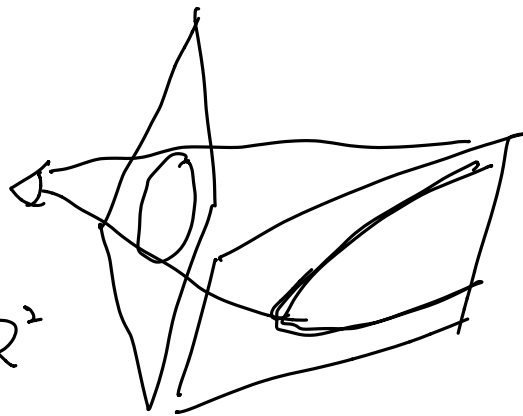
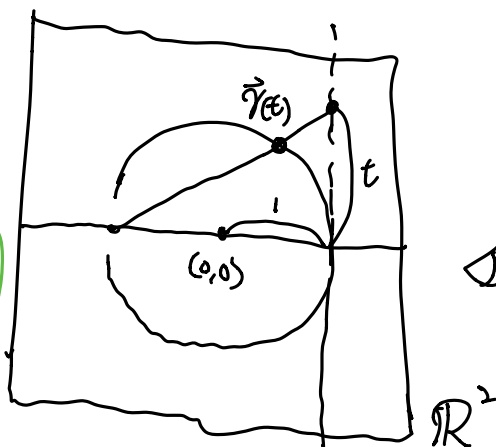
• circle

$$\vec{\gamma}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$



$$\begin{pmatrix} \frac{1-t^2}{1+t^2} \\ \frac{2t}{1+t^2} \end{pmatrix}$$

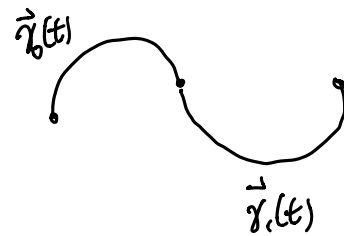
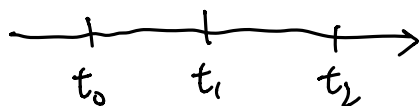
$$\begin{pmatrix} \frac{1-t^2}{1+t^2} \\ \frac{2t}{1+t^2} \\ 1 \end{pmatrix} \sim \begin{pmatrix} 1-t^2 \\ 2t \\ 1+t^2 \end{pmatrix}$$



$$(at^p)' = ap t^{p-1}$$

Piecewise defined parametric curves

$$\vec{r}(t) = \begin{cases} \vec{r}_0(t) & \text{if } t \in [t_0, t_1) \\ \vec{r}_1(t) & \text{if } t \in [t_1, t_2) \\ \vdots \end{cases}$$

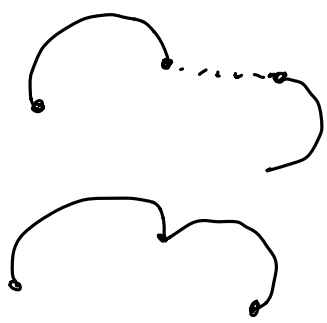


We may get

Terminology:

We say $\vec{r}(t)$ is C^0 (zeroth order continuous)

if $\vec{r}(t)$ is a continuous function



We say $\vec{r}(t)$ is C^1 (1st order continuous)

if $\vec{r}(t)$ is C^0 and $\vec{r}'(t)$ is also C^0 .

We say $\vec{\gamma}(t)$ is C^2

if $\vec{\gamma}(t), \vec{\gamma}'(t), \vec{\gamma}''(t)$ are C^0

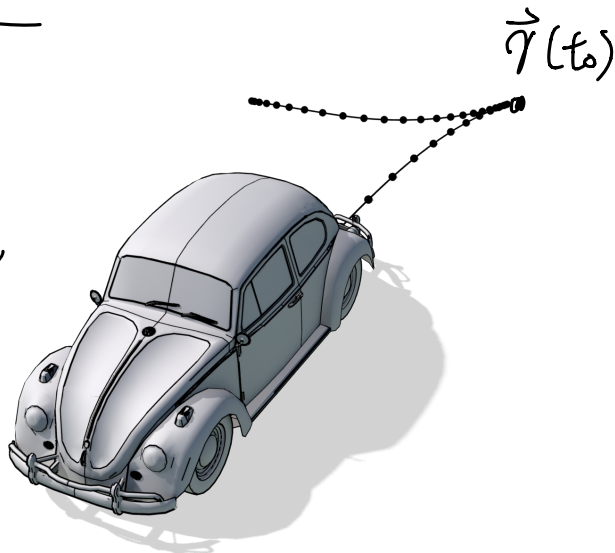
$\vdots$

If, however, $\vec{\gamma}'(t_0) = 0$

then even if $\vec{\gamma} \in C^k$ $k > 1$,

we may have a "kink"
or
"cusp"

i.e. the tangent $\frac{\vec{\gamma}'}{|\vec{\gamma}'|}$ may still be discontinuous.

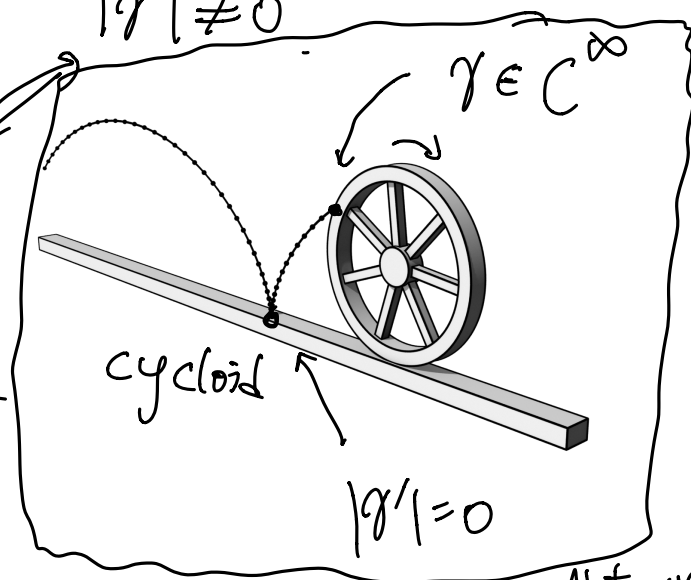
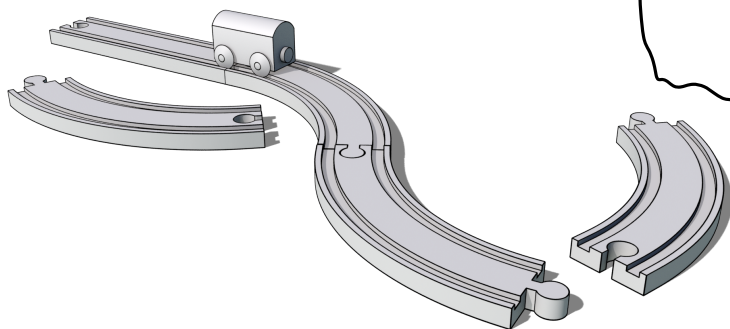


We say $\vec{\gamma}$ is G^k (geometric continuity of order k)

if $\vec{\gamma}$ is C^k and $|\vec{\gamma}'| \neq 0$

$$\gamma_x'^2 + \gamma_y'^2 \neq 0$$

G^1 but not G^2



Not even G^1 .

Splines (piecewise) $\underbrace{\text{polynomial}}_{\text{(rational)}} \vec{\gamma}(t)$.

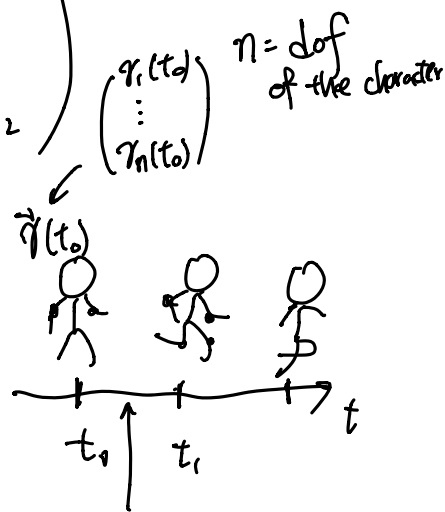
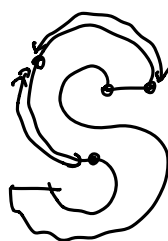
(for rational $\vec{\gamma}(t) = \begin{pmatrix} \gamma_x(t) \\ \gamma_y(t) \end{pmatrix} = \begin{pmatrix} \frac{p_1(t)}{q_1(t)} \\ \frac{p_2(t)}{q_2(t)} \end{pmatrix}$ p_1, p_2, q_1, q_2 polynomials)

$\Leftrightarrow \vec{\gamma}(t) = \begin{pmatrix} p_1/q_1 \\ p_2/q_2 \\ 1 \end{pmatrix} \sim \begin{pmatrix} p_1 q_2 \\ p_2 q_1 \\ q_1 q_2 \end{pmatrix}$

application: Vector graphics (fonts)

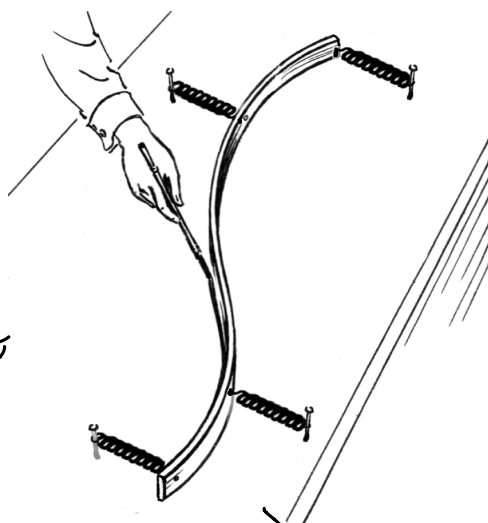
keyframe animation

historically

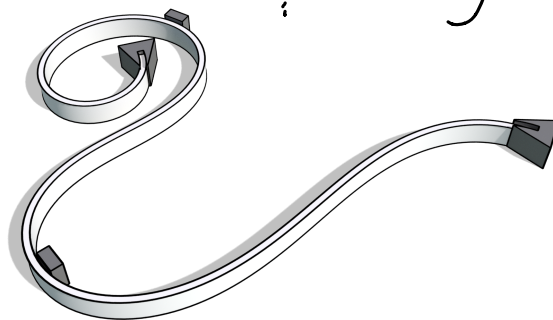


elastic spline

(The curve $\vec{\gamma}(t)$ would "minimize" $\int \left| \frac{(\vec{\gamma}'(t))'}{|\vec{\gamma}'(t)|} \right|^2 dt$



subject to $\left. \begin{aligned} \vec{\gamma}(t_0) &= \vec{p}_0 \\ \vec{\gamma}(t_1) &= \vec{p}_1 \\ &\vdots \end{aligned} \right\} \frac{\vec{\gamma}'(t_0)}{|\vec{\gamma}'(t_0)|} = \vec{d}$



Common misconception: elastic splines are described by polynomials ~~X~~

A good approximation to the elastic curve equation

is that $\vec{\gamma}'''' = 0$ (applies only when the wood is deformed slightly)

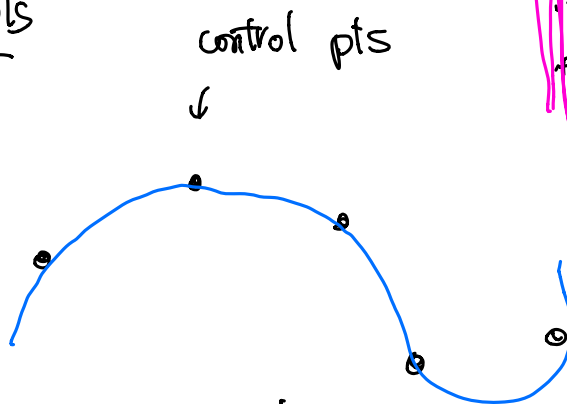
$\vec{\gamma}$ is a cubic polynomial

$\begin{pmatrix} \gamma_x(t) \\ \gamma_y(t) \end{pmatrix}$

$$\gamma_x \text{ or } \gamma_y = at^3 + bt^2 + ct + d.$$

Spline controls

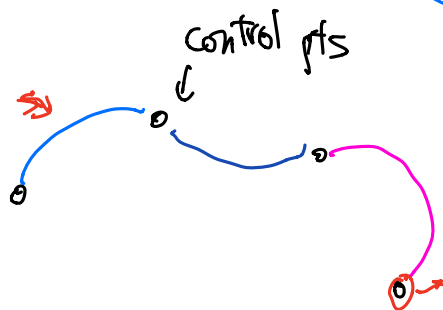
(1)



$\vec{\gamma}$ polynomial passing thru all pts

NOT STABLE

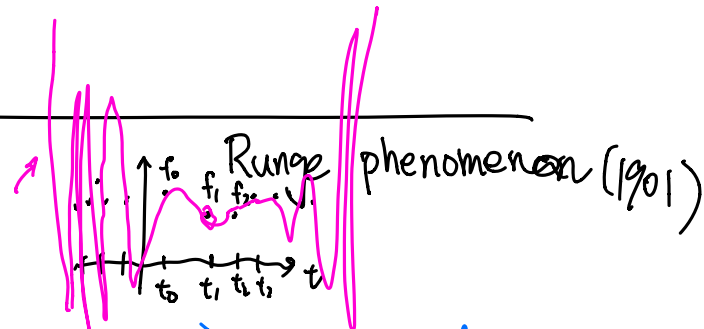
(2)



piecewise polynomials with certain

G^k continuity at the junction

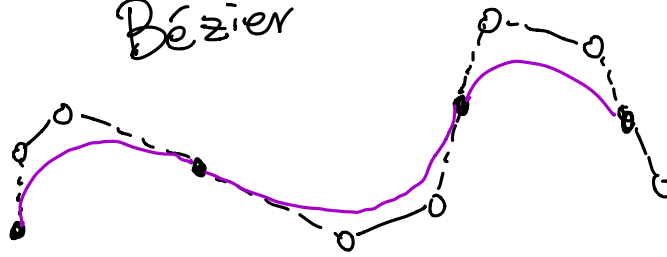
(NOT FREQUENTLY USED)
require solving some big



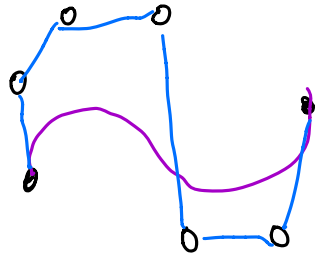
linear system
global effect during editing!

(3)

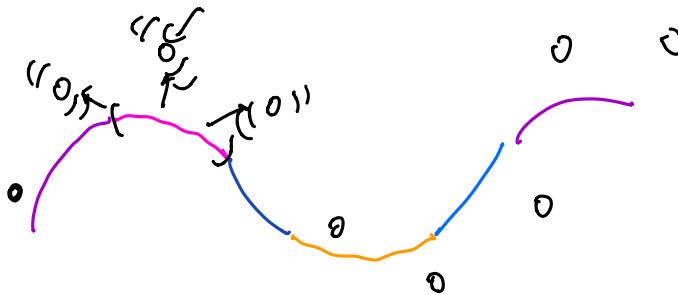
Bézier



The curve only passes through a subset of control pts



(4)



B-spline

The curve does not pass through the control pts,
defined piecewise,
local control.

(Non-uniform
Rational B-spline)
NURBS