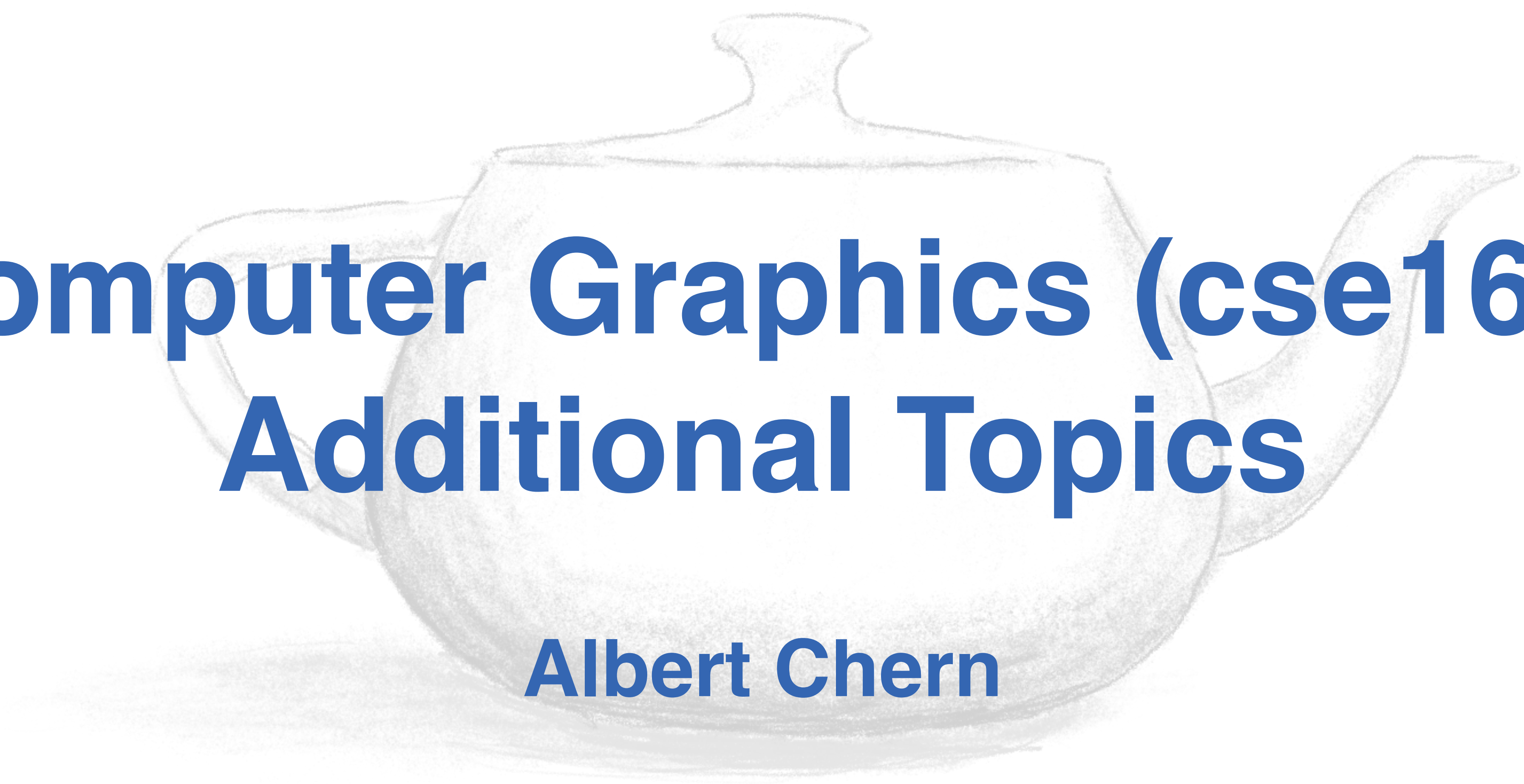


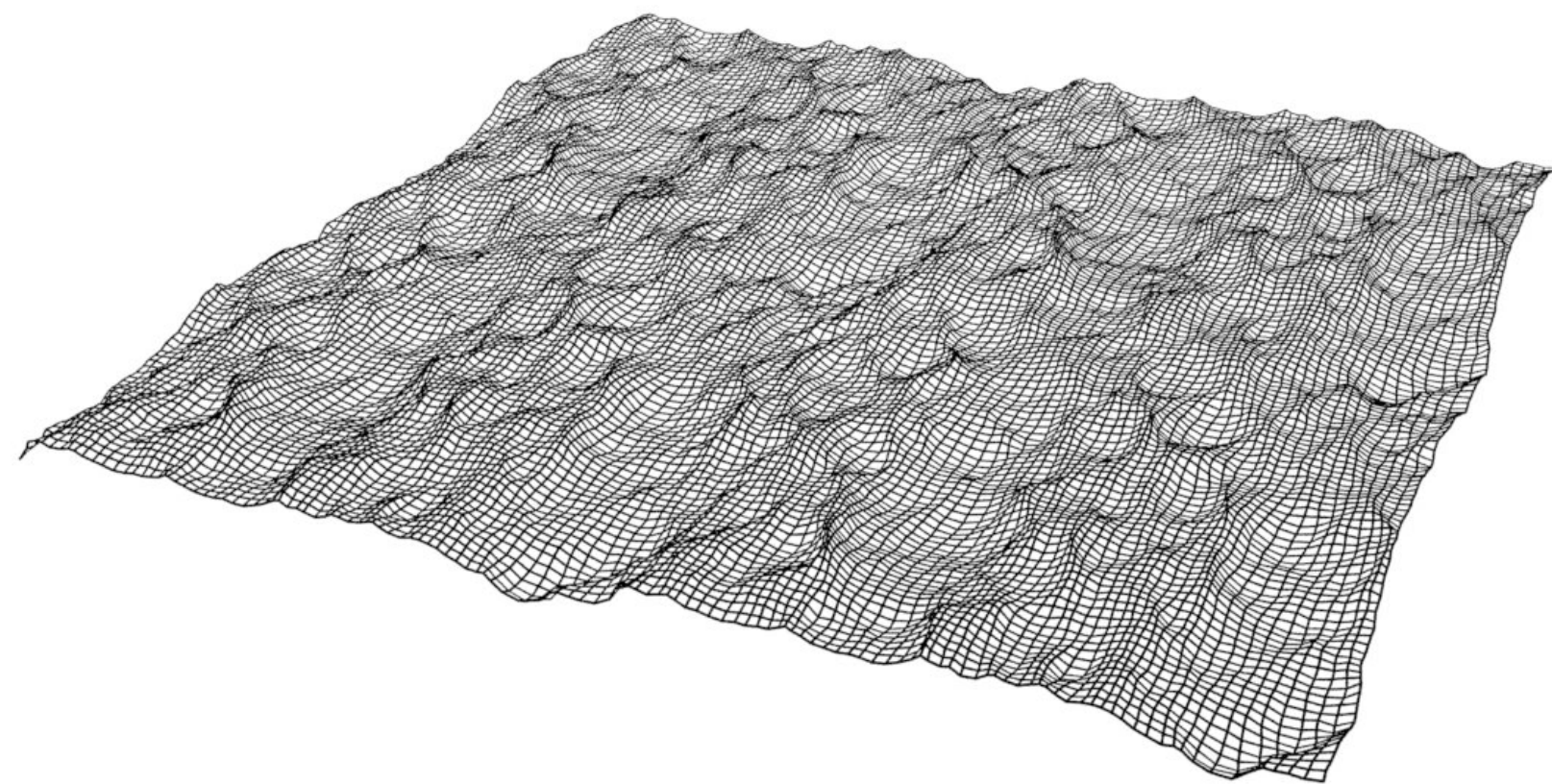


Computer Graphics (cse167) Additional Topics

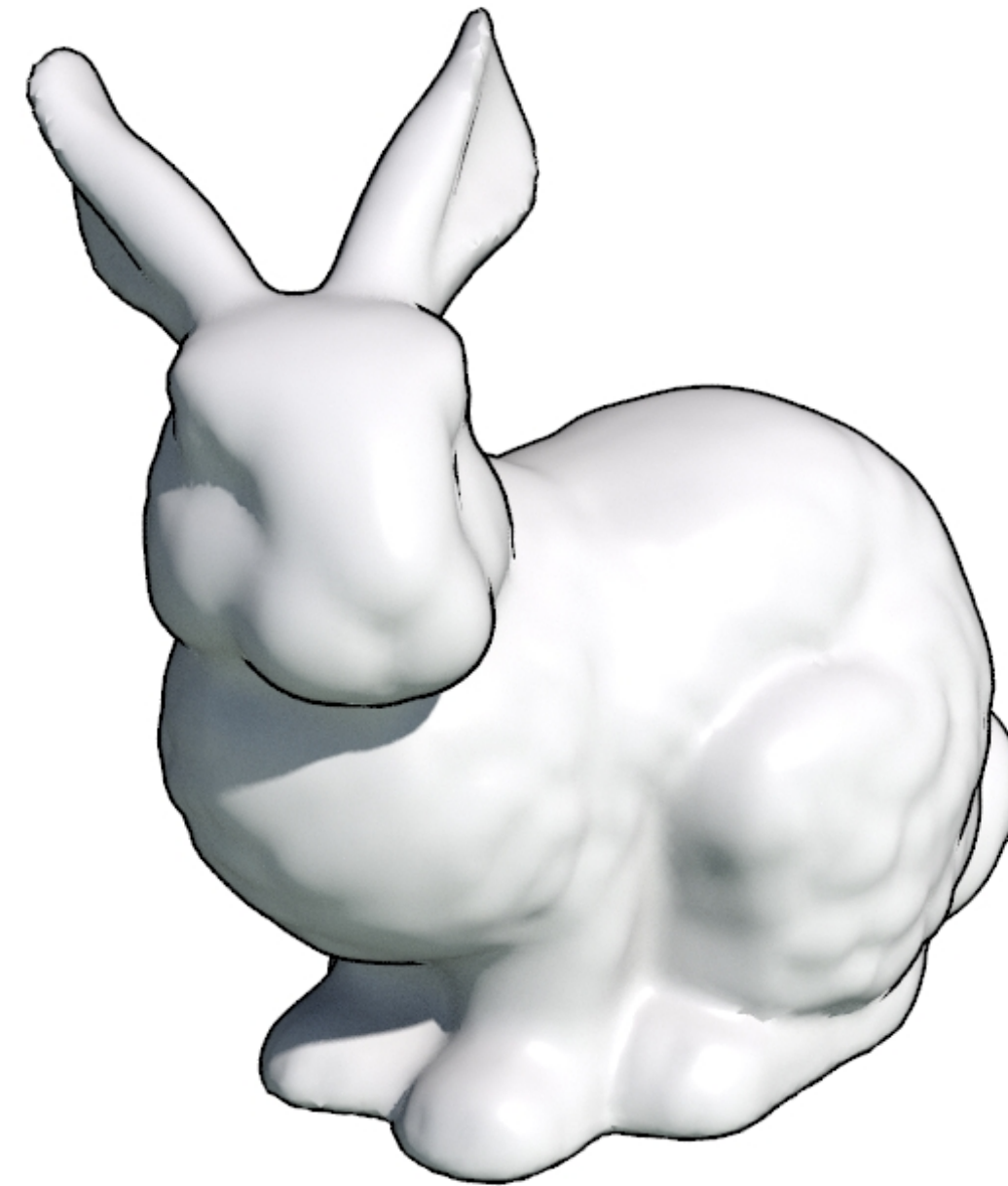
Albert Chern

Computer Graphics

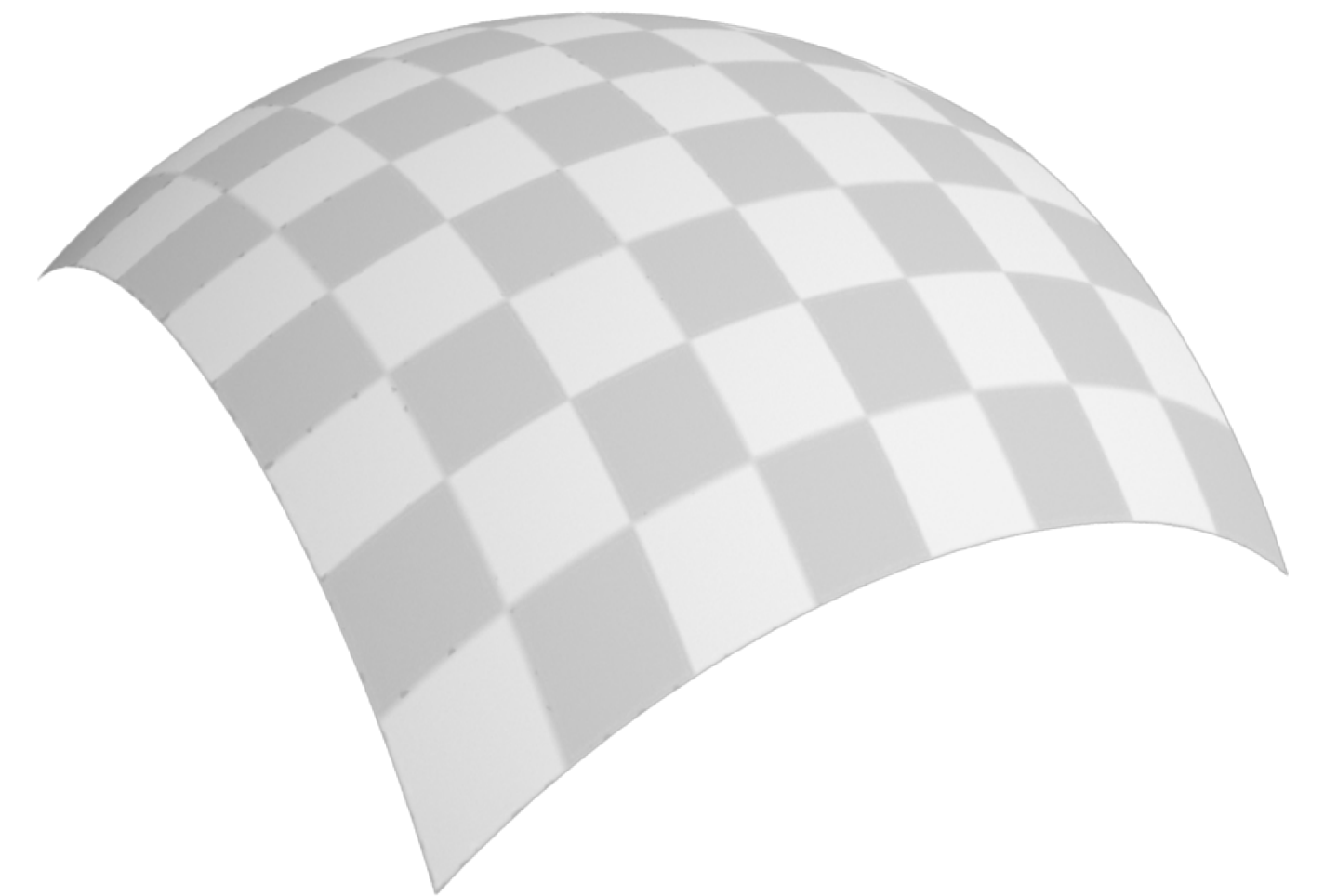
Physics simulation



Geometry processing

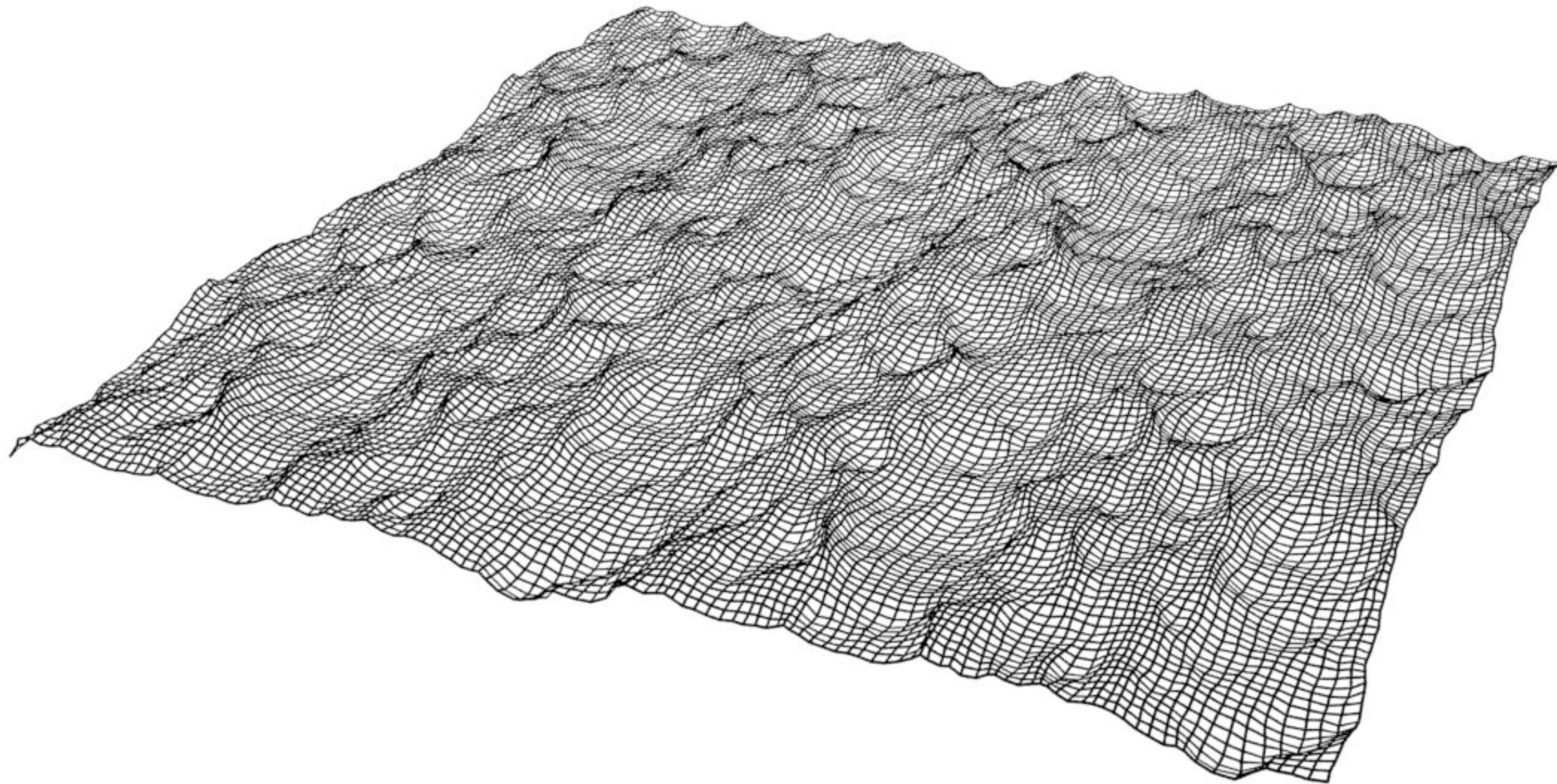


Differential geometry



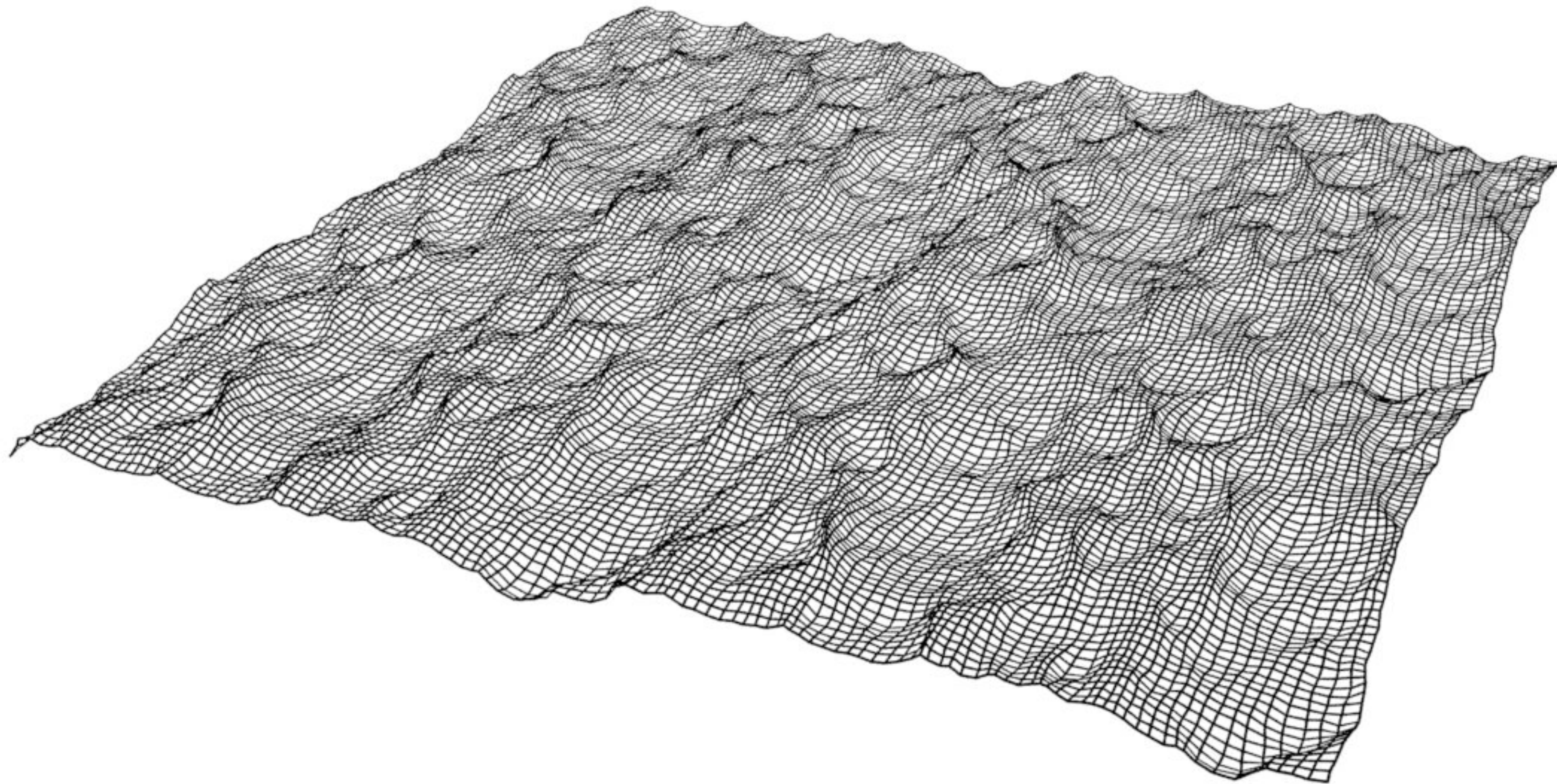
Computer Graphics

Physics simulation



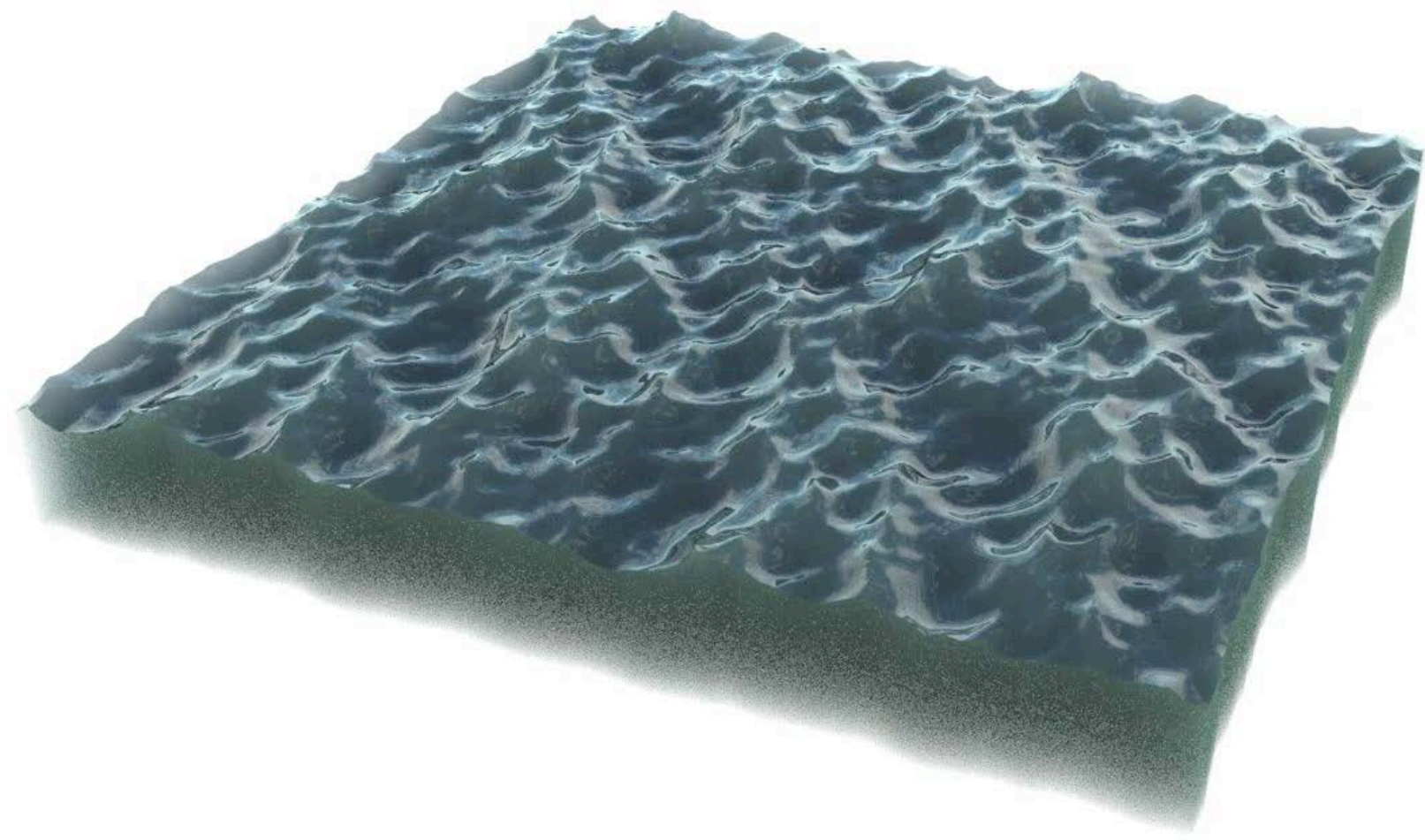
Computer Graphics

Physics simulation

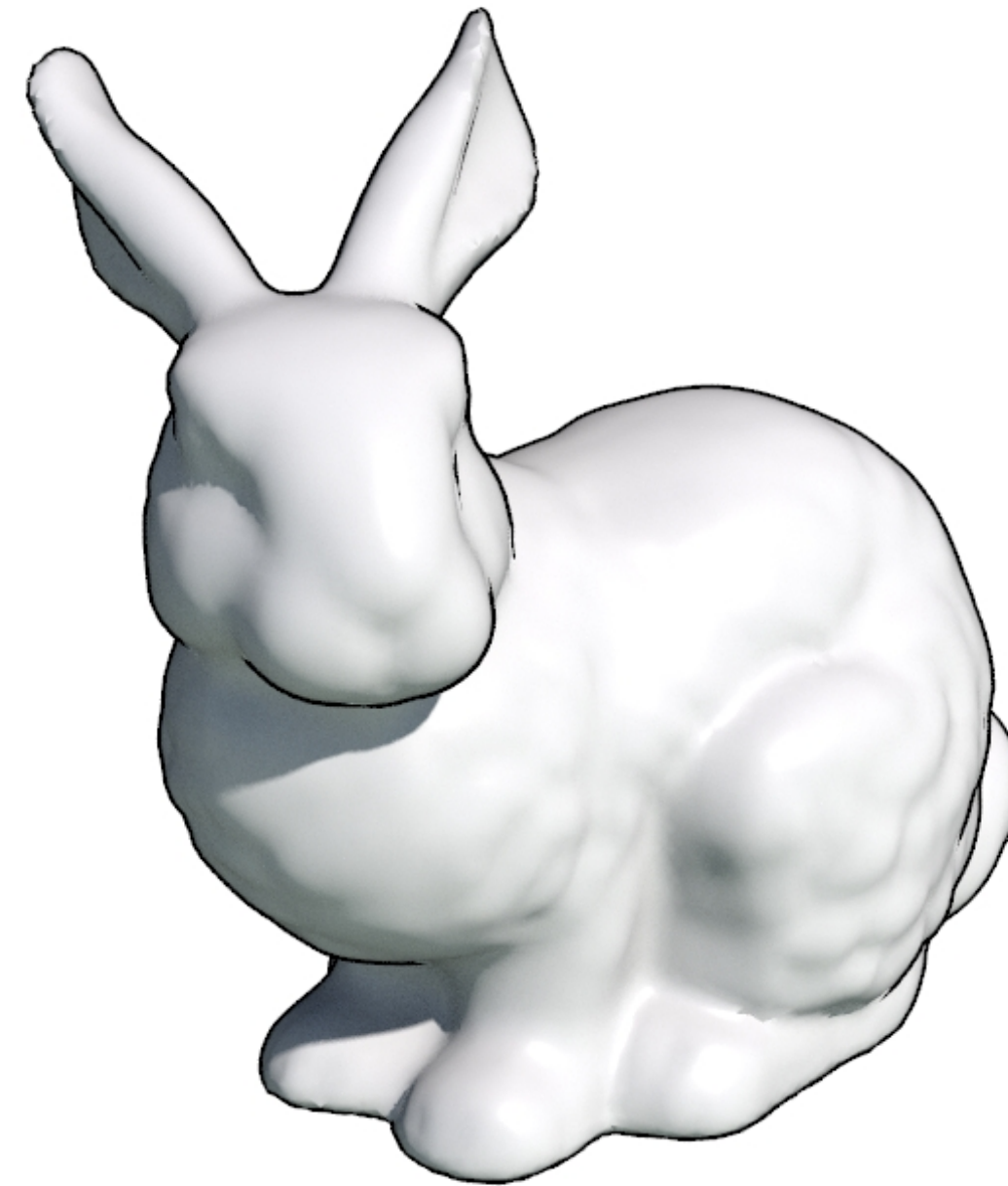


Computer Graphics

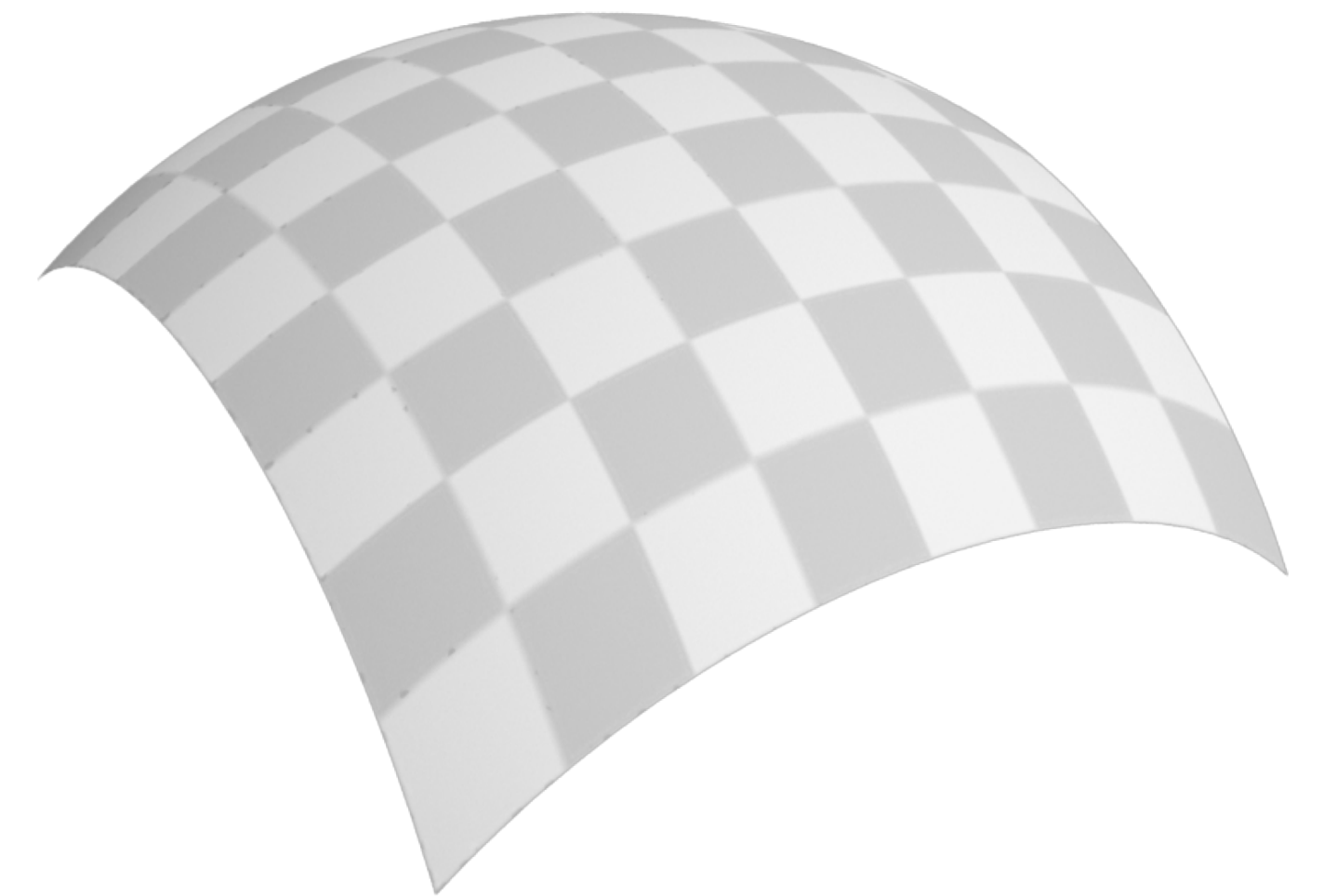
Physics simulation



Geometry processing



Differential geometry



Computer Graphics

Geometry processing



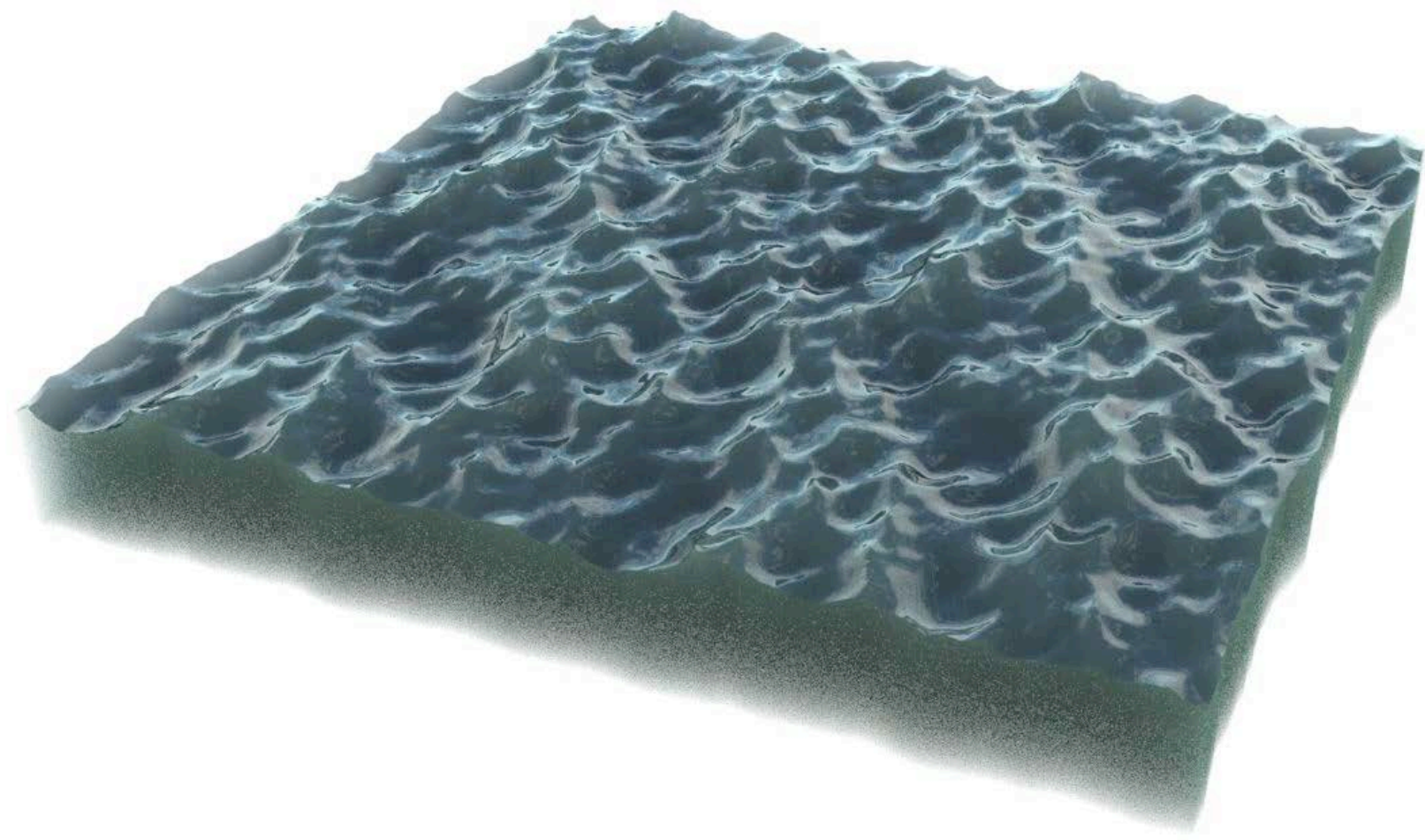
Computer Graphics

Geometry processing



Computer Graphics

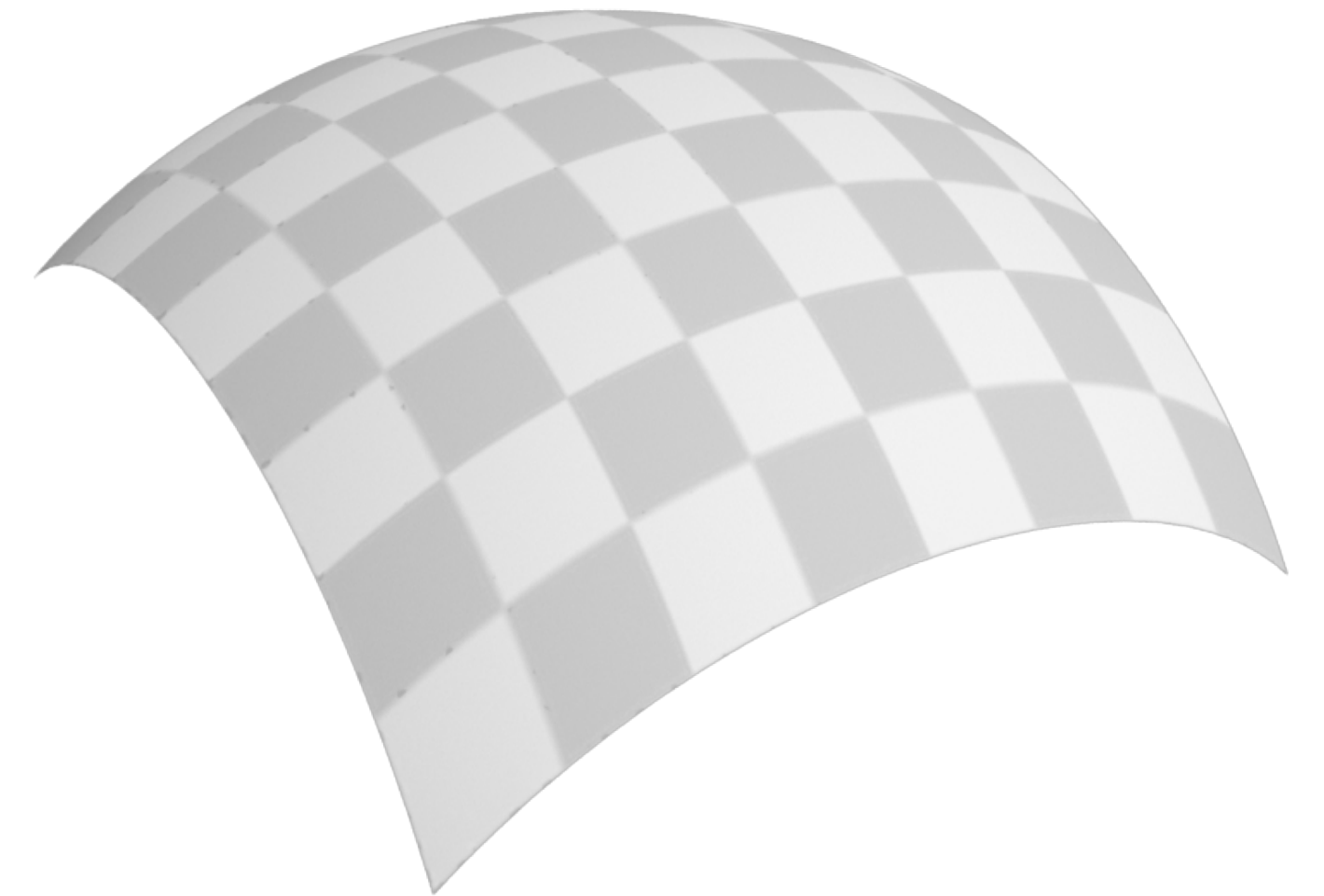
Physics simulation



Geometry processing

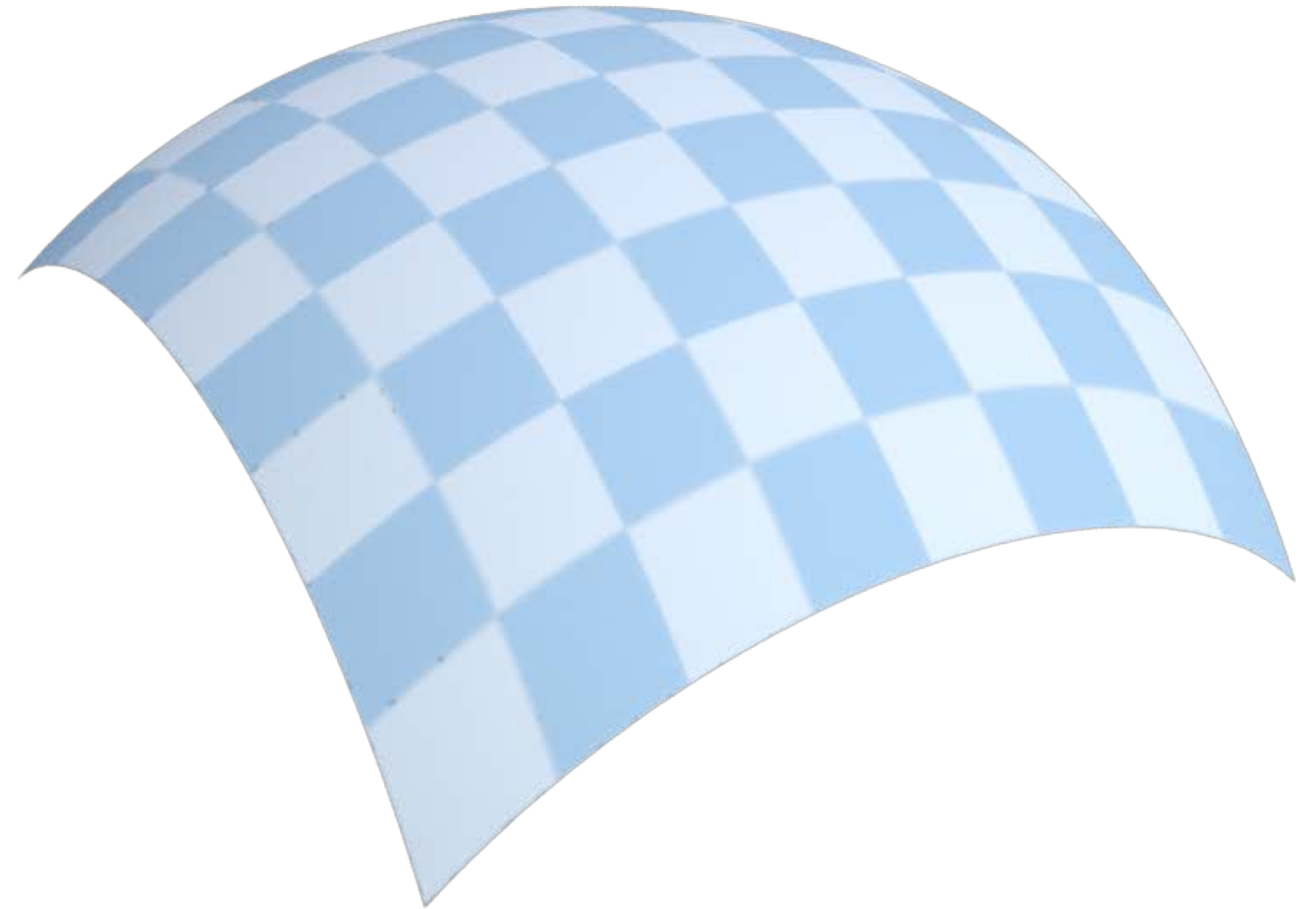


Differential geometry



Computer Graphics

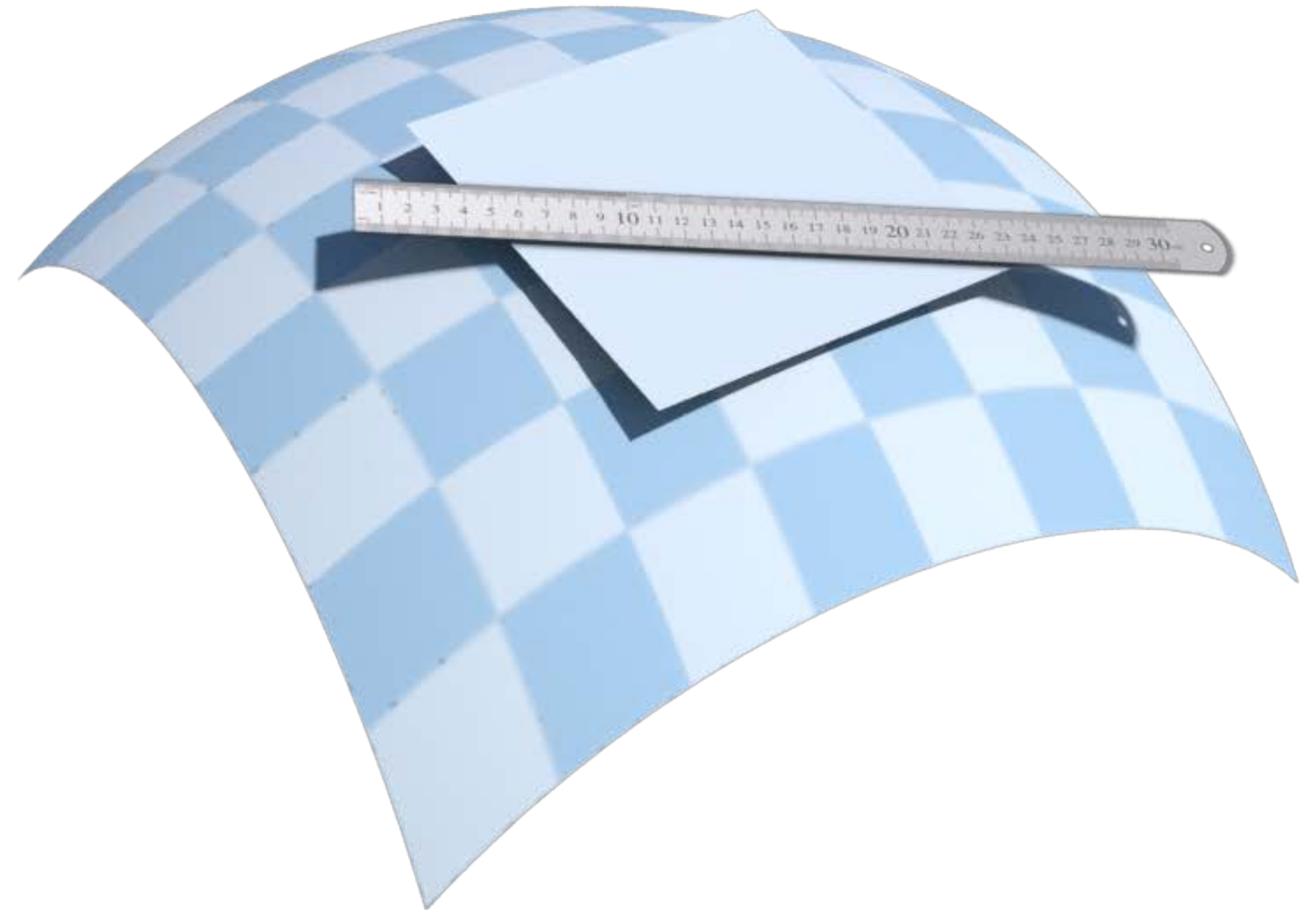
Differential geometry



Computer Graphics

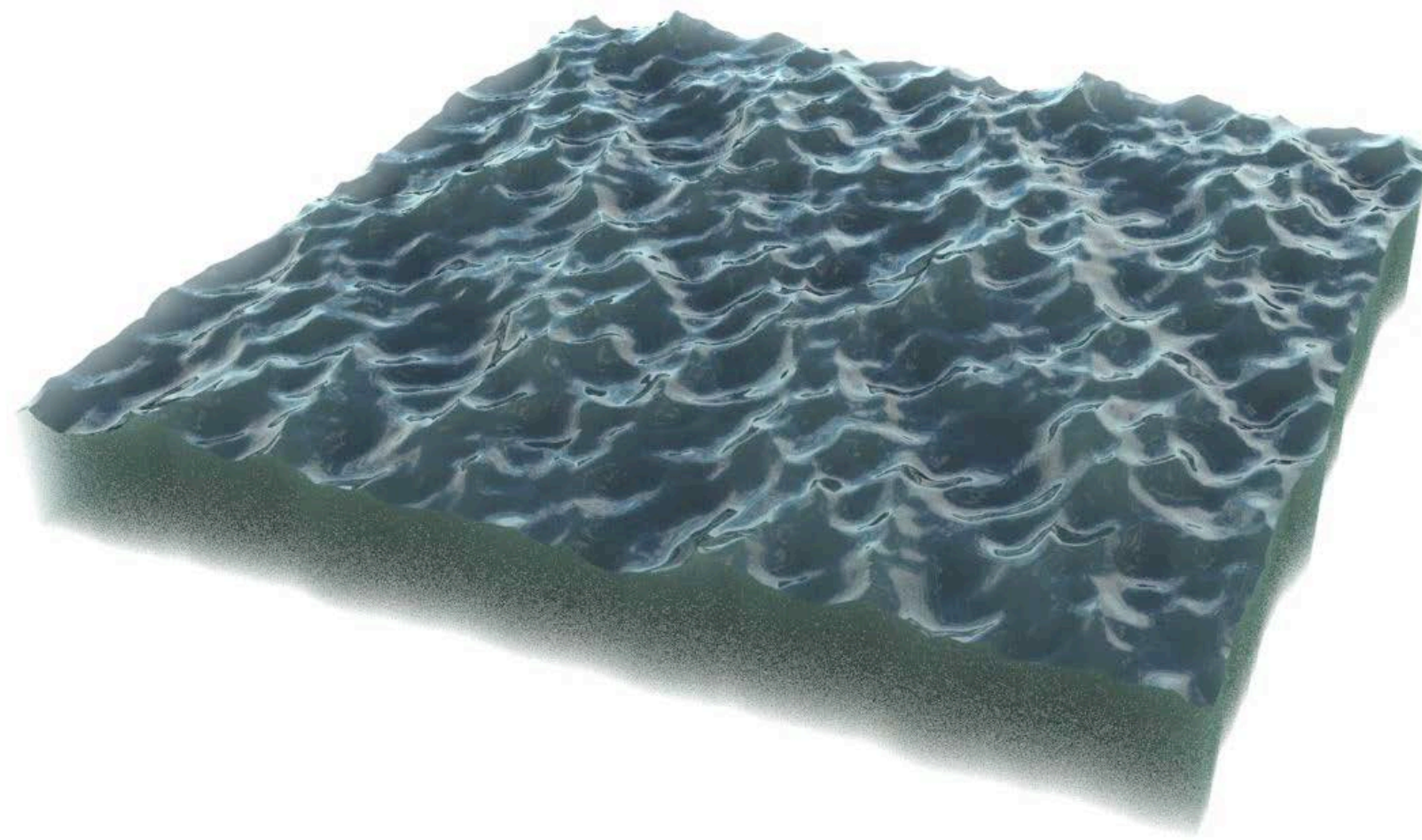
Differential geometry

- measurements on curved spaces
- abstraction and structures



Computer Graphics

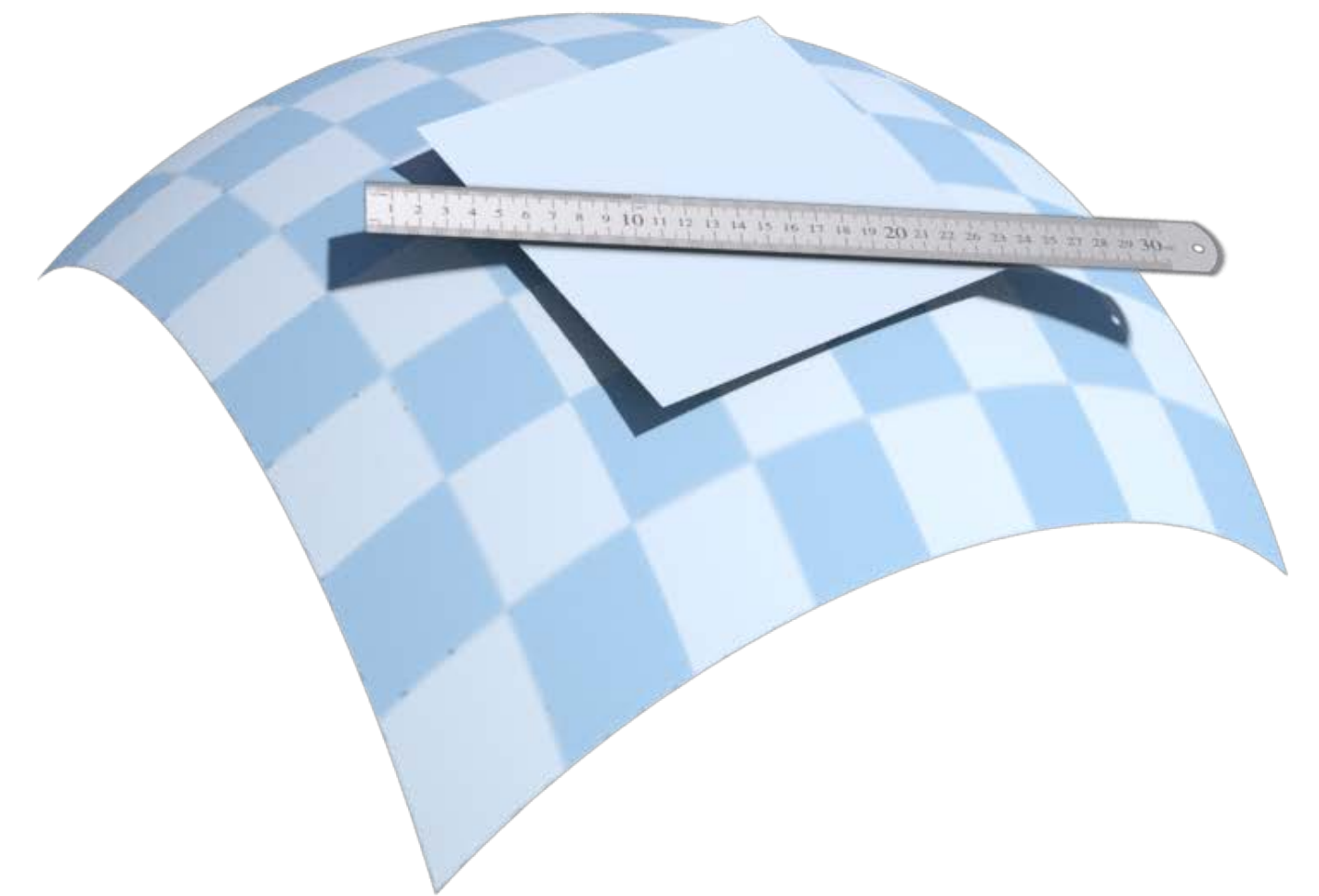
Physics simulation



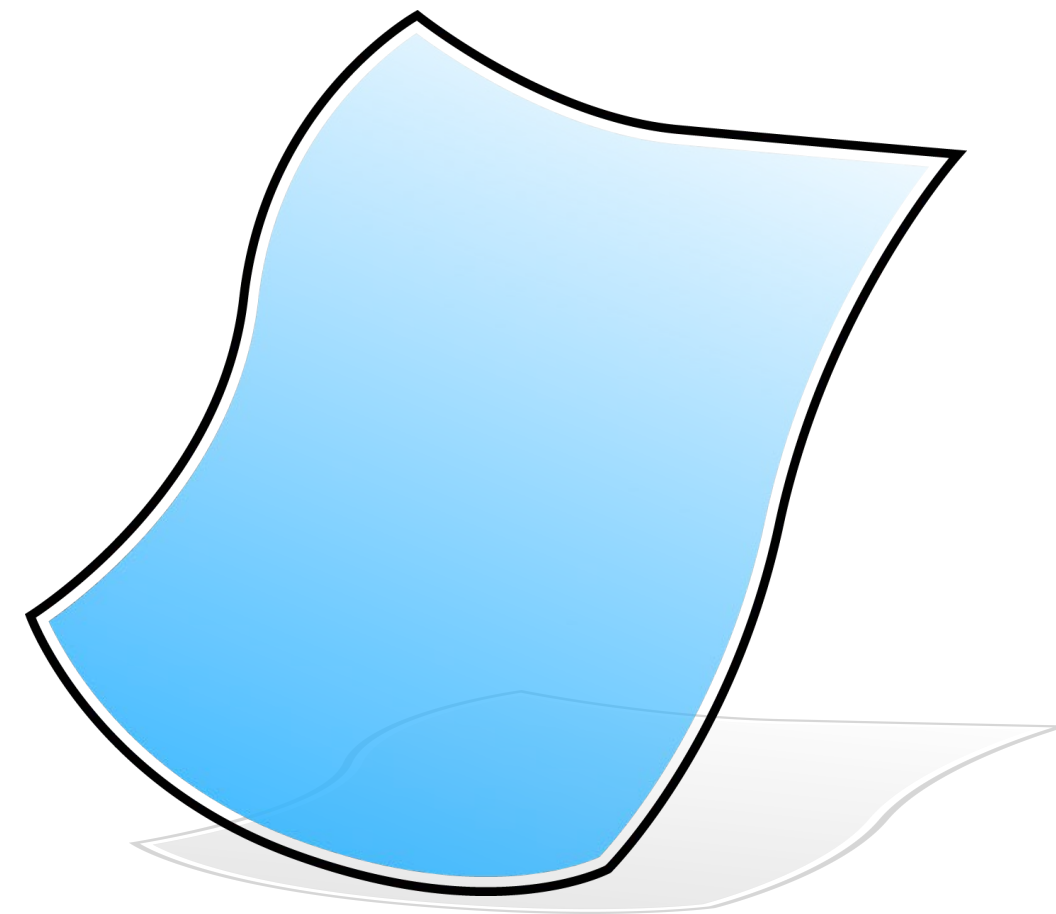
Geometry processing



Differential geometry



Physical world
as a continuum

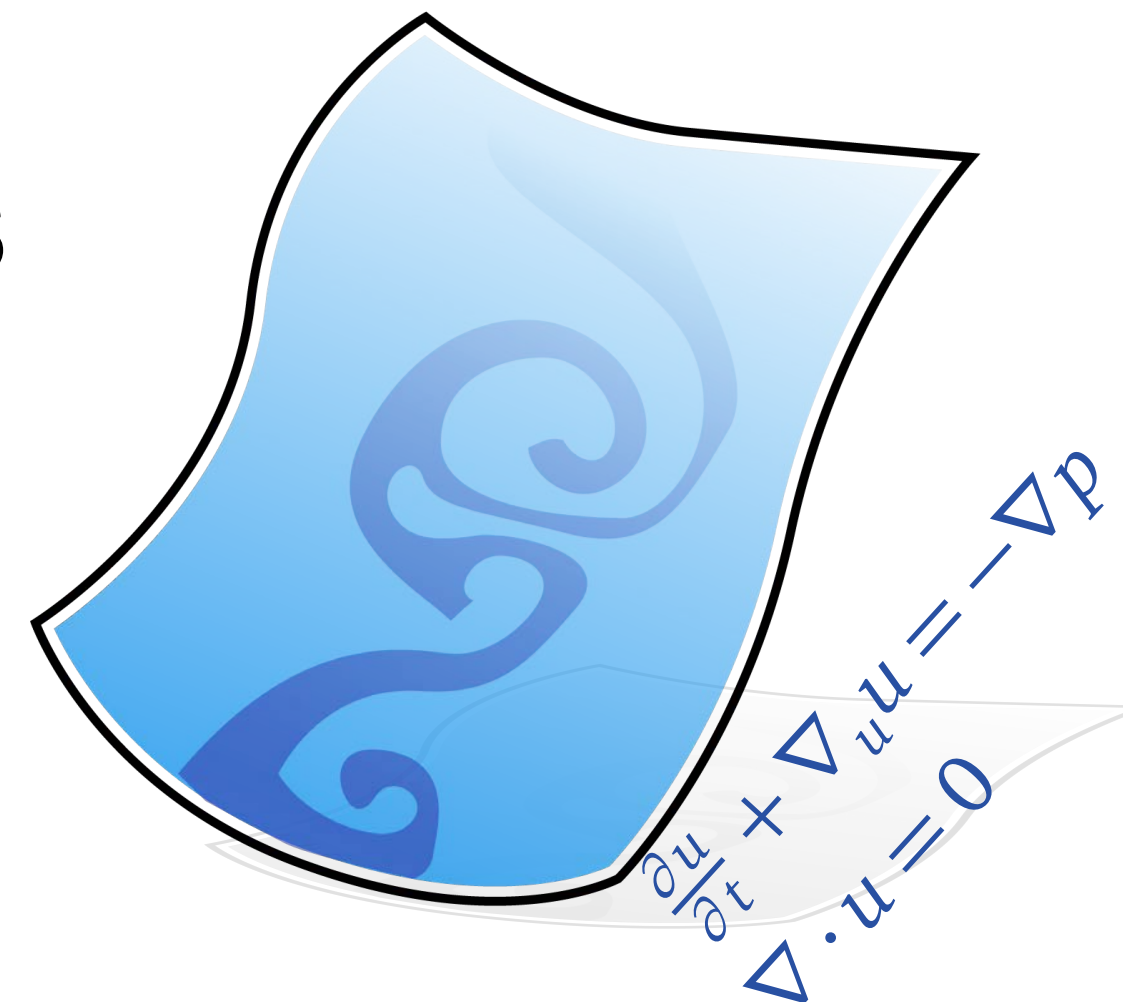


$\mathbb{R}^2, \mathbb{R}^3$,
manifolds etc.

modeling
& analysis

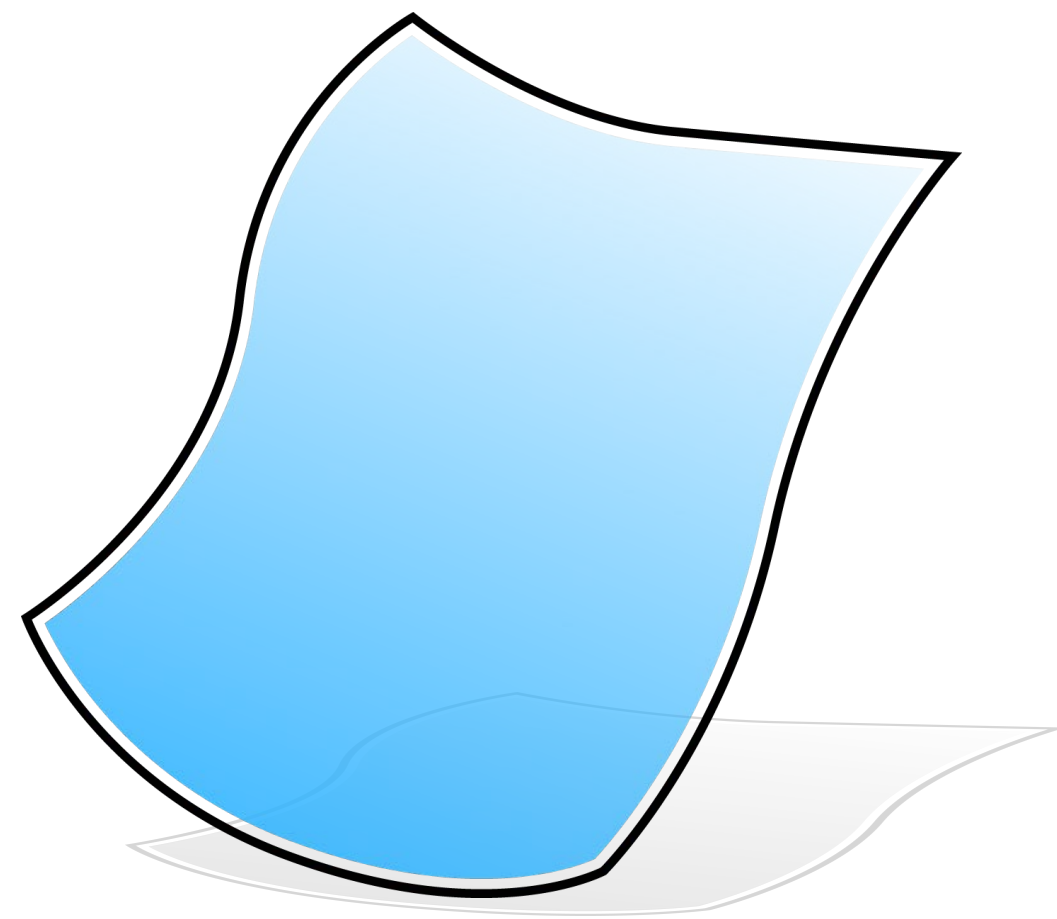


Differential
equations



Maxwell equations,
Navier–Stokes equations
etc.

Physical world
as a continuum



$\mathbb{R}^2, \mathbb{R}^3$,
manifolds etc.

modeling
& analysis



Differential
equations



Maxwell equations,
Navier–Stokes equations
etc.

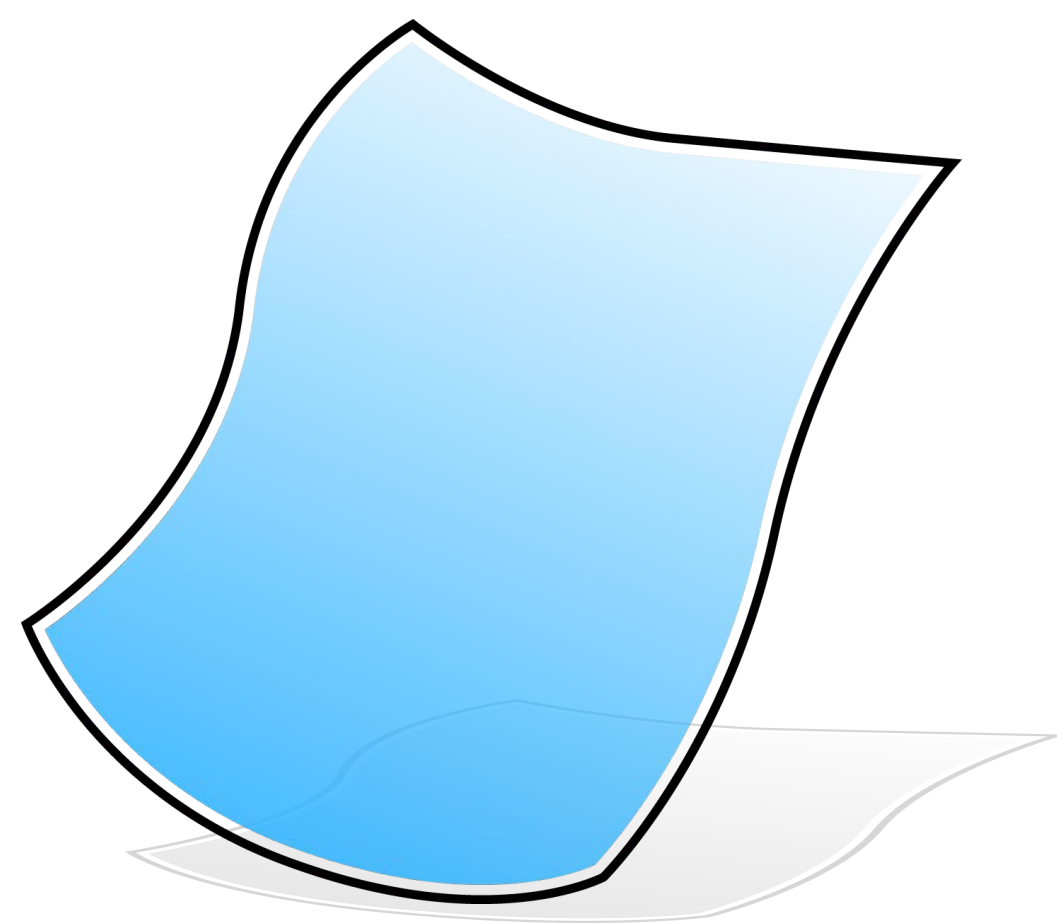
discretization



Numerical
methods



Finite difference,
finite element
etc.

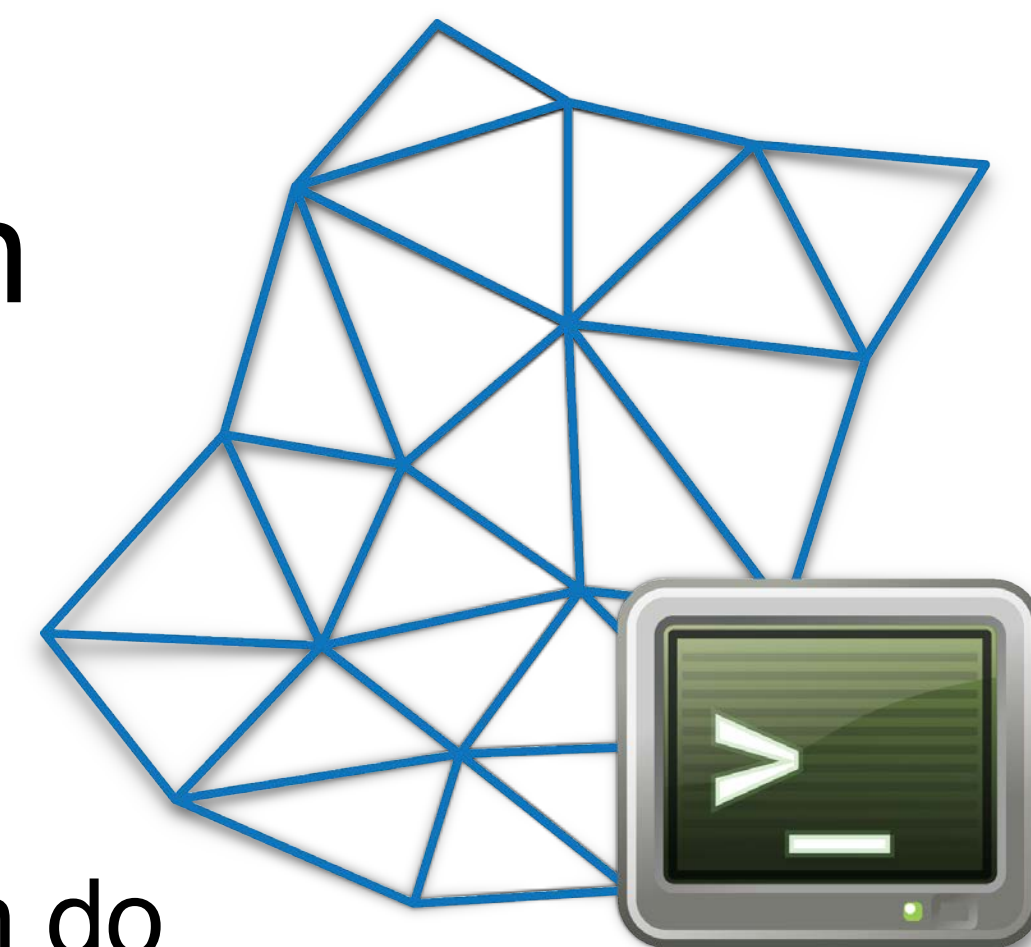


modeling
& analysis



Relations among
objects in
**differential
calculus**

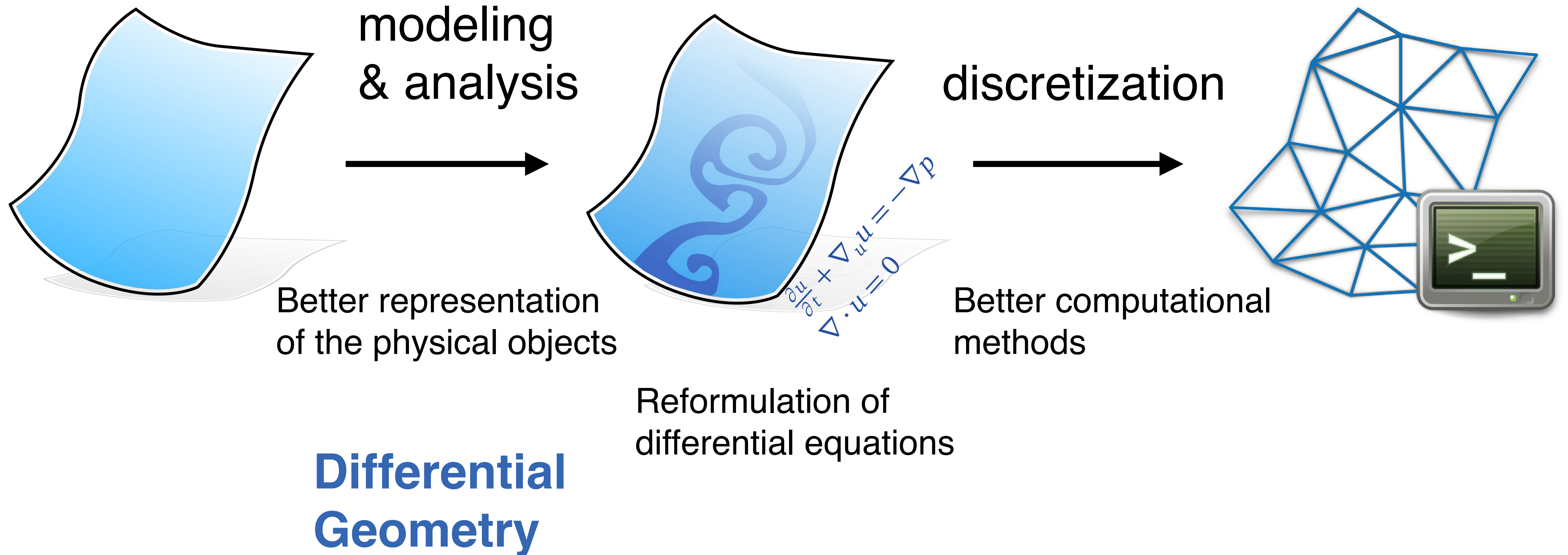
discretization

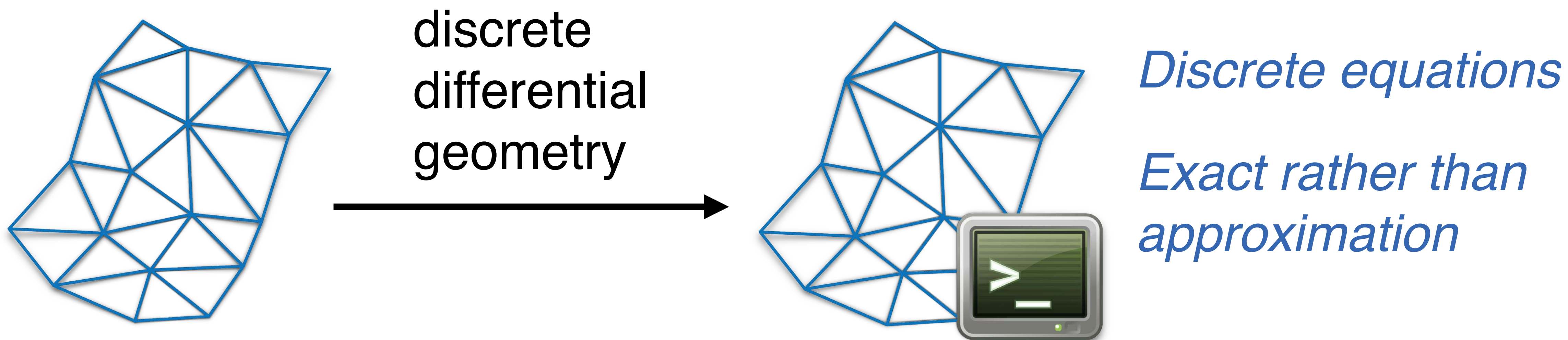
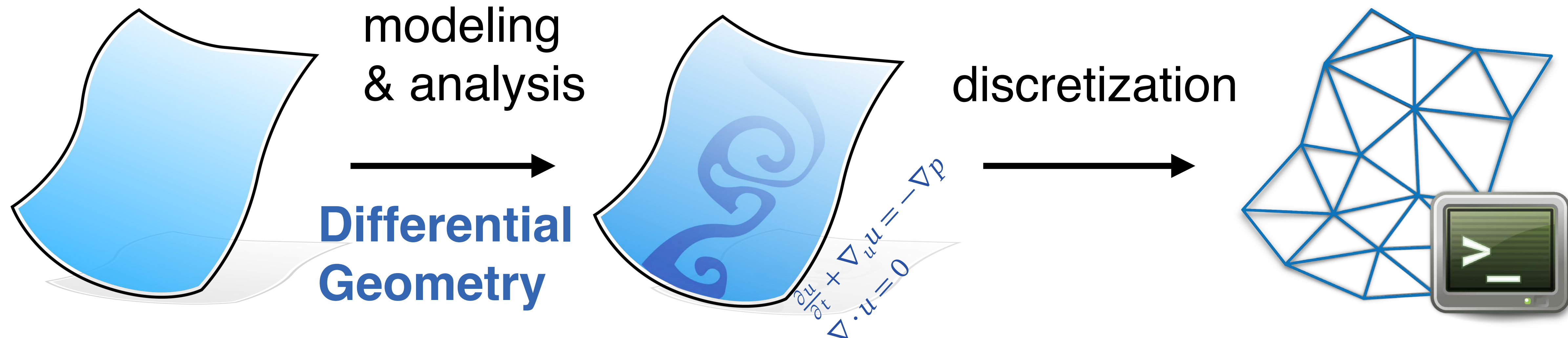


The best one can do
to discretize calculus is

**approximation
theory**

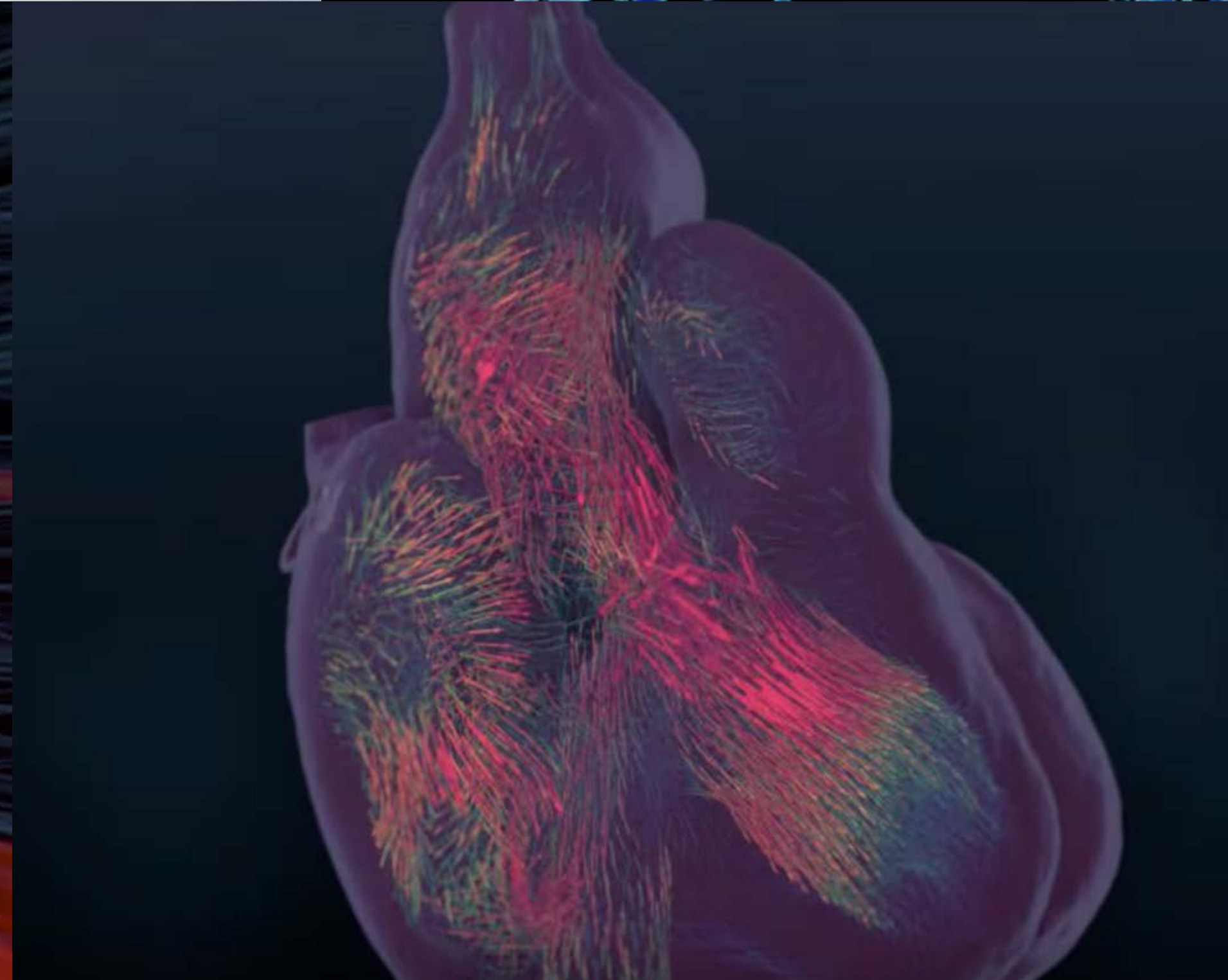
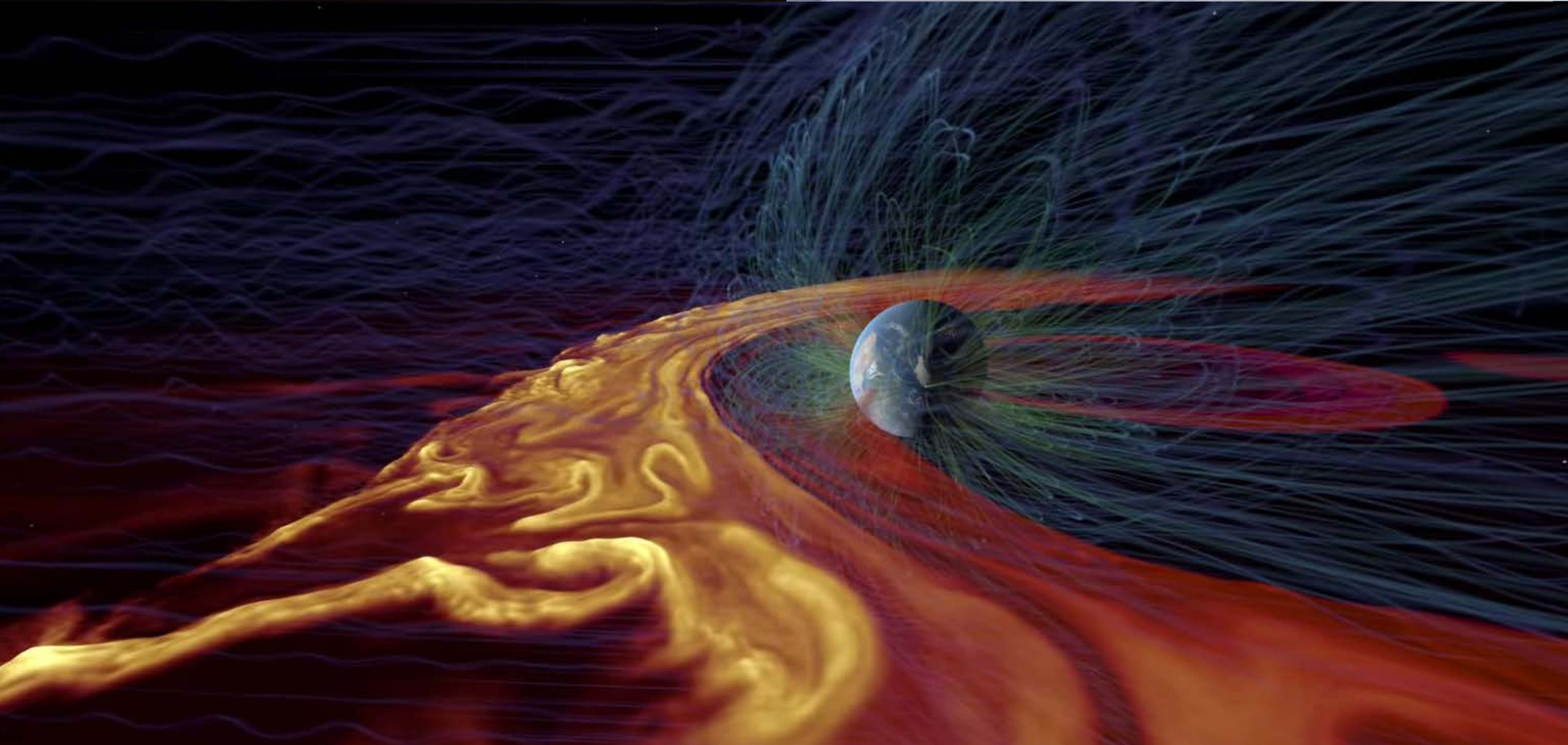
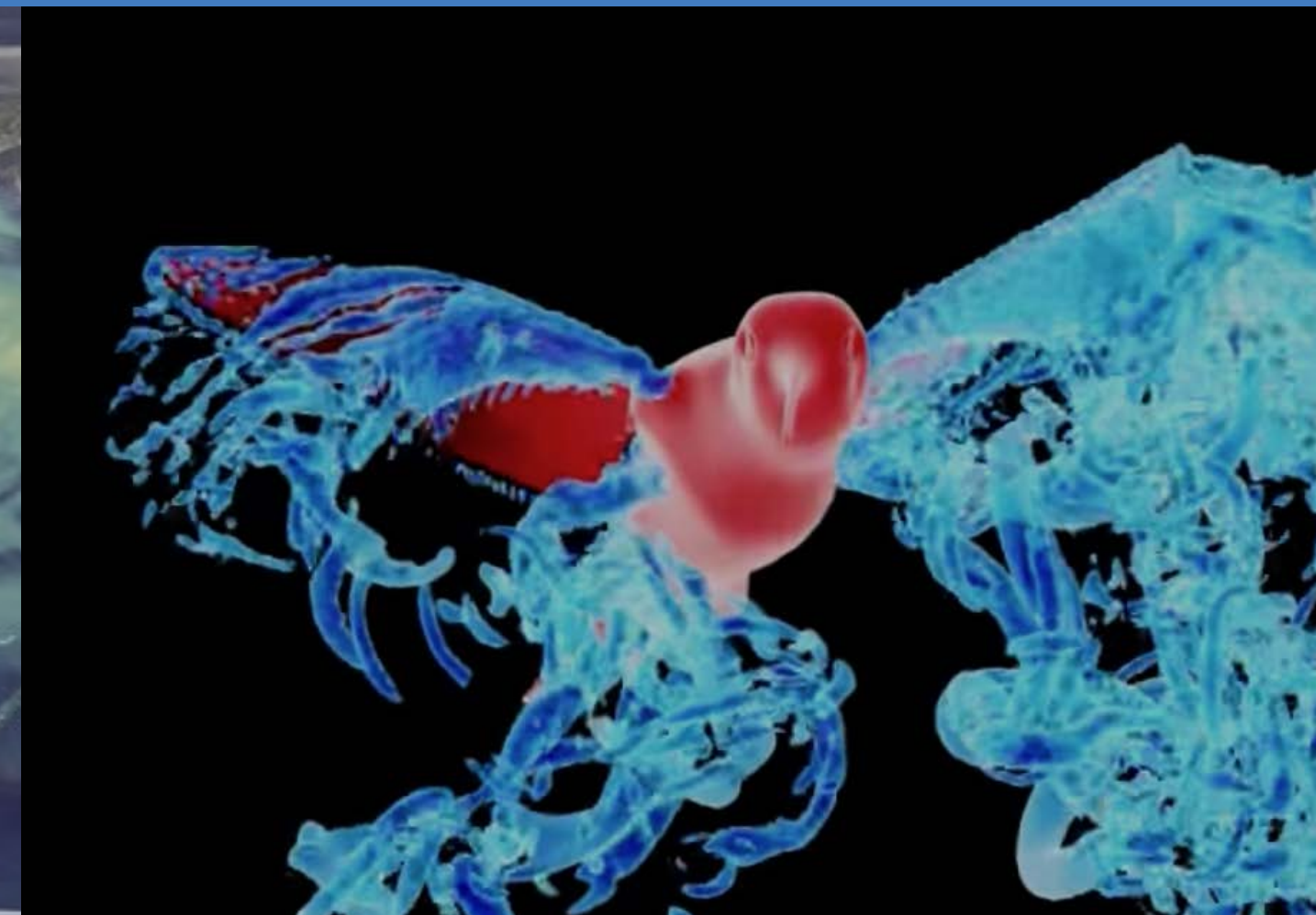
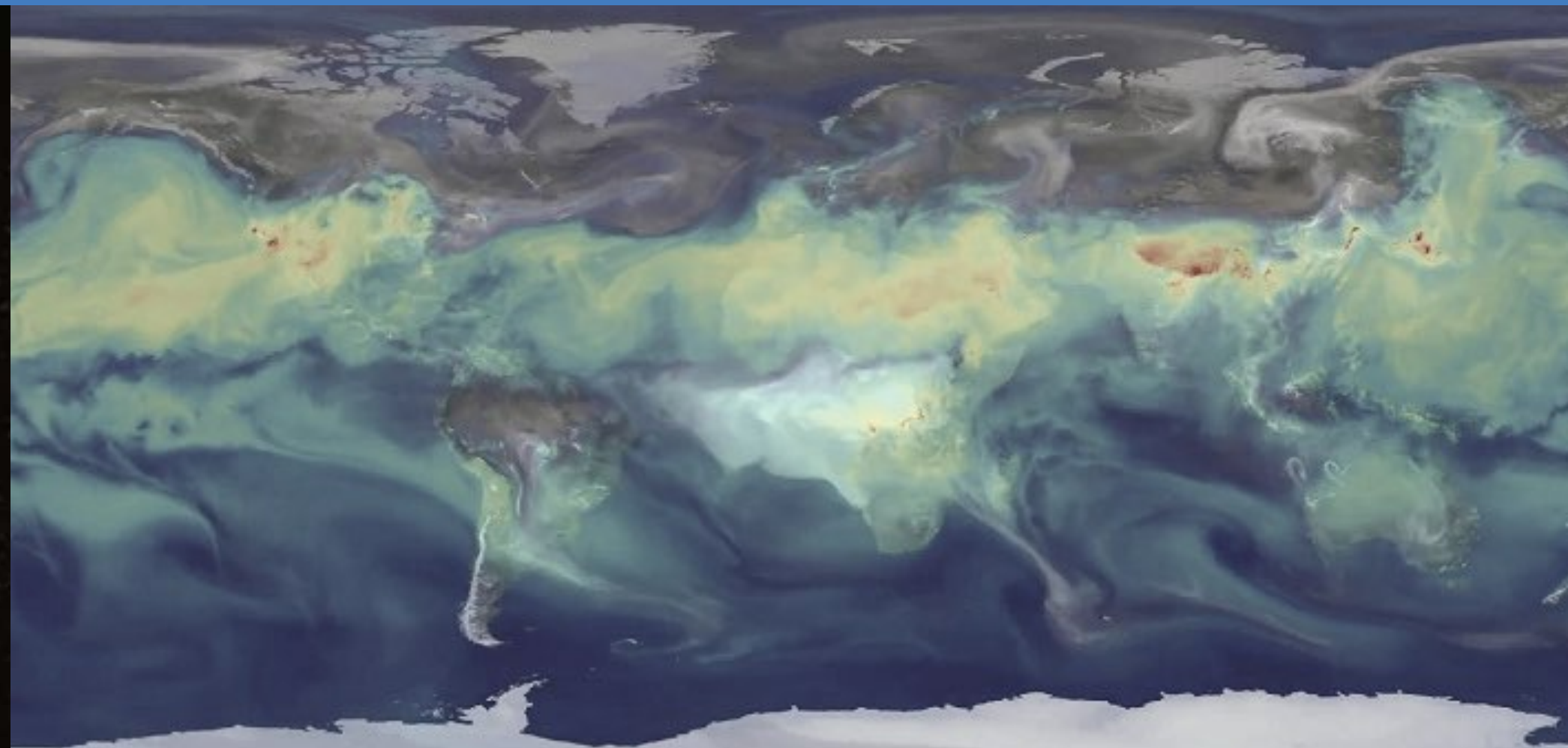
Our approach:

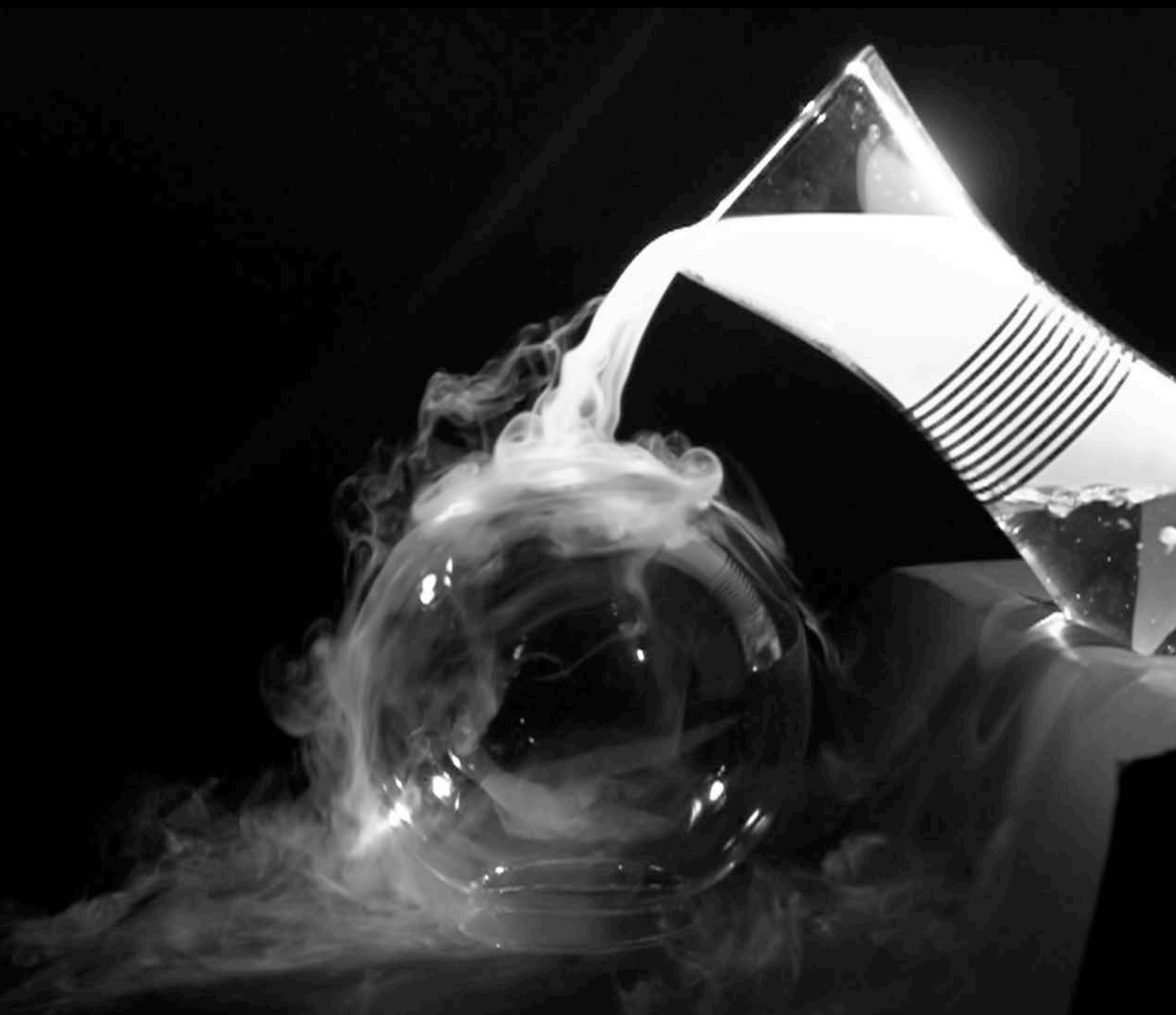




Fluid Dynamics

Simulation of fluids





real footage



simulation



A. Chern, F. Knöppel, U. Pinkall, P. Schröder, S. Weißmann
Schrödinger's Smoke
SIGGRAPH 2016

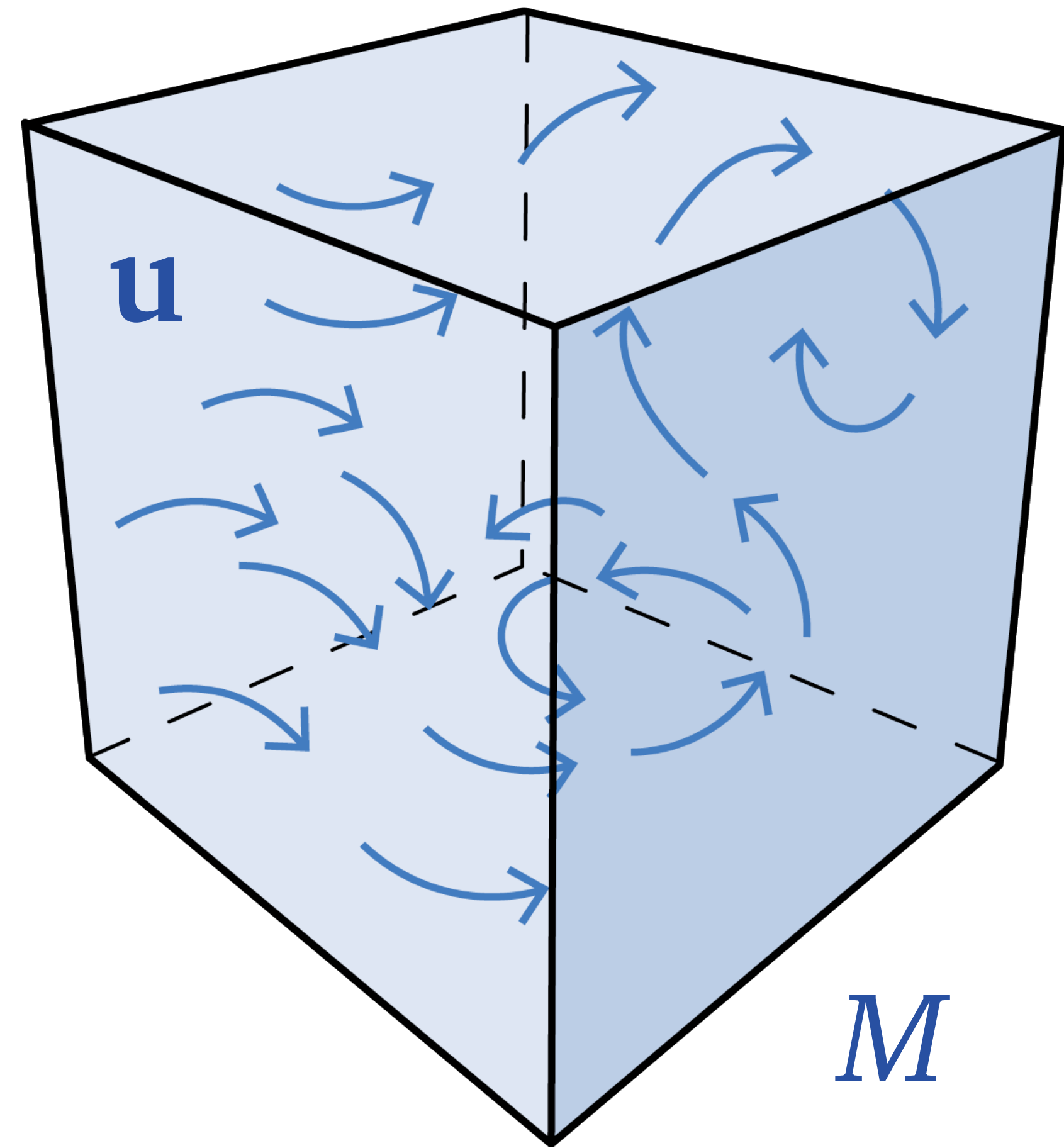


Classical Fluids

Fluid state is described by

velocity field

$$\mathbf{u}: M \rightarrow \mathbb{R}^3$$

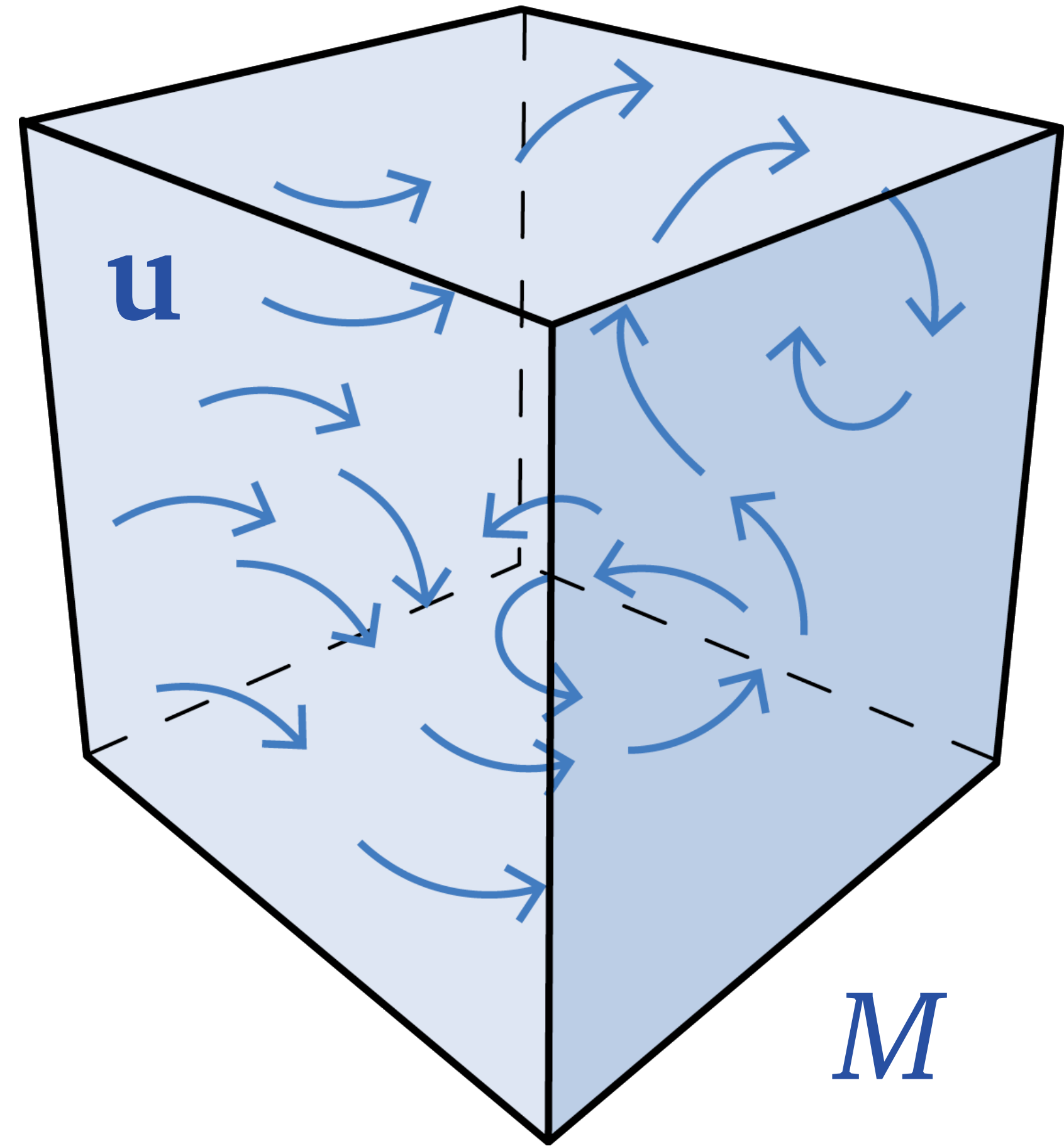


Classical Fluids

- Law of inertia

$$\frac{\partial}{\partial t} \mathbf{u} + \nabla_{\mathbf{u}} \mathbf{u} = \mathbf{0}$$

velocity transported by velocity

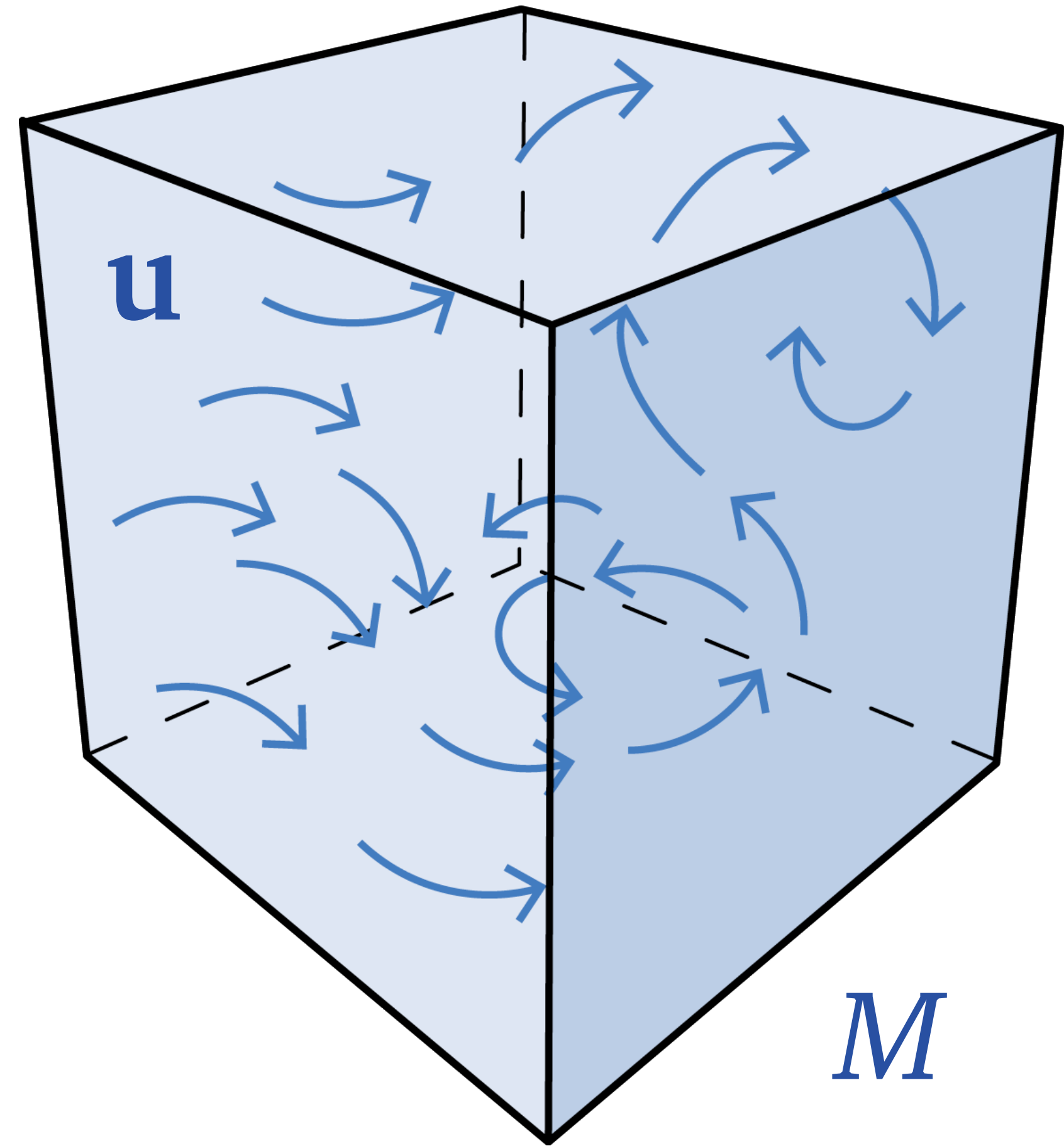


Classical Fluids

- Law of inertia

$$\frac{\partial}{\partial t} \mathbf{u} + \boxed{\nabla_{\mathbf{u}} \mathbf{u}} = \mathbf{0}$$

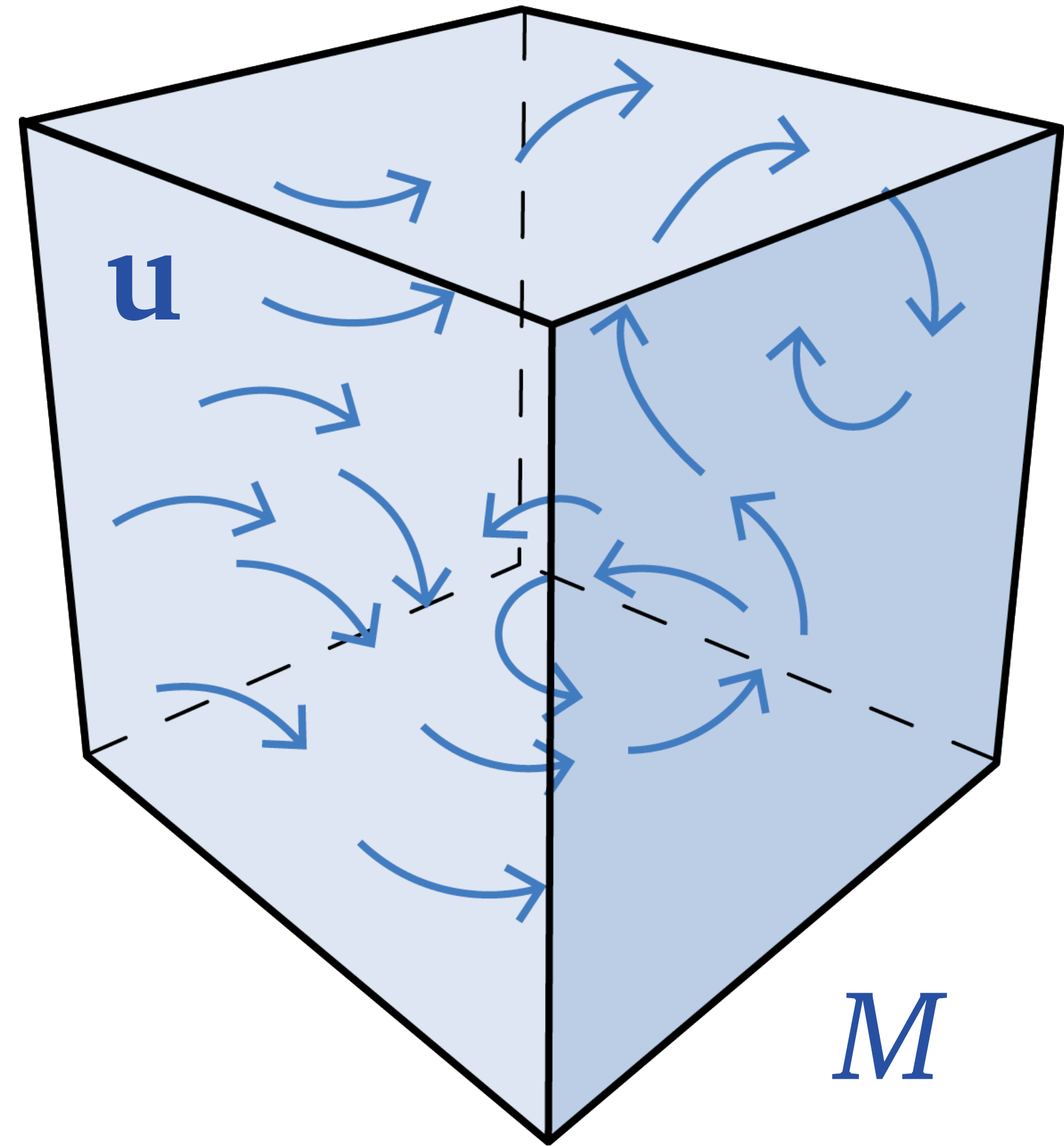
Nonlinear!!!



Classical Fluids

- Law of inertia

$$\frac{\partial}{\partial t} \mathbf{u} + \nabla_{\mathbf{u}} \mathbf{u} = \mathbf{0}$$

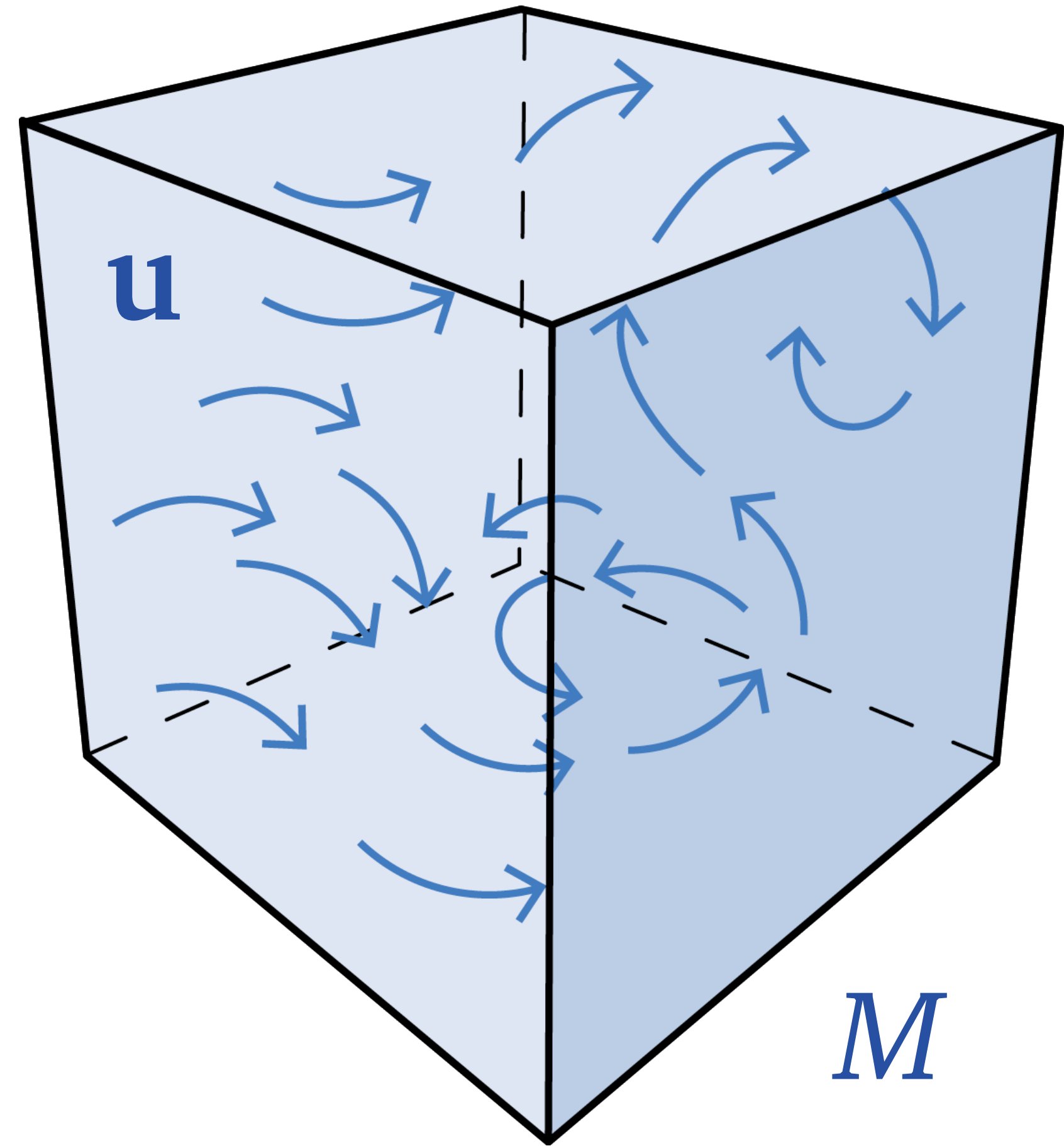


Classical Fluids

- Law of inertia
- Internal force is a gradient

$$\frac{\partial}{\partial t} \mathbf{u} + \nabla_{\mathbf{u}} \mathbf{u} = -\nabla p$$

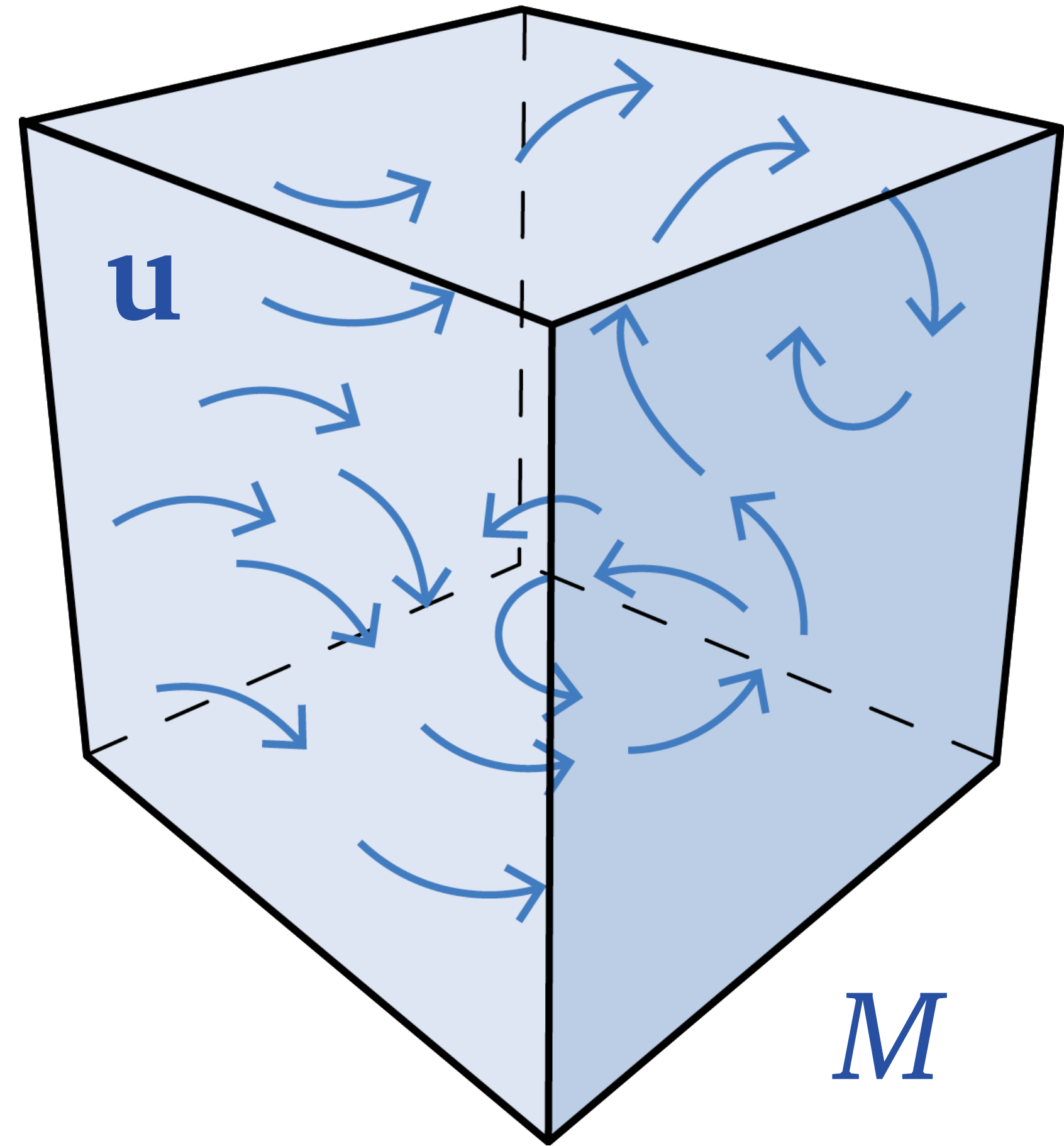
*pressure
(enthalpy)*



Classical Fluids

- Law of inertia
- Internal force is a gradient

$$\frac{\partial}{\partial t} \mathbf{u} + \nabla_{\mathbf{u}} \mathbf{u} = -\nabla p$$

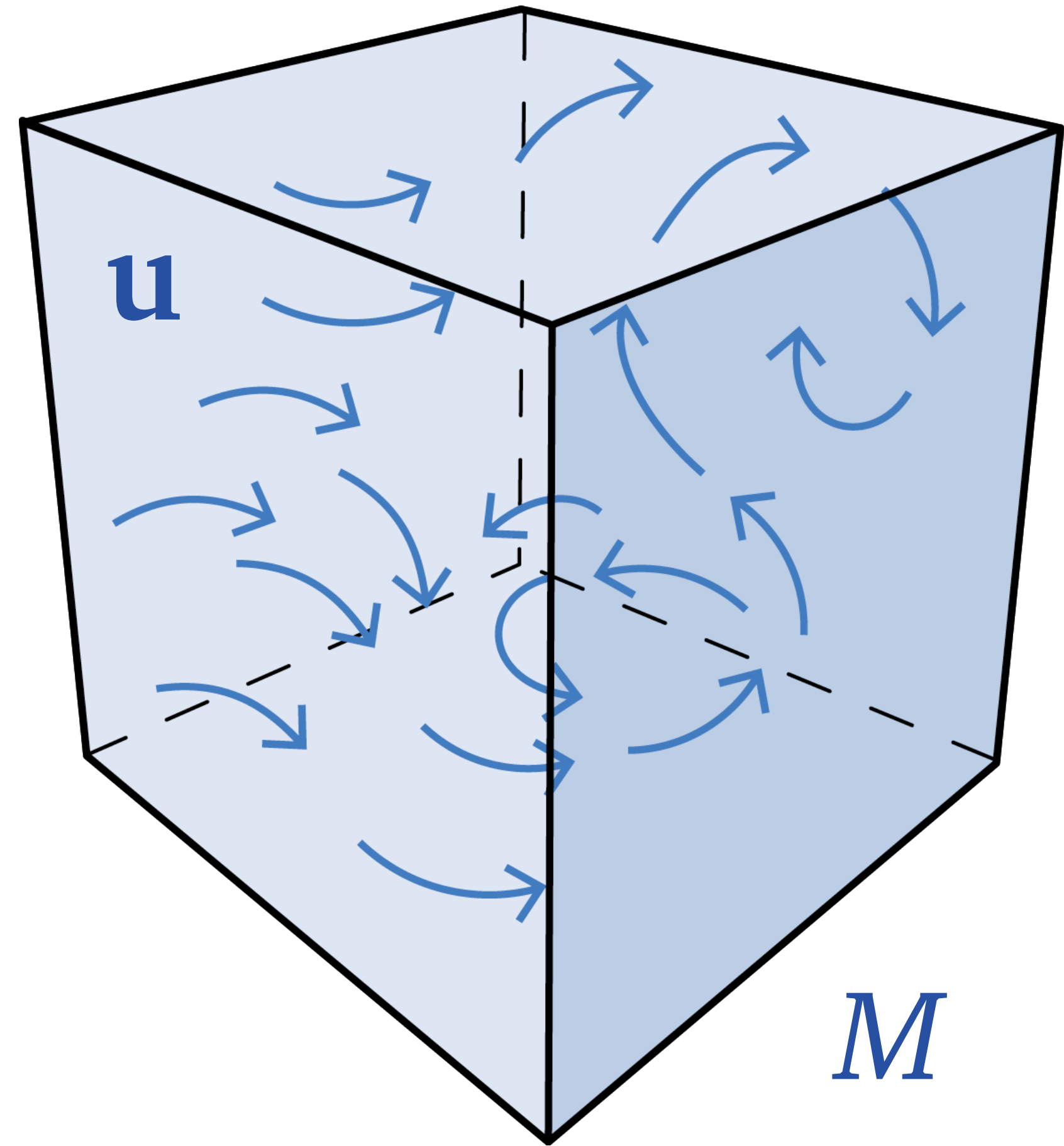


Classical Fluids

- Law of inertia
- Internal force is a gradient
- Fluid is incompressible

$$\begin{cases} \frac{\partial}{\partial t} \mathbf{u} + \nabla_{\mathbf{u}} \mathbf{u} = -\nabla p \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

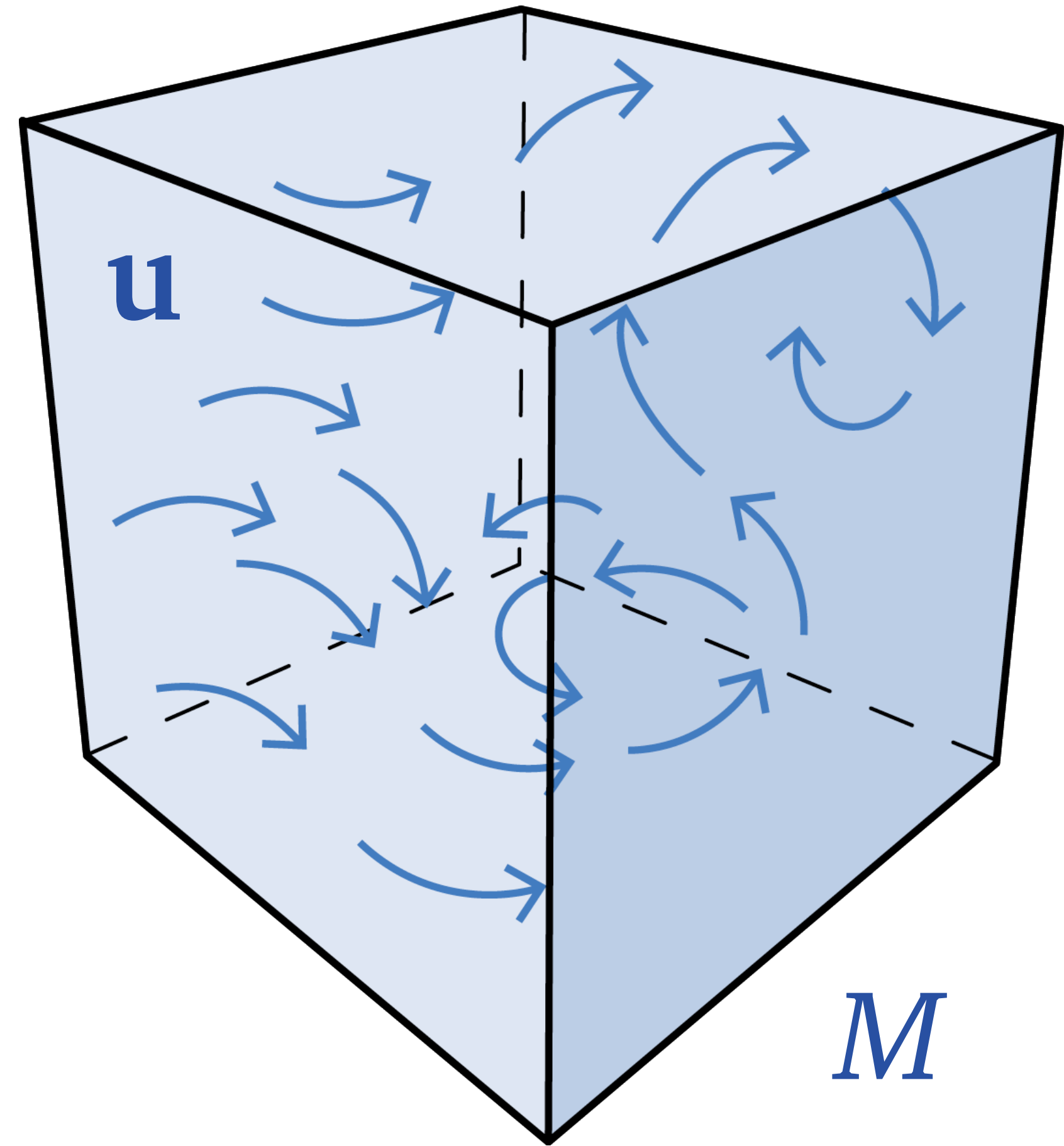
divergence free



Classical Fluids

- Law of inertia
- Internal force is a gradient
- Fluid is incompressible

$$\begin{cases} \frac{\partial}{\partial t} \mathbf{u} + \nabla_{\mathbf{u}} \mathbf{u} = -\nabla p \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

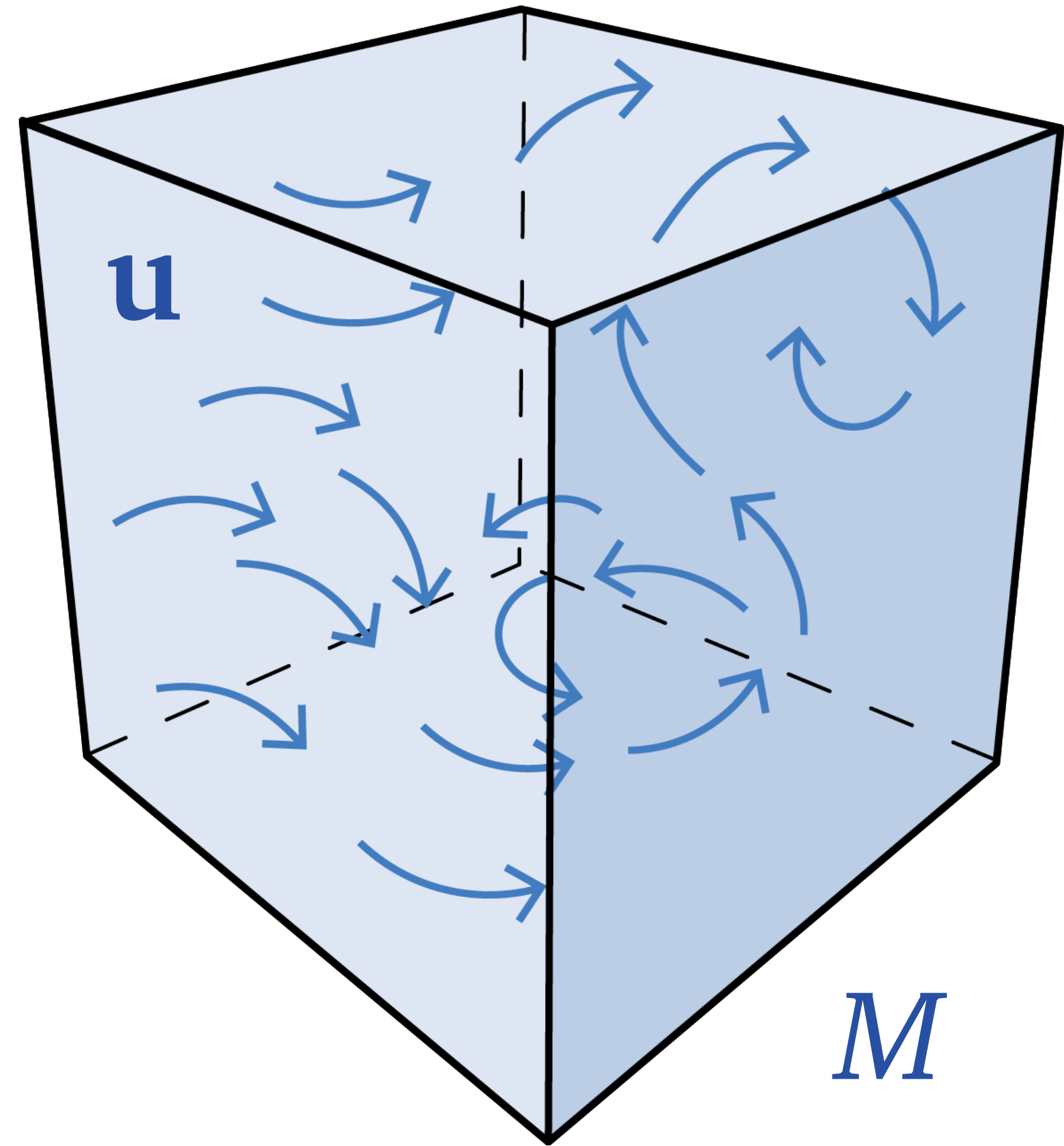


Classical Fluids

- Law of inertia
- Internal force is a gradient
- Fluid is incompressible

$$\begin{cases} \frac{\partial}{\partial t} \mathbf{u} + \nabla_{\mathbf{u}} \mathbf{u} = -\nabla p \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

Incompressible Euler Equation



Classical Fluids

$$\begin{cases} \frac{\partial}{\partial t} \mathbf{u} + \nabla_{\mathbf{u}} \mathbf{u} = -\nabla p \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

Classical Fluids

$$\left\{ \frac{\partial}{\partial t} \mathbf{u} + \nabla_{\mathbf{u}} \mathbf{u} = \mathbf{0} \right.$$

Nonlinear advection

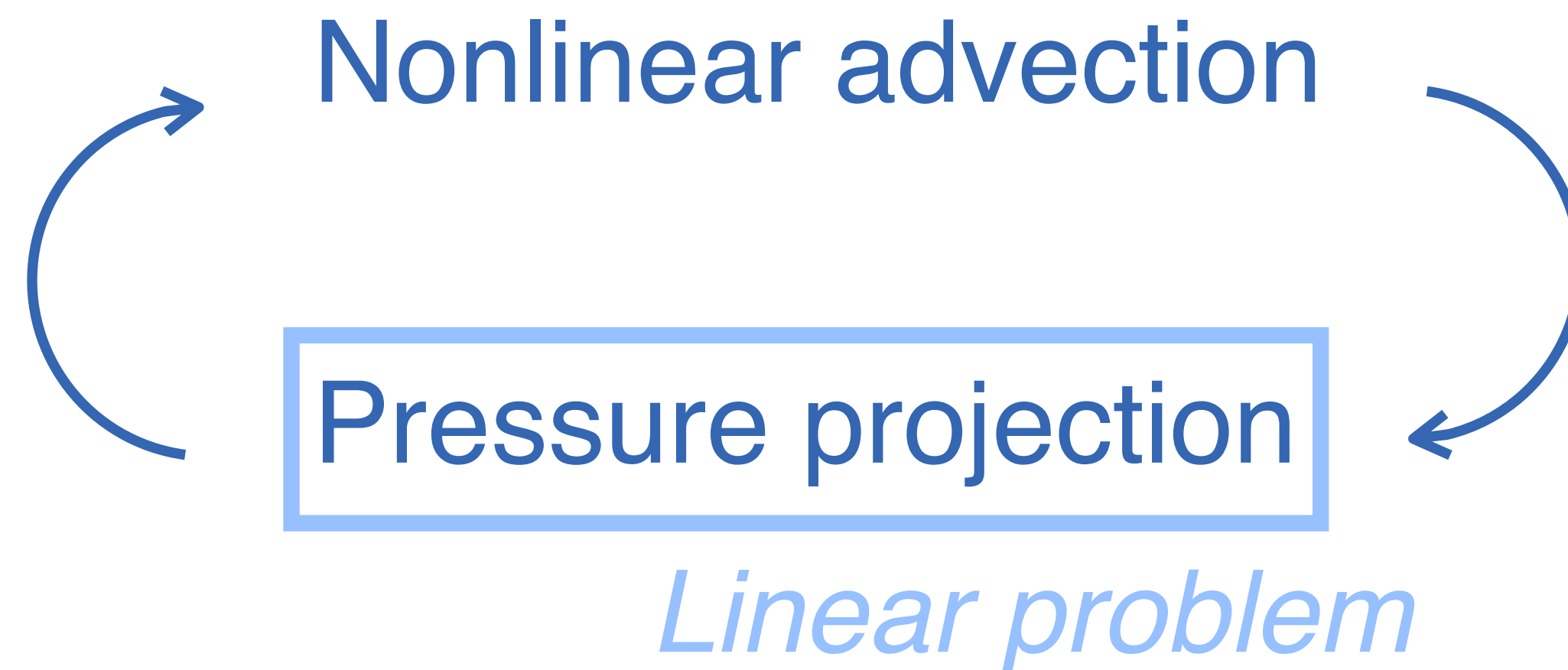
Classical Fluids

$$\begin{cases} \frac{\partial}{\partial t} \mathbf{u} &= -\nabla p \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

Pressure projection

Classical Fluids

$$\begin{cases} \frac{\partial}{\partial t} \mathbf{u} + \nabla_{\mathbf{u}} \mathbf{u} = -\nabla p \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$



Classical Fluids

$$\begin{cases} \frac{\partial}{\partial t} \mathbf{u} + \boxed{\nabla_{\mathbf{u}} \mathbf{u}} = -\nabla p \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

Nonlinear advection

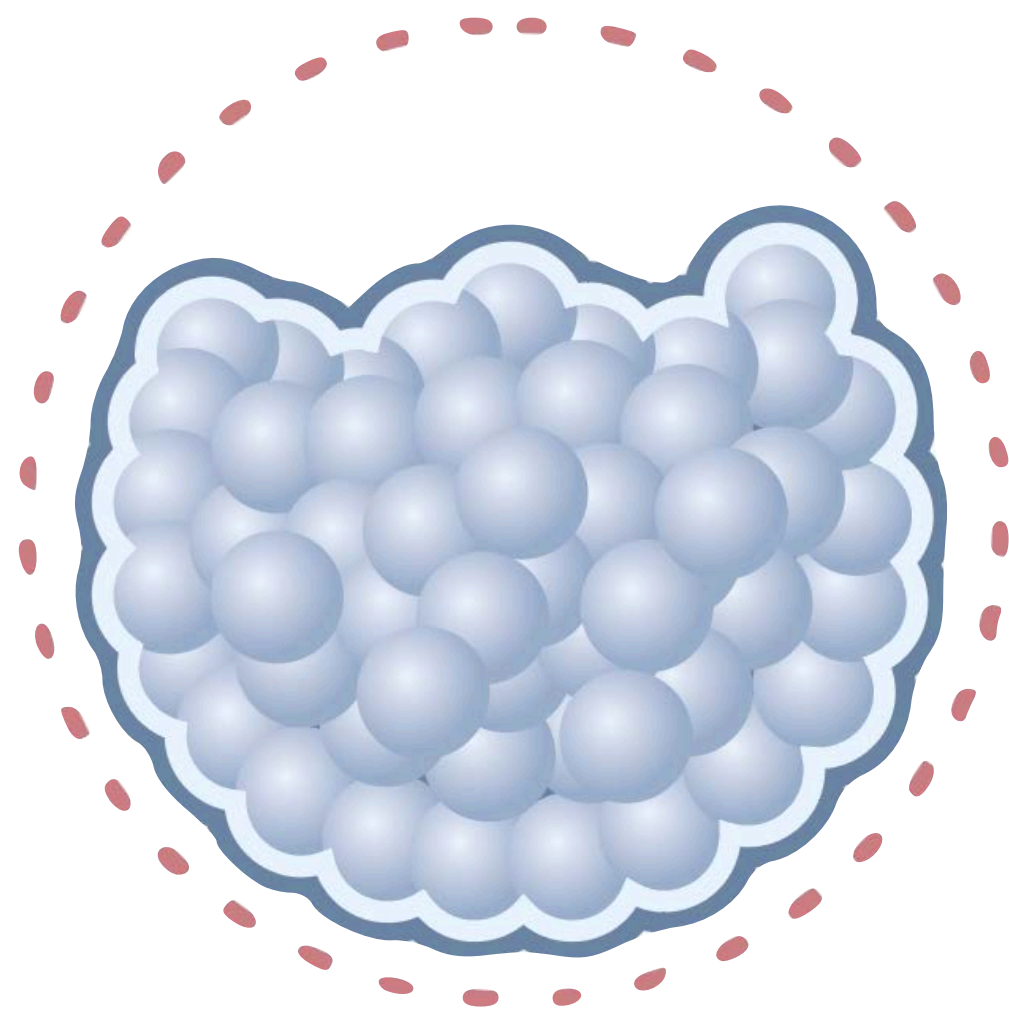
Could be challenging

Pressure projection

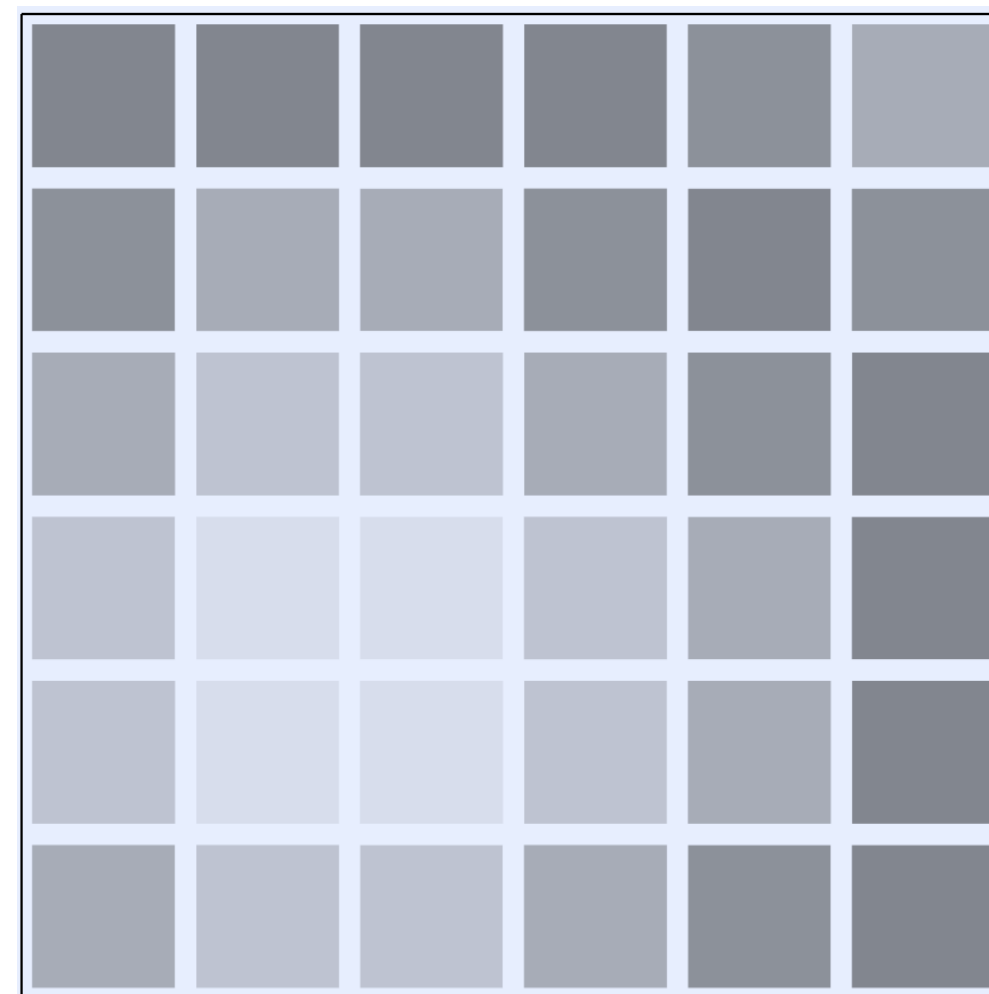


Classical Fluids

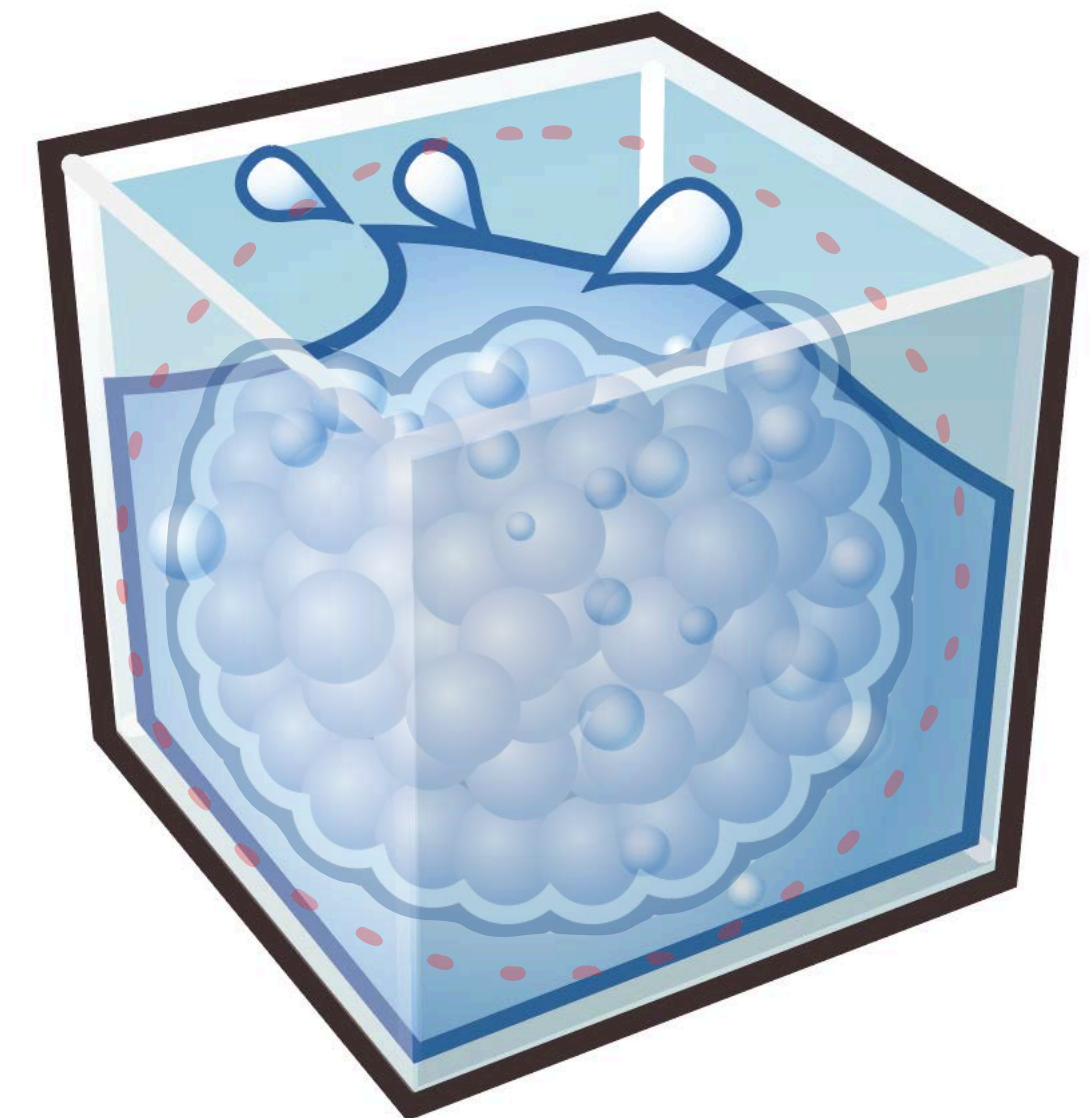
Nonlinear advection



particle methods

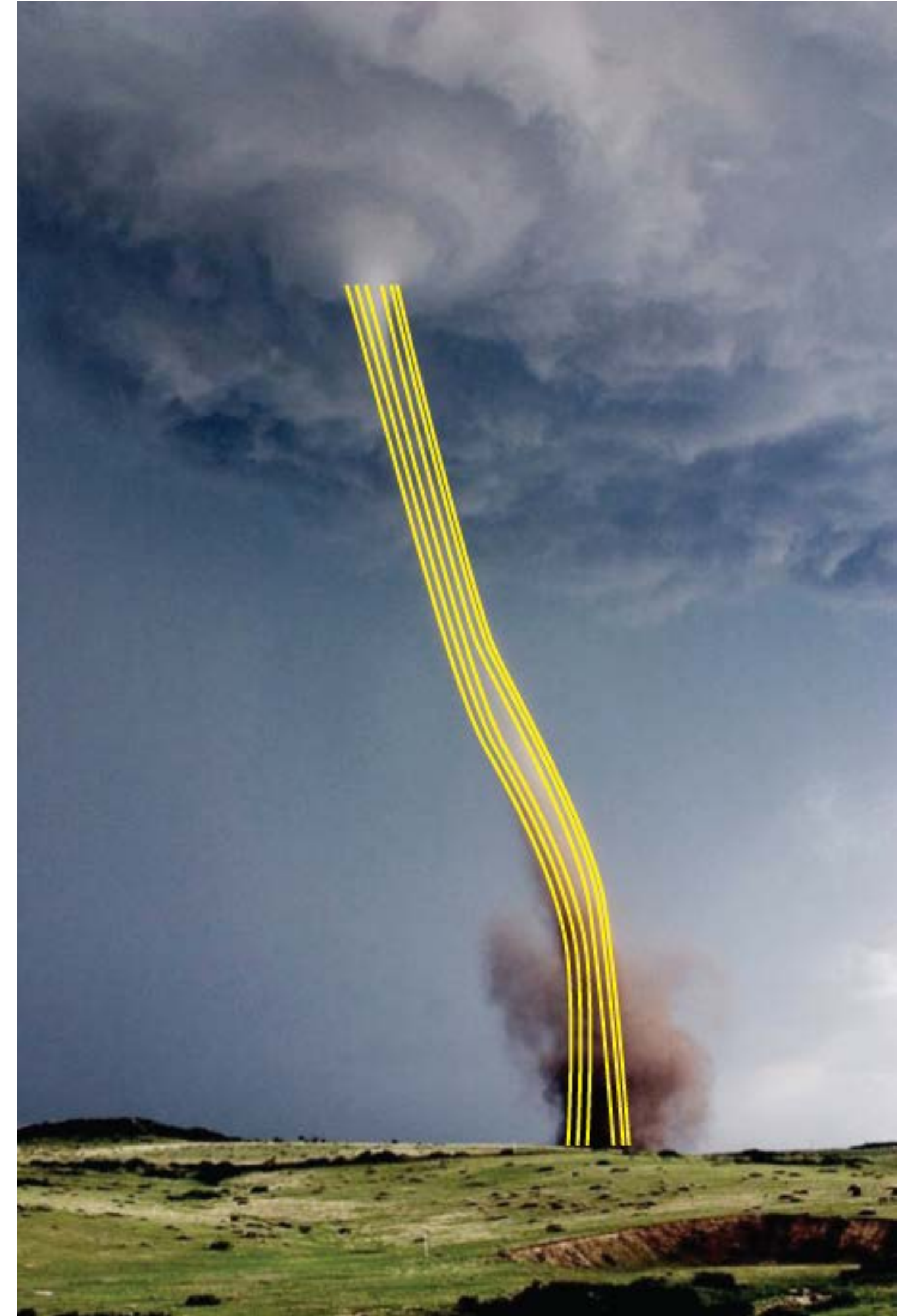


grid-based methods

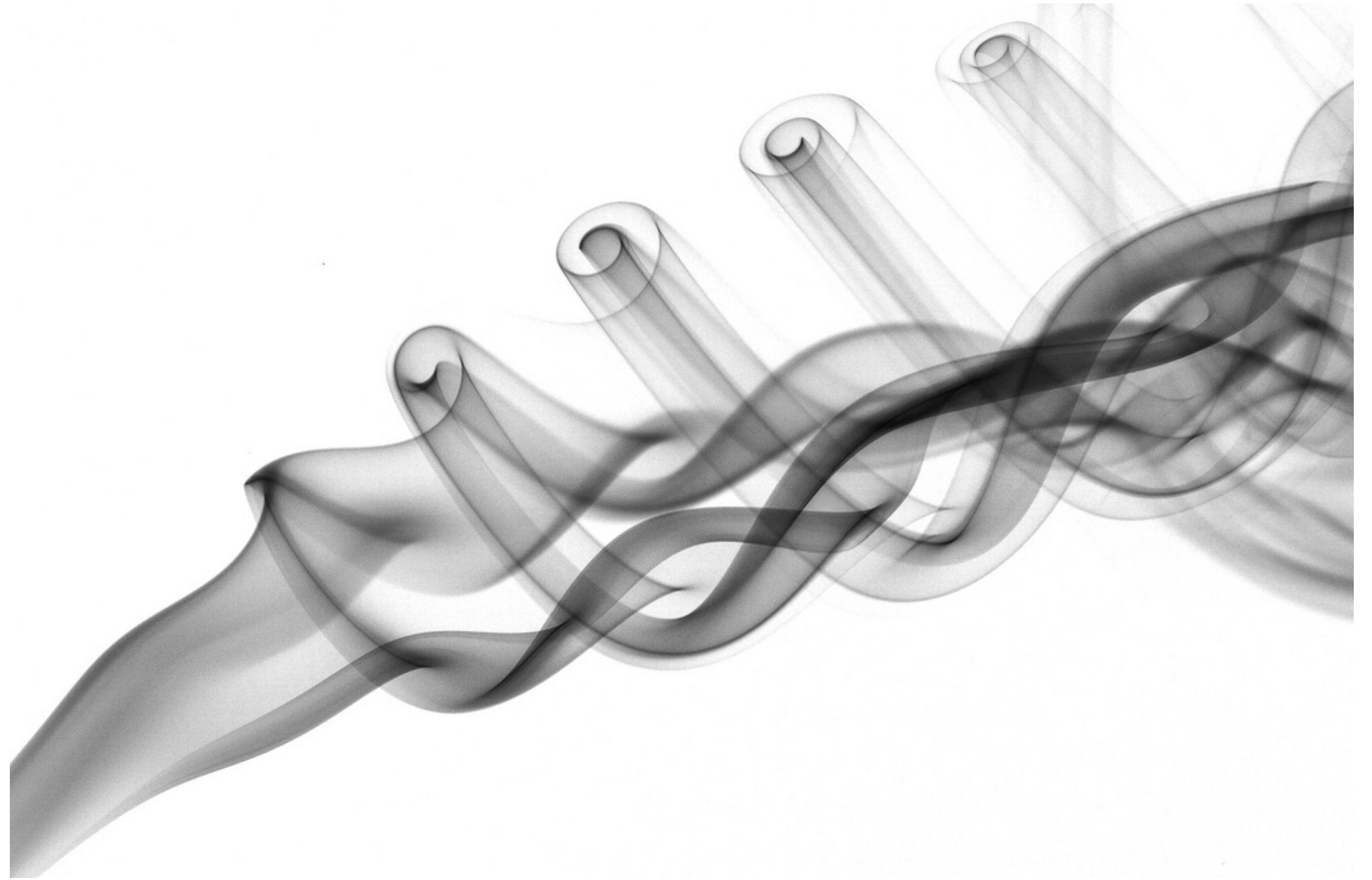


hybrid methods

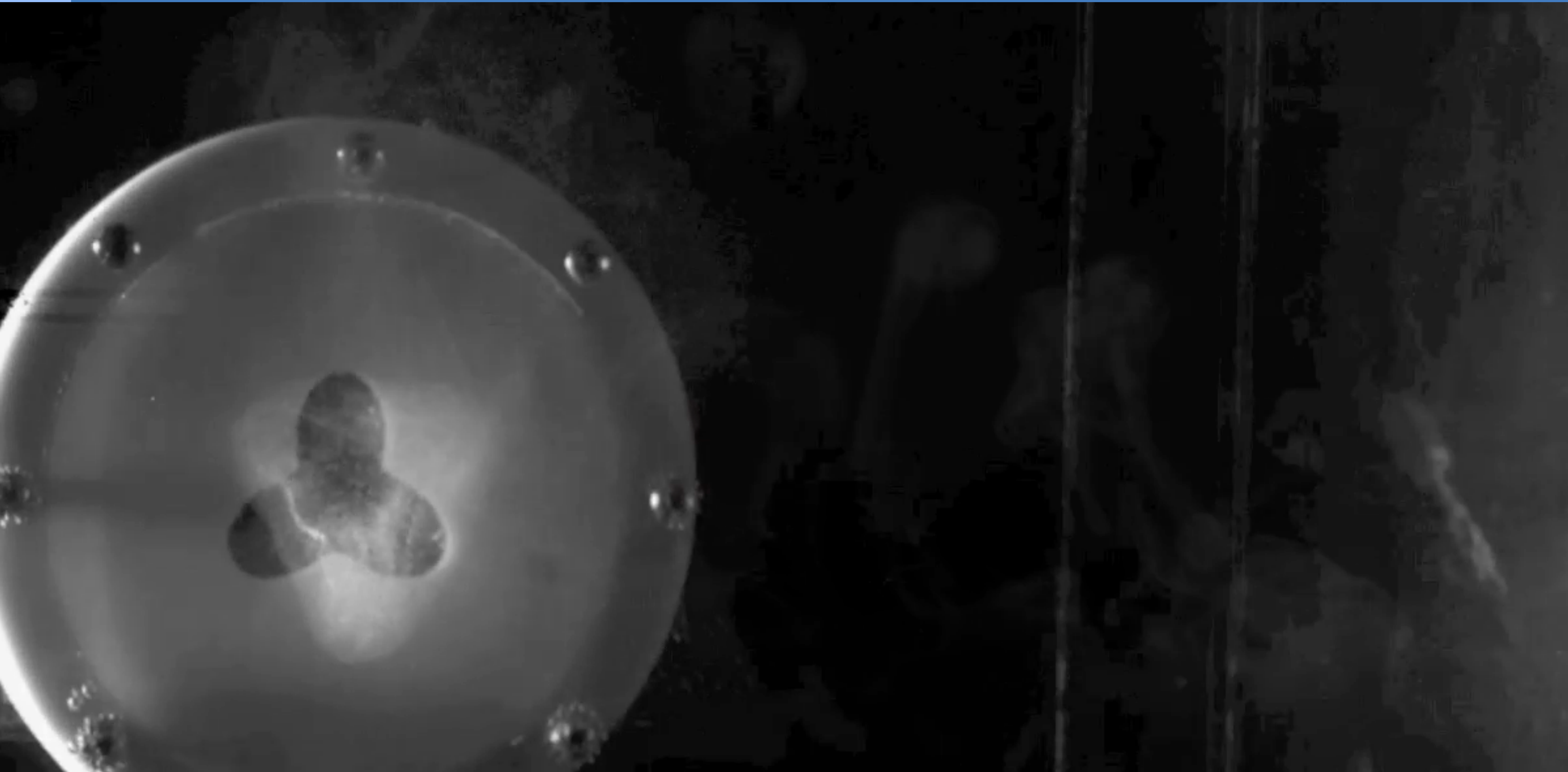
Vortices in Fluids



Vortices in Fluids



Vortices in Fluids

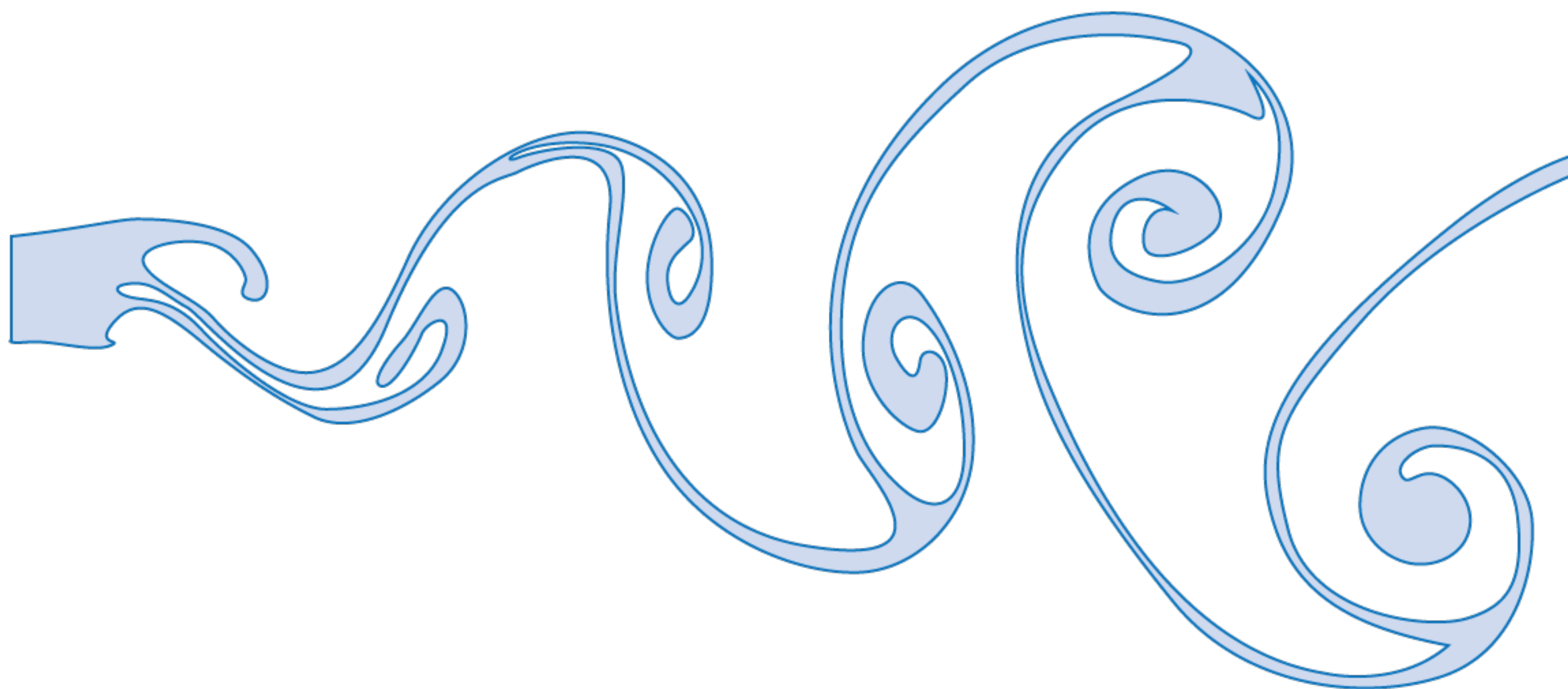


Vortices in Fluids

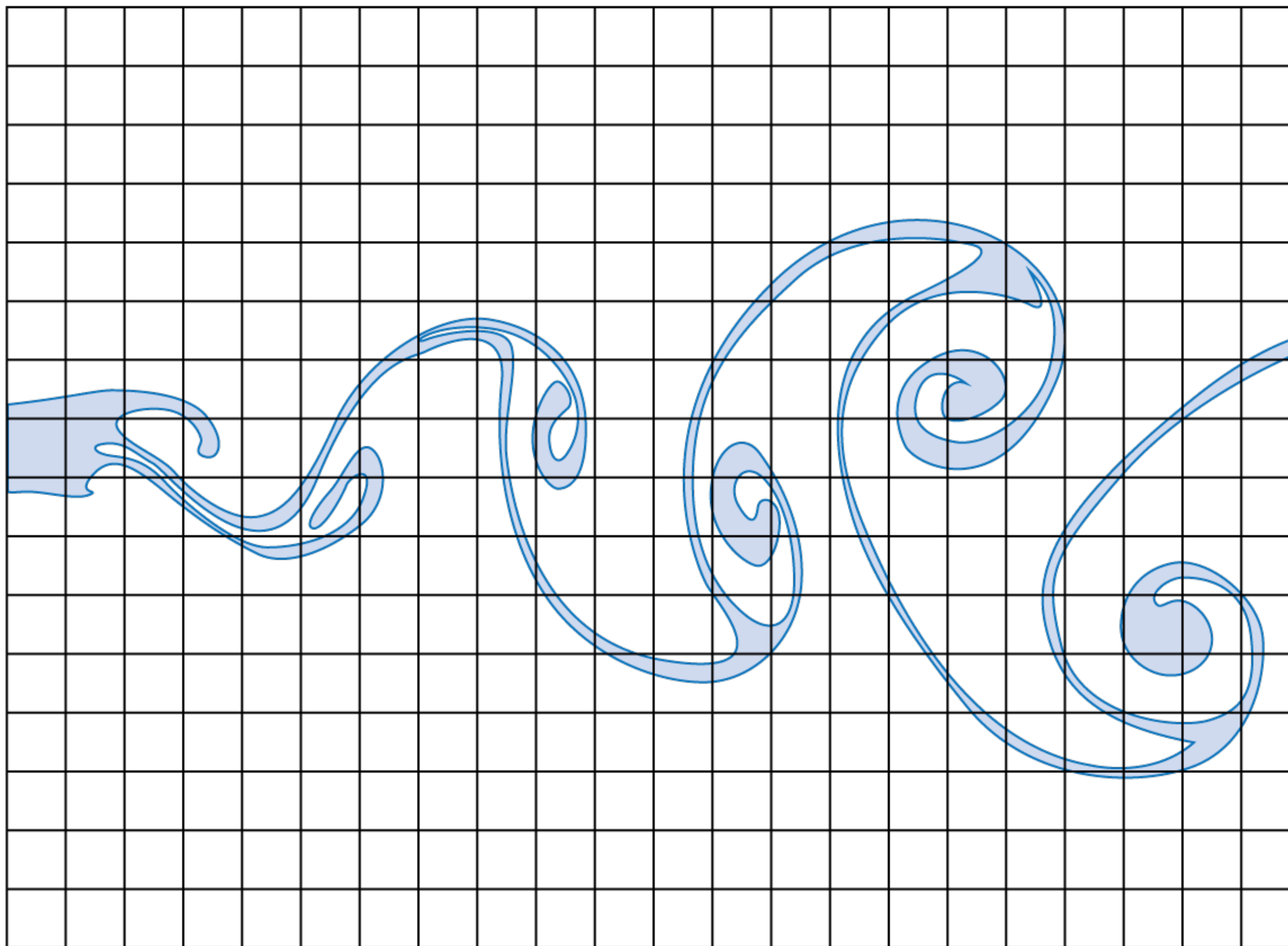
- Most interesting flows are dominated by **thin vortices**.
- The global flow behavior is sensitive to the **thickness** of vortices.
- Decent fluid simulations need **very high resolutions**.

- Challenge: reproduce sharp features

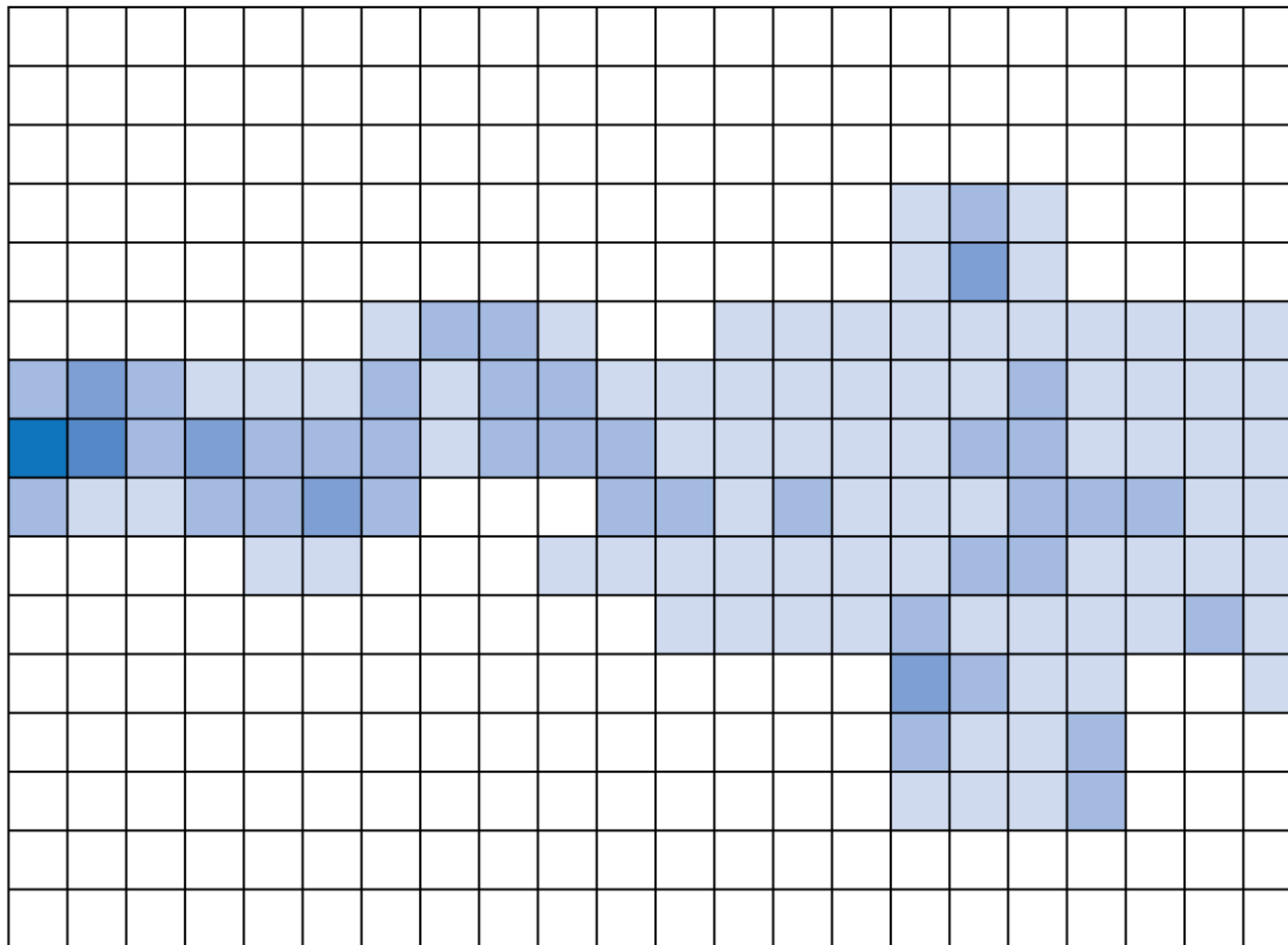
- Challenge: reproduce sharp features

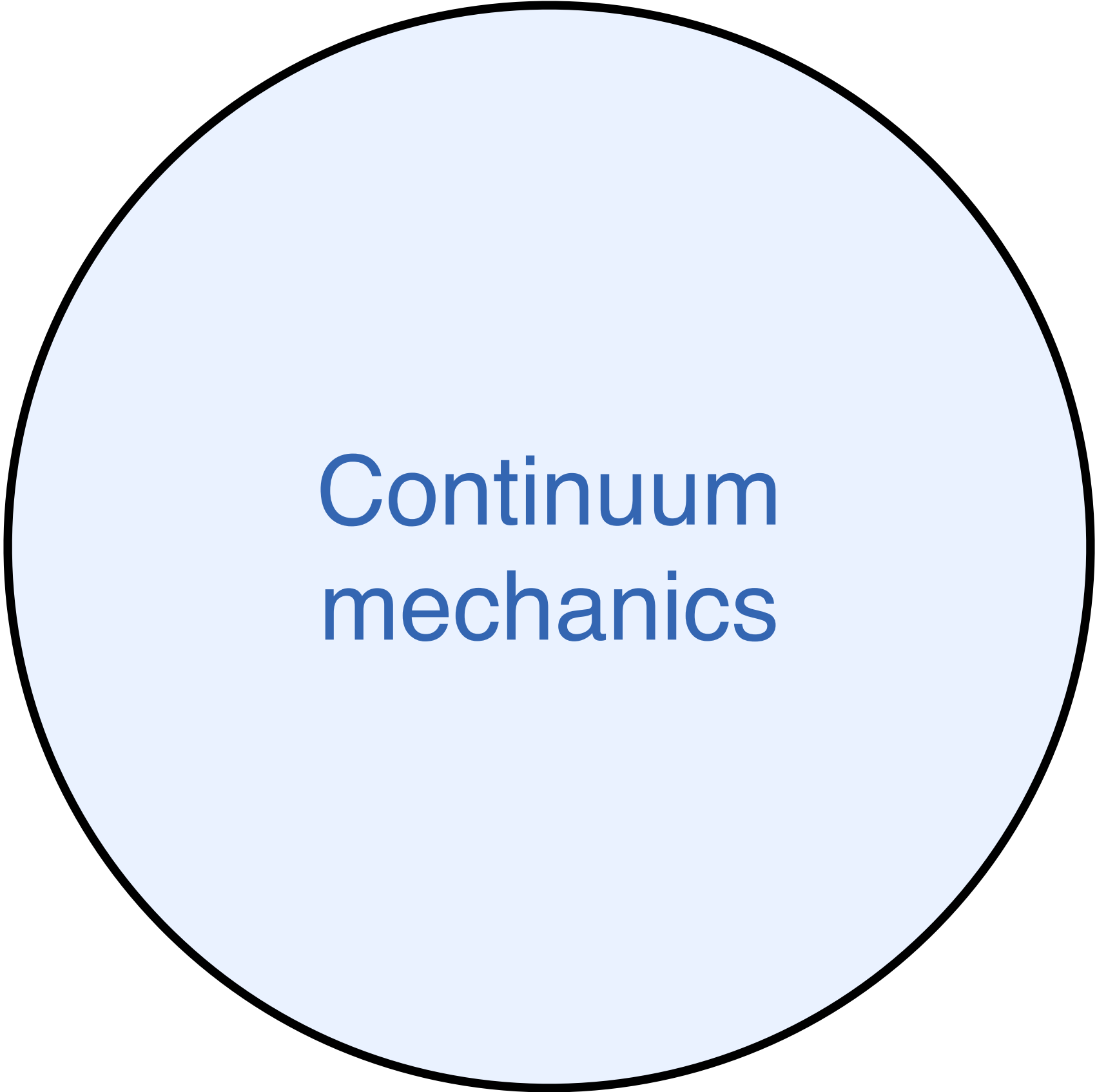


- Challenge: reproduce sharp features

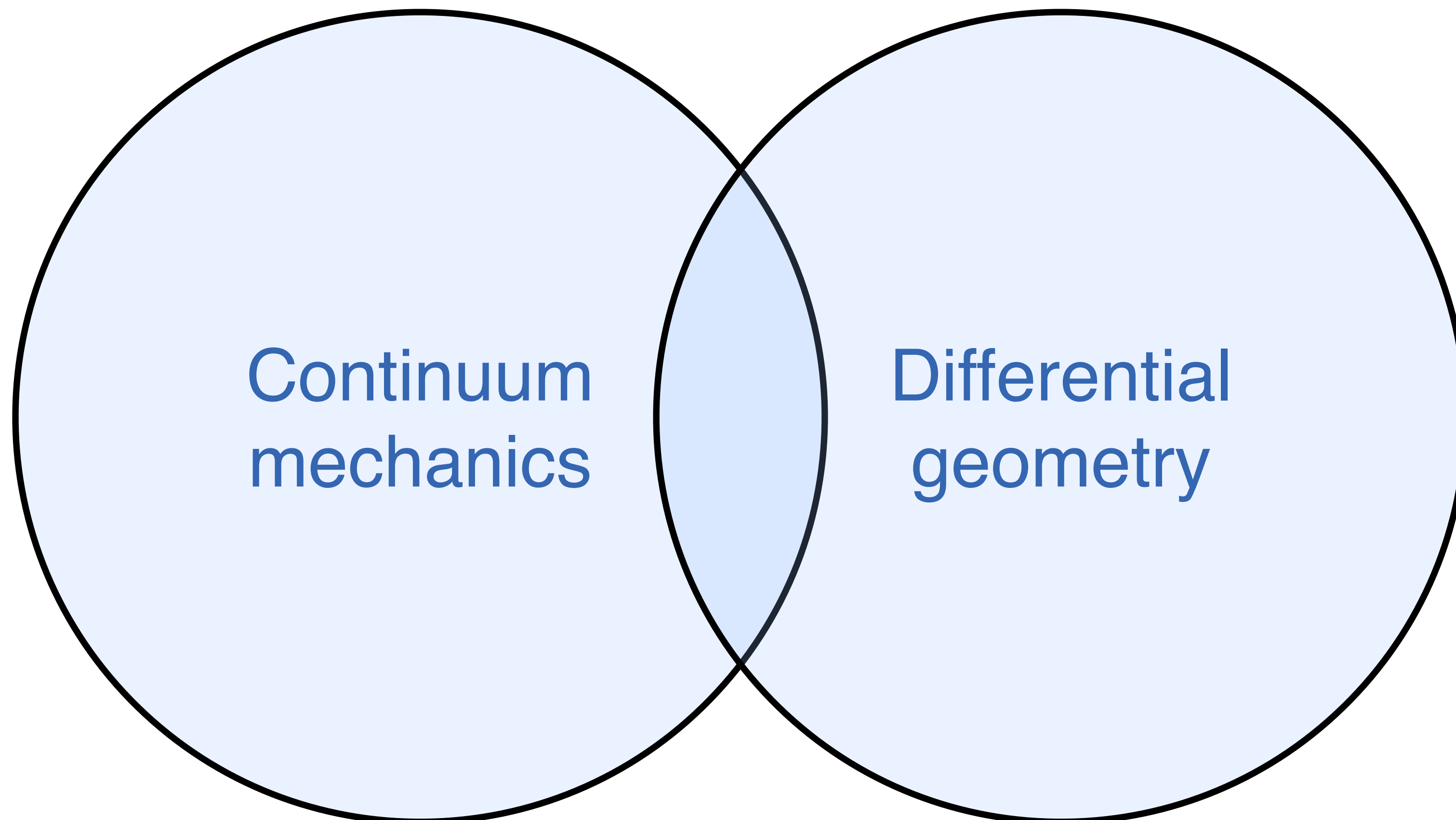


- Challenge: reproduce sharp features



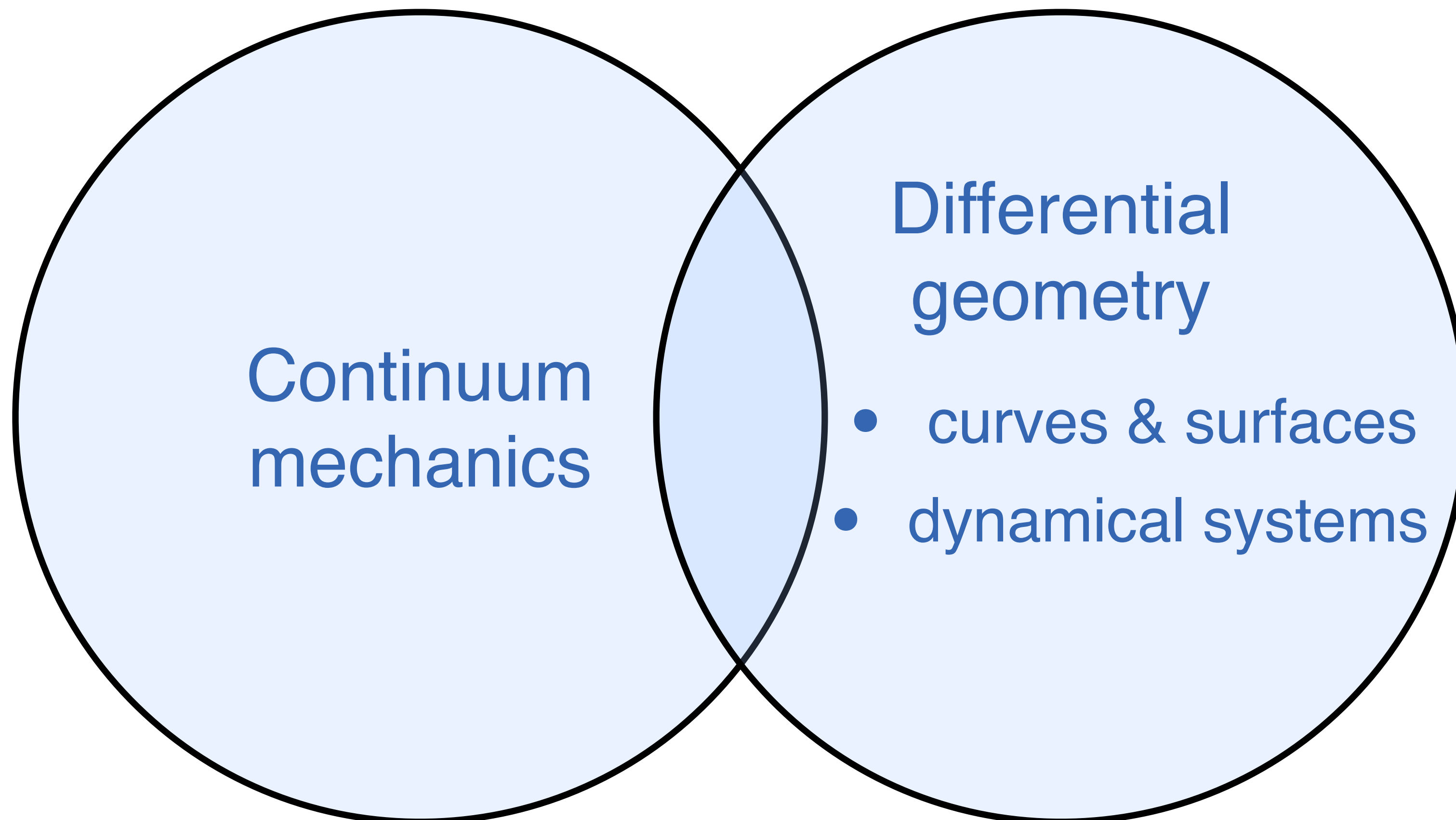


Continuum
mechanics



Continuum
mechanics

Differential
geometry

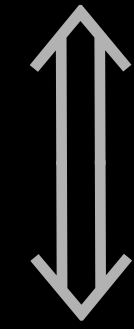


Continuum
mechanics

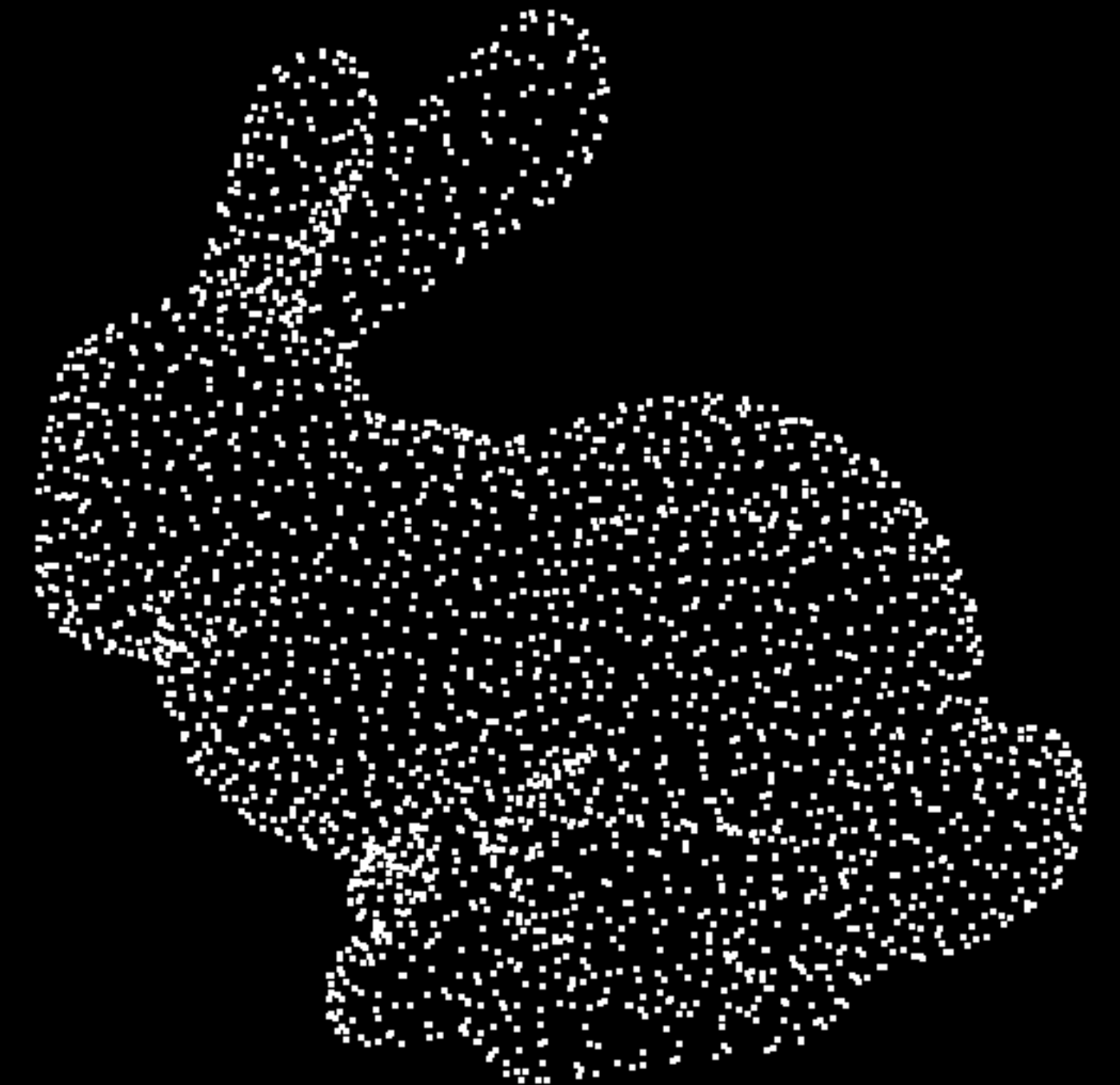
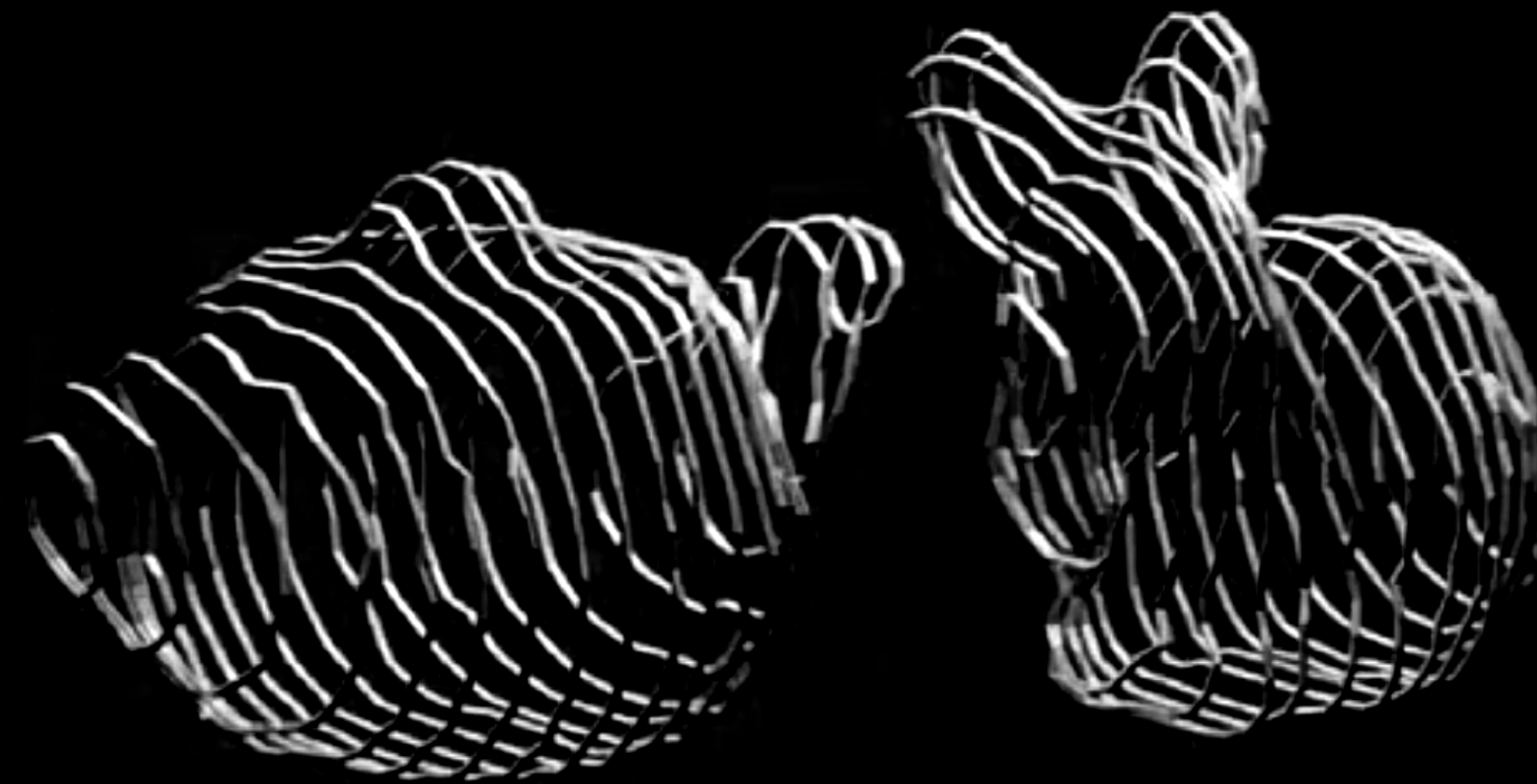
Differential
geometry

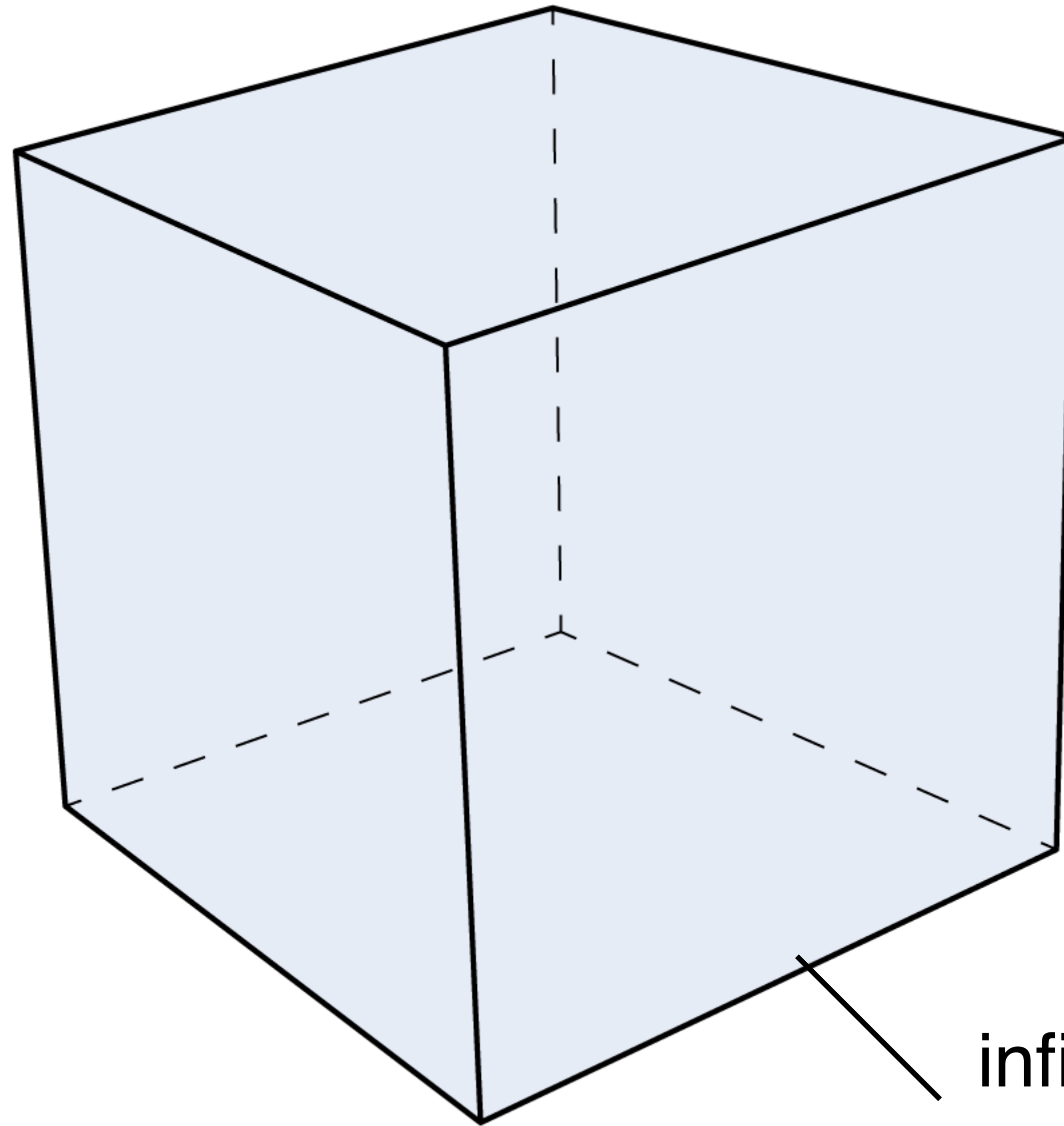
- curves & surfaces
- dynamical systems

Discretization



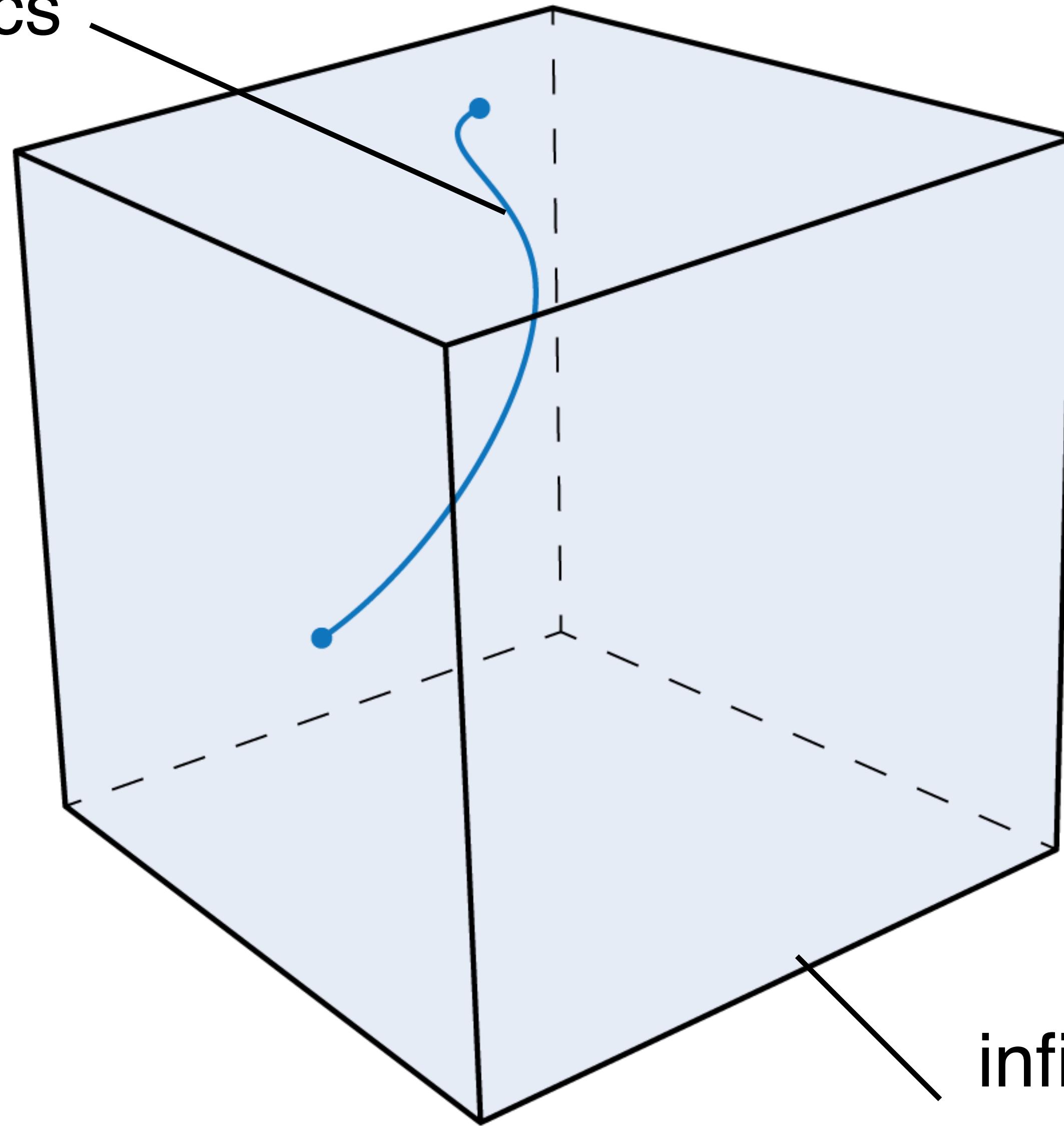
Dimension reduction





infinite dimensional
space of all fluid states

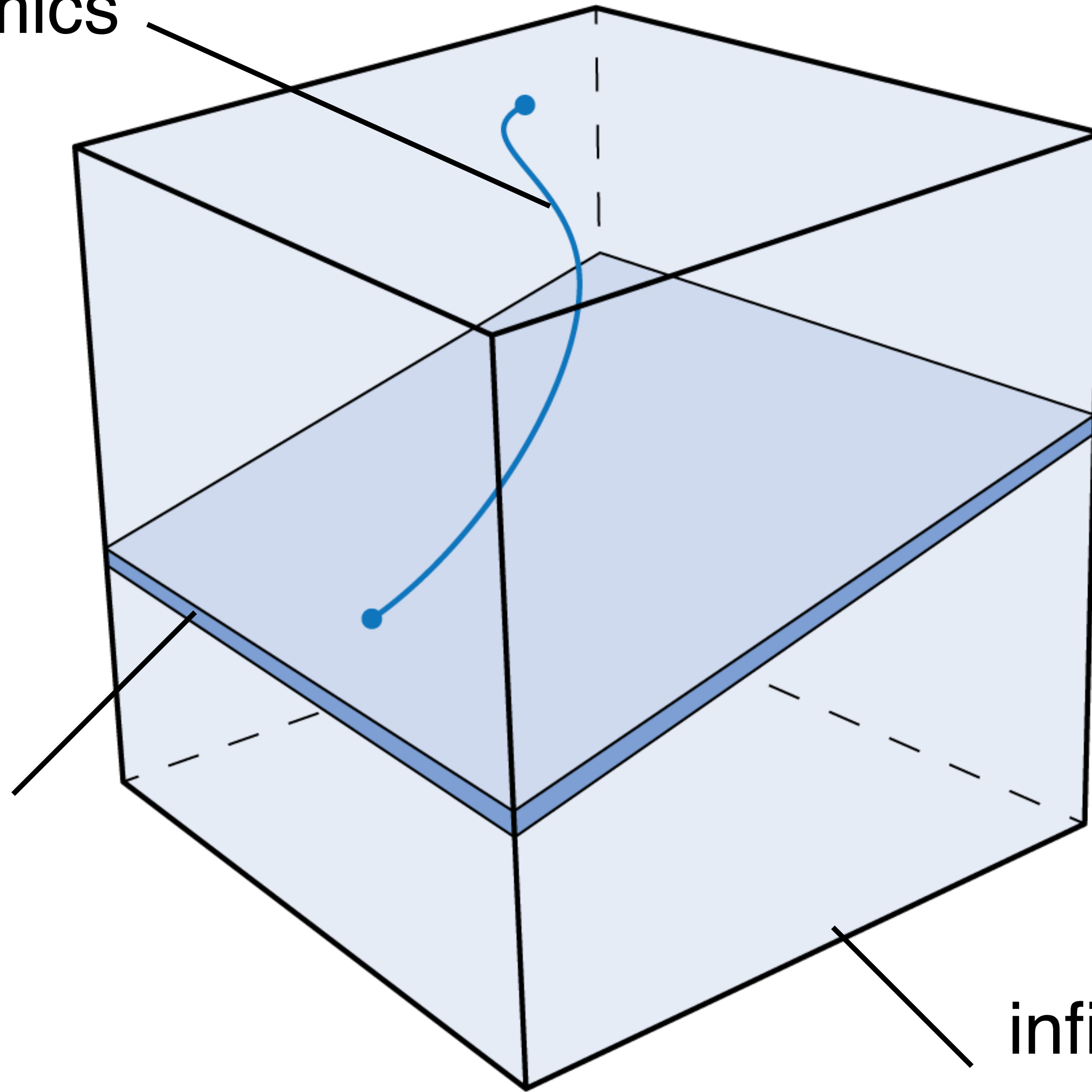
real fluid dynamics



infinite dimensional
space of all fluid states

real fluid dynamics

fluid domain is
discretized into
a grid

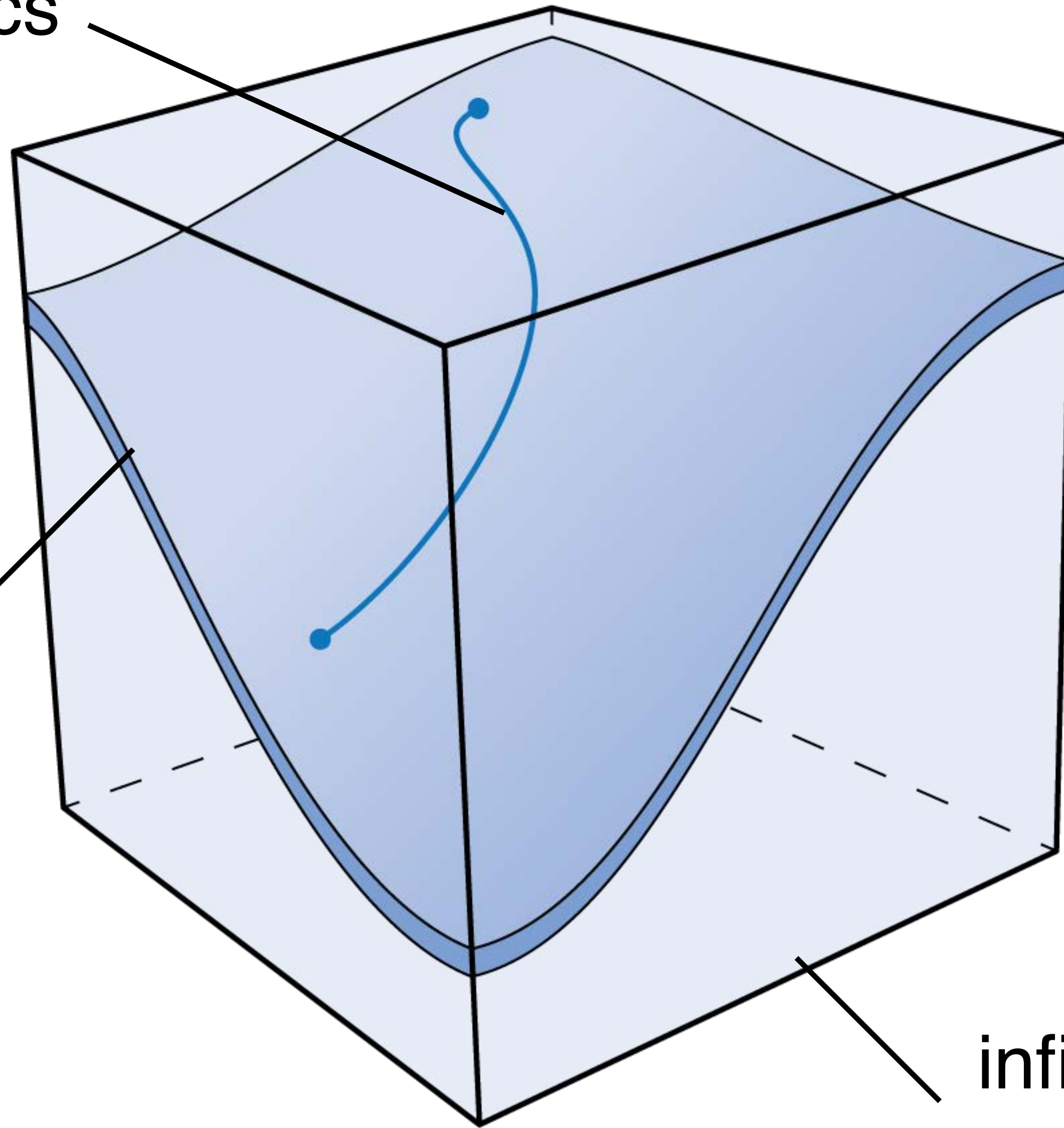


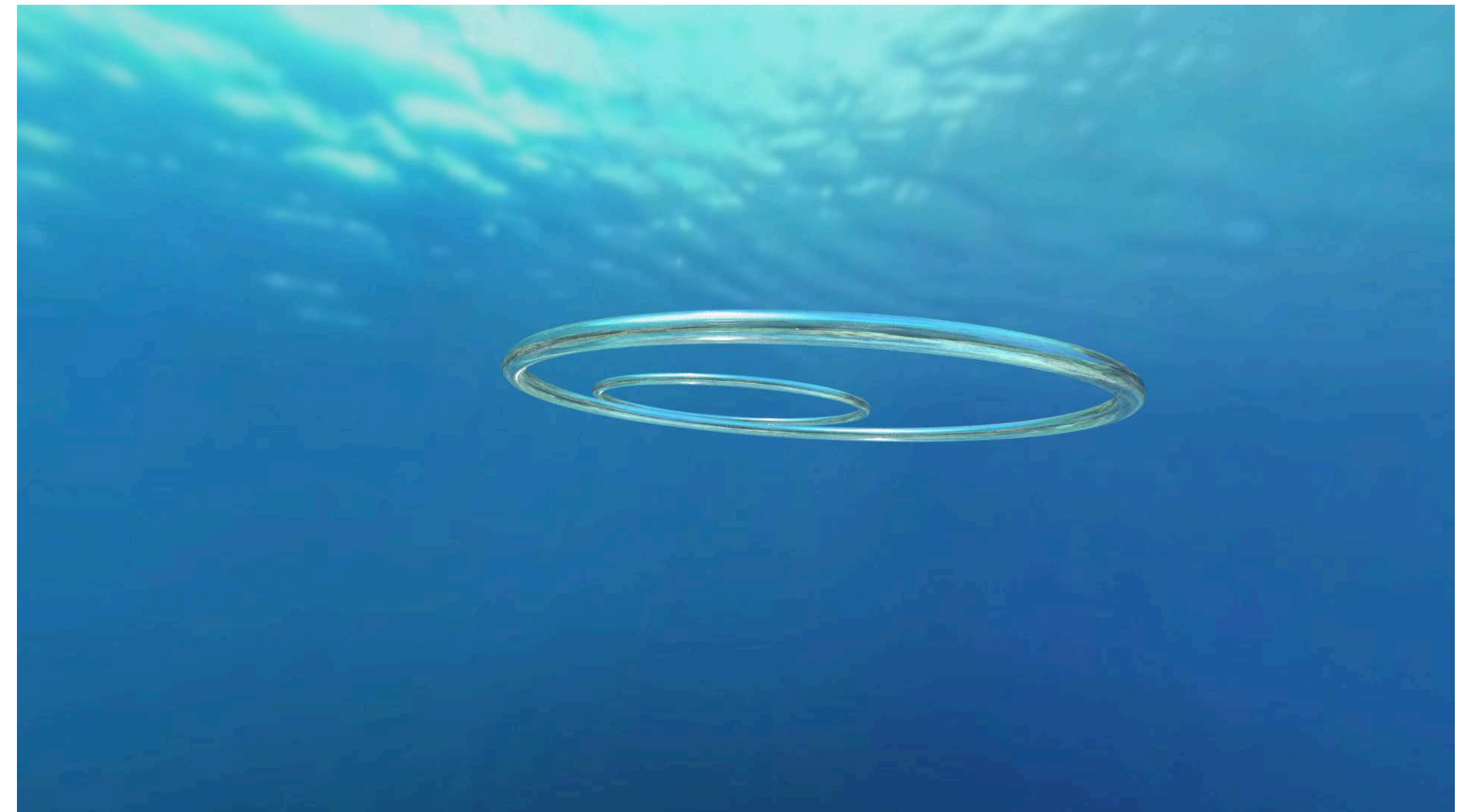
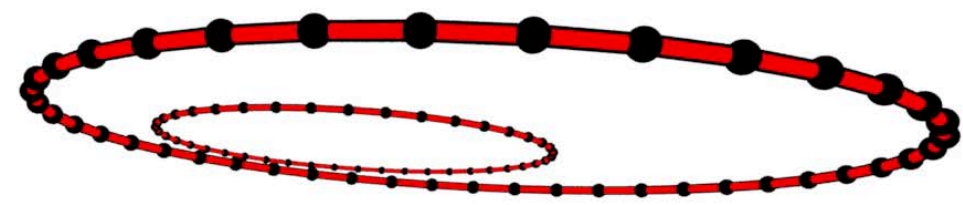
infinite dimensional
space of all fluid states

real fluid dynamics

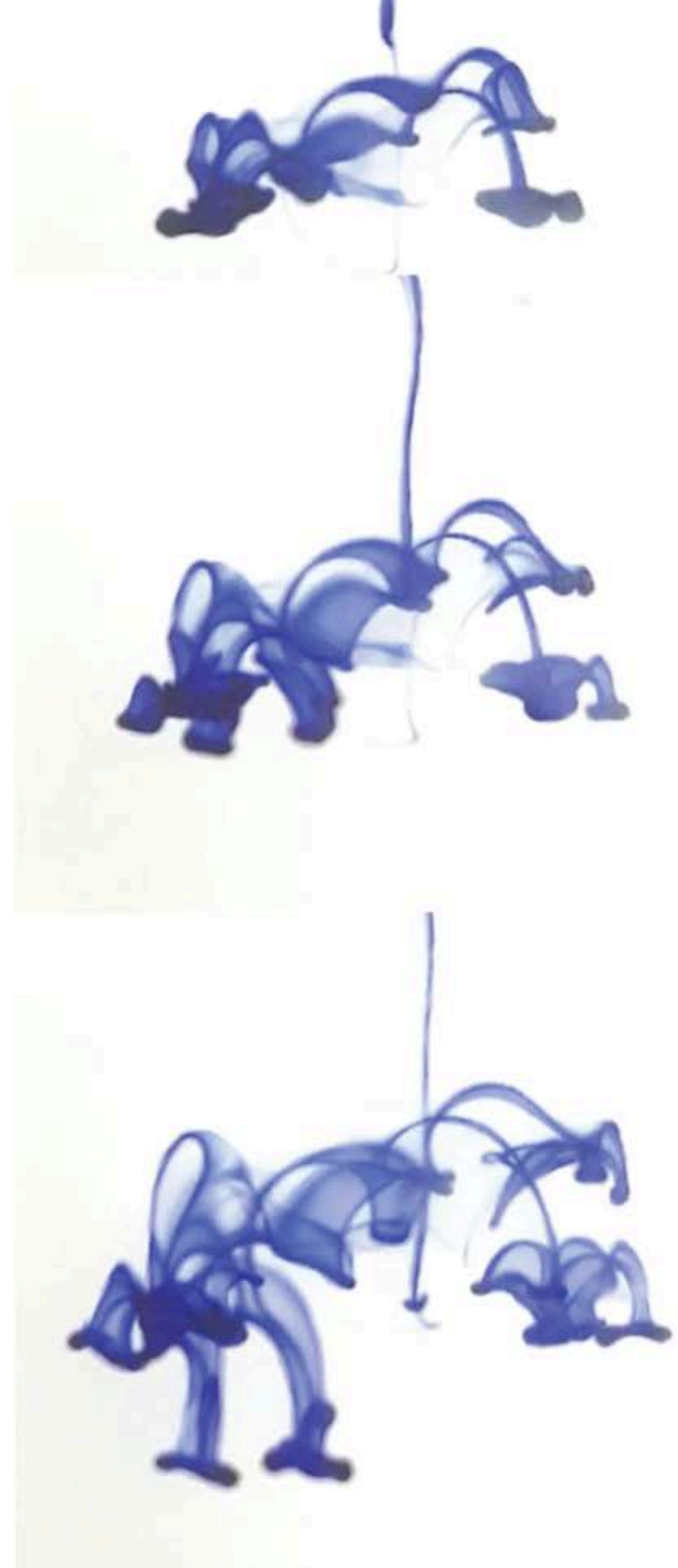
space of vortex
filaments

infinite dimensional
space of all fluid states





~ 85 vertices



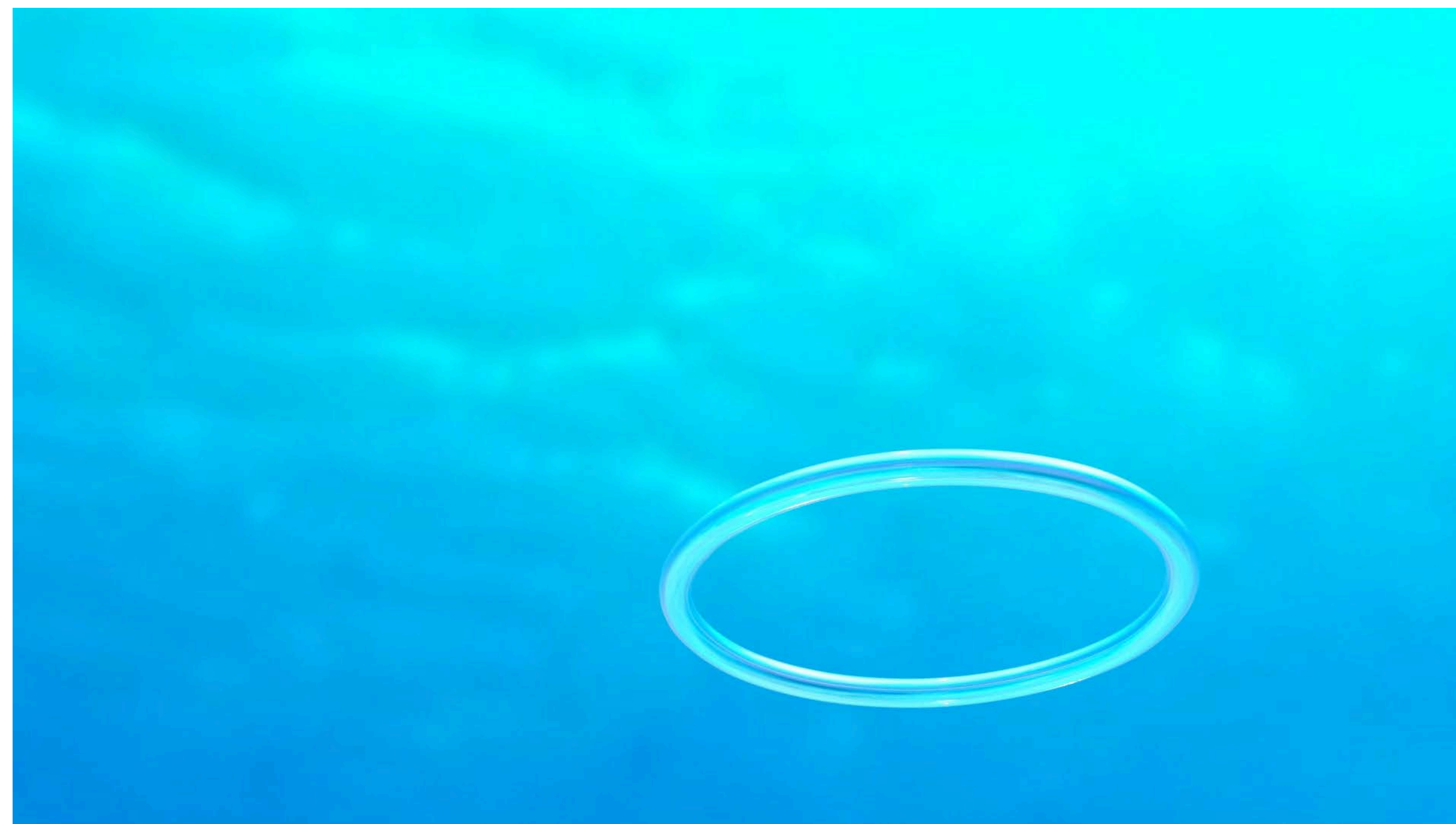
Real footage



Simulation



Real footage



Simulation

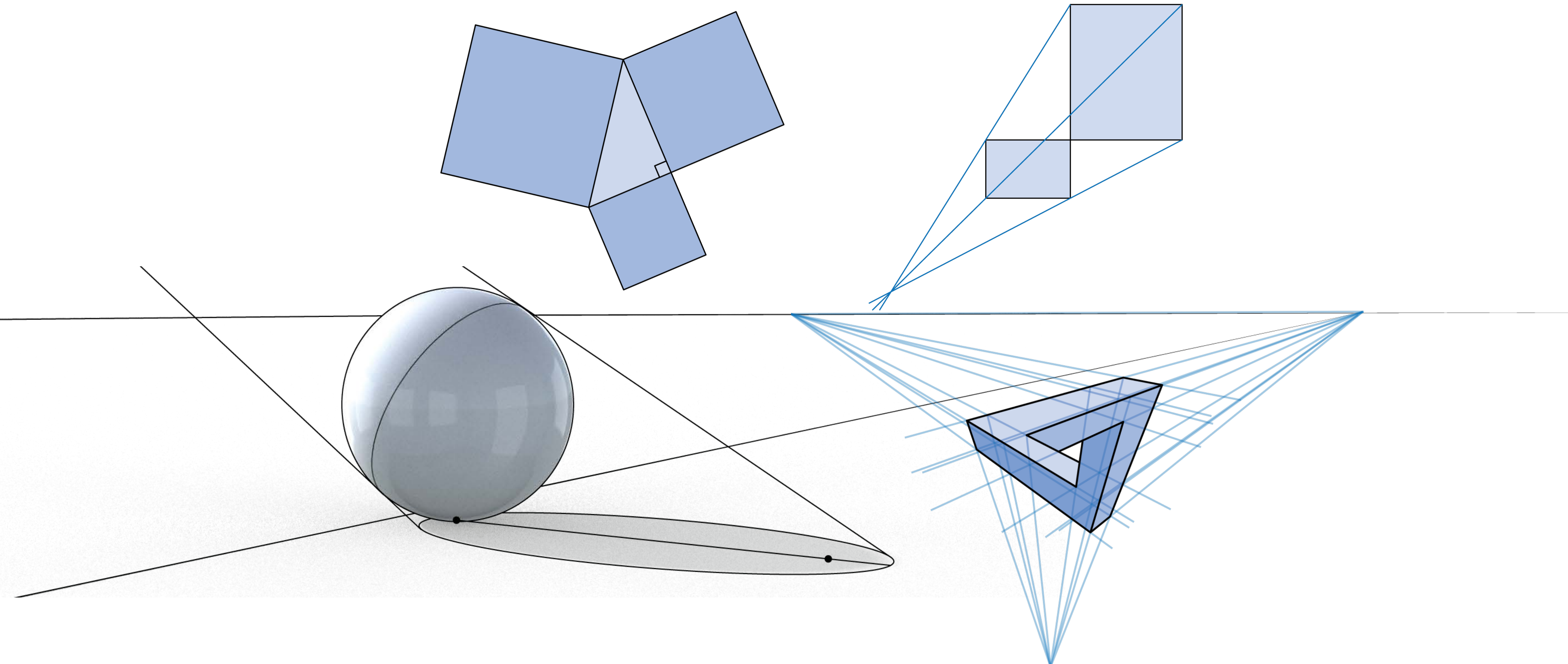
Discrete Differential Geometry

What is Geometry?

- Relationships between shapes, order of spaces
- Symmetries and invariants

What is Geometry?

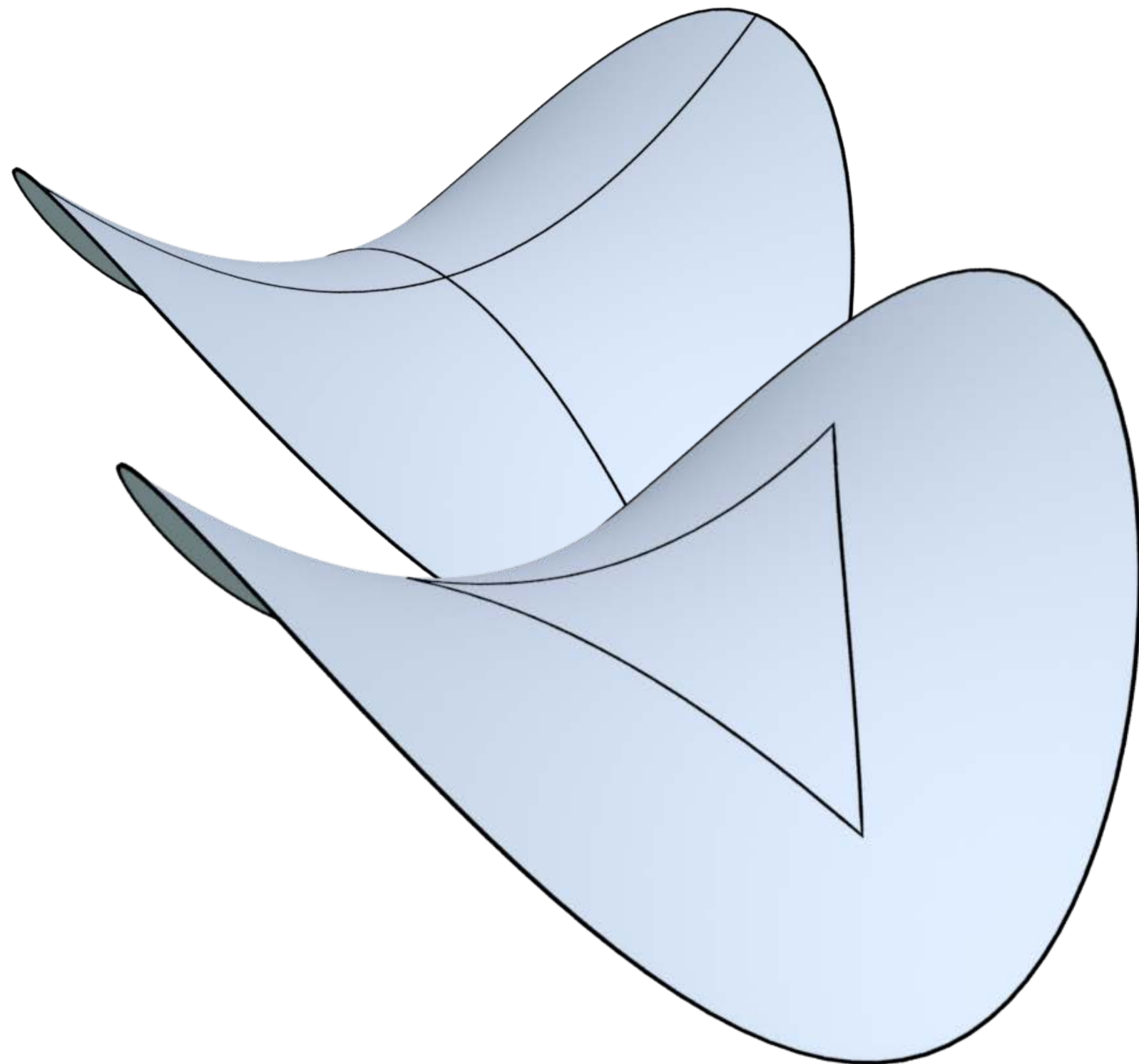
- Relationships between shapes, order of spaces



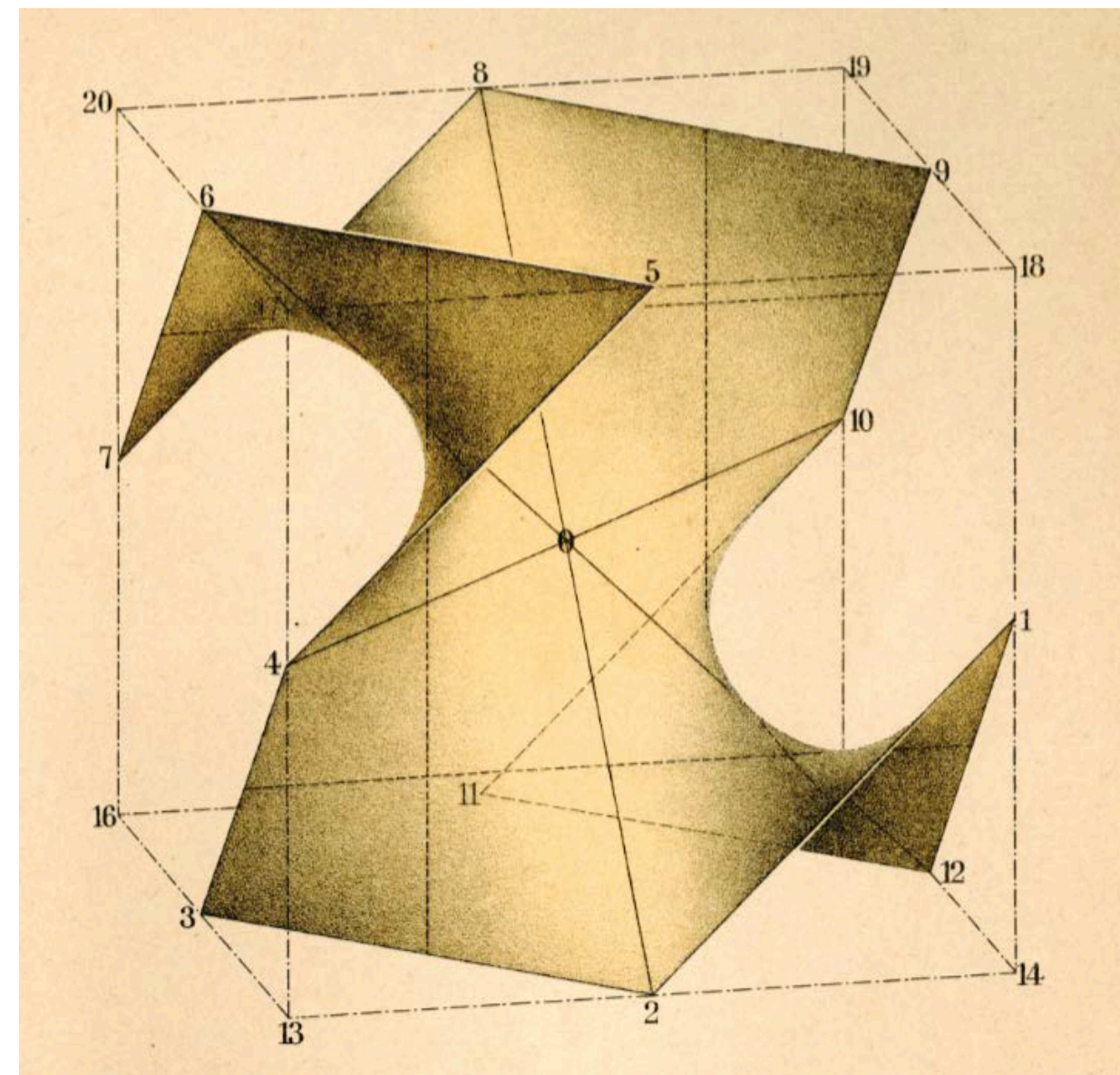
What is Geometry?

- Relationships between shapes, order of spaces (18th, 19th century)

Differential geometry



Gauss, Riemann, Poincaré, ...



Schwarz 1890

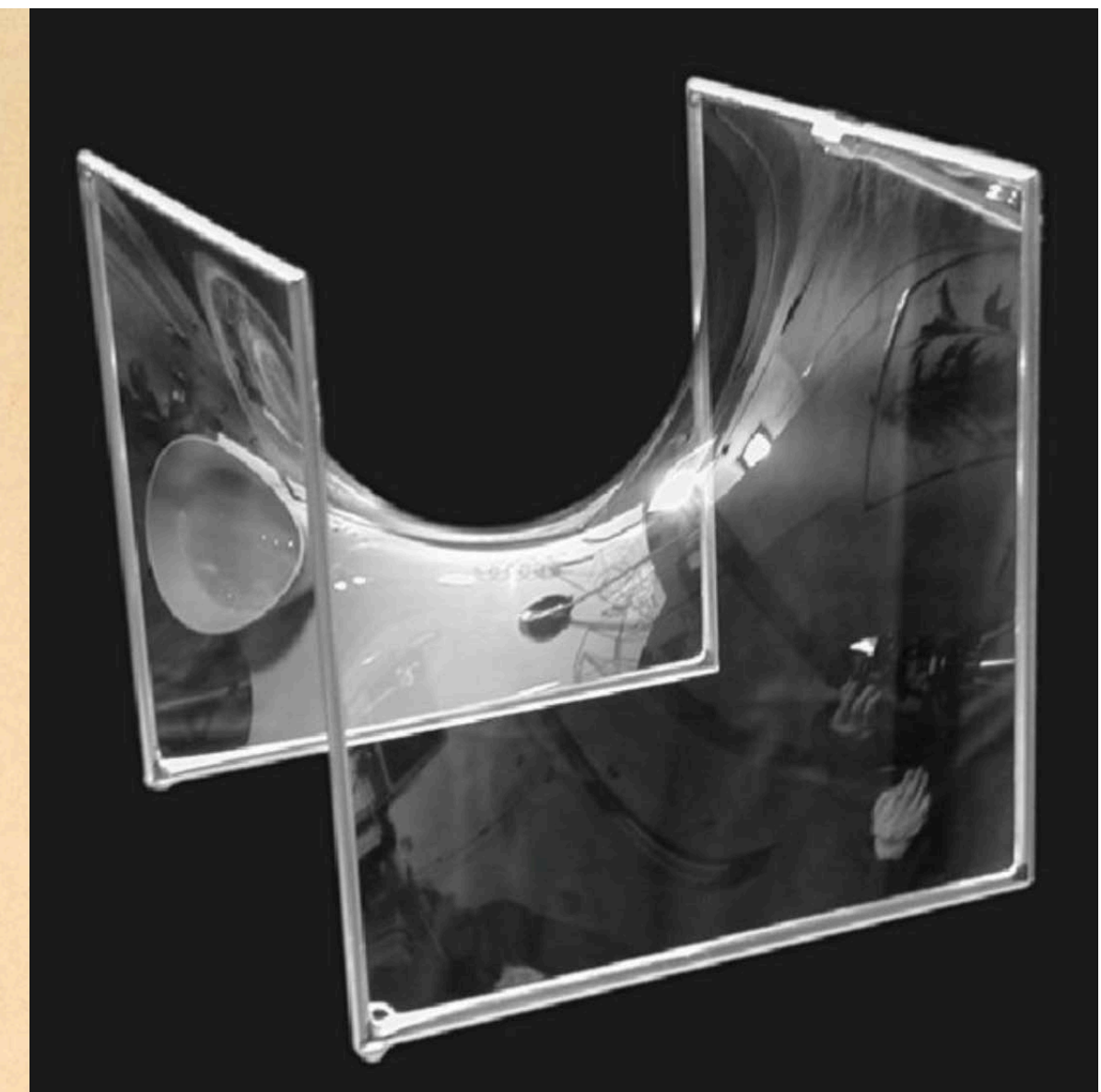


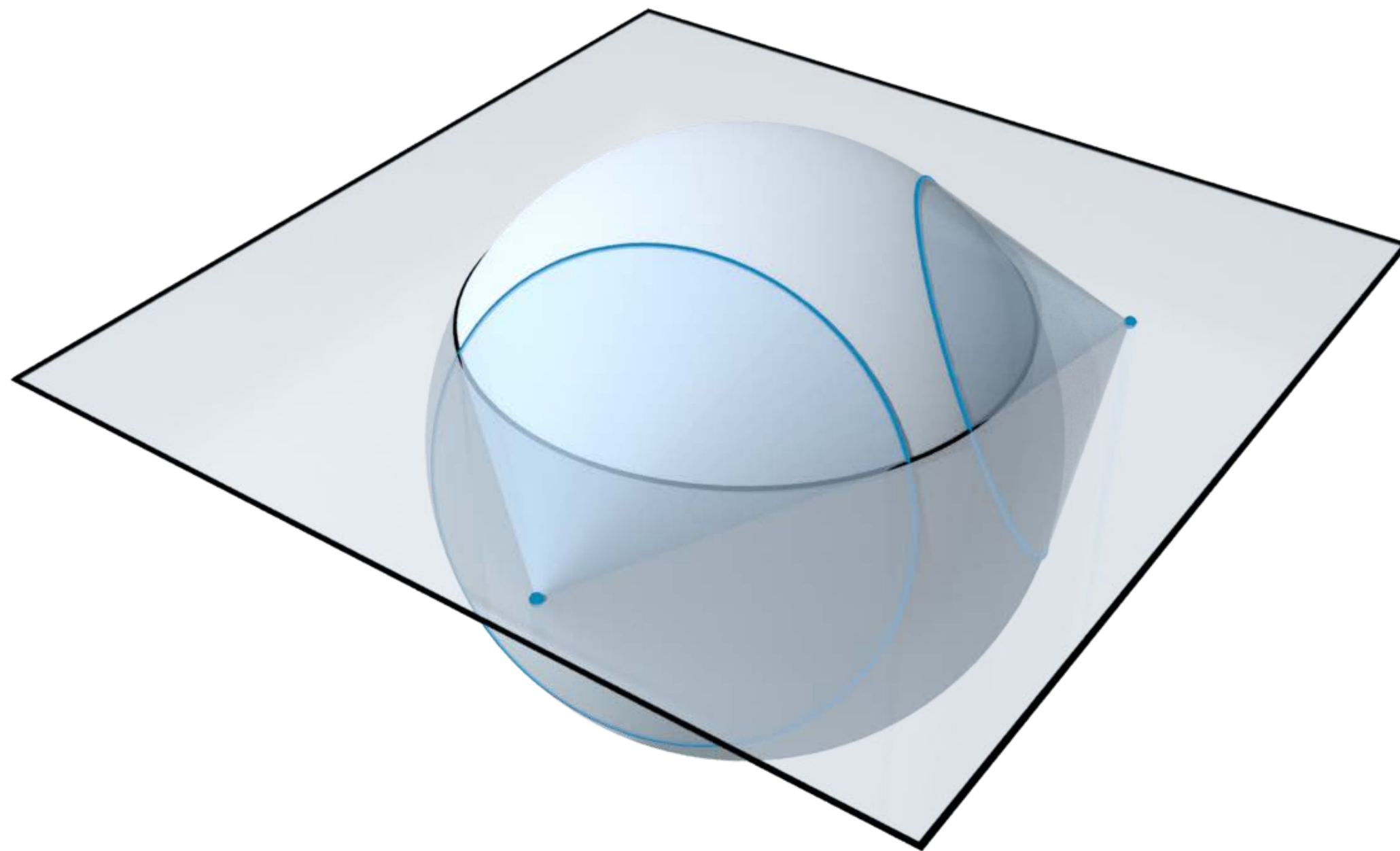
photo courtesy Capanna

- Engineering design
- General relativity
- Computer graphics

What is Geometry?

- Relationships between shapes, order of spaces (18th, 19th century)

Non-Euclidean projective geometry



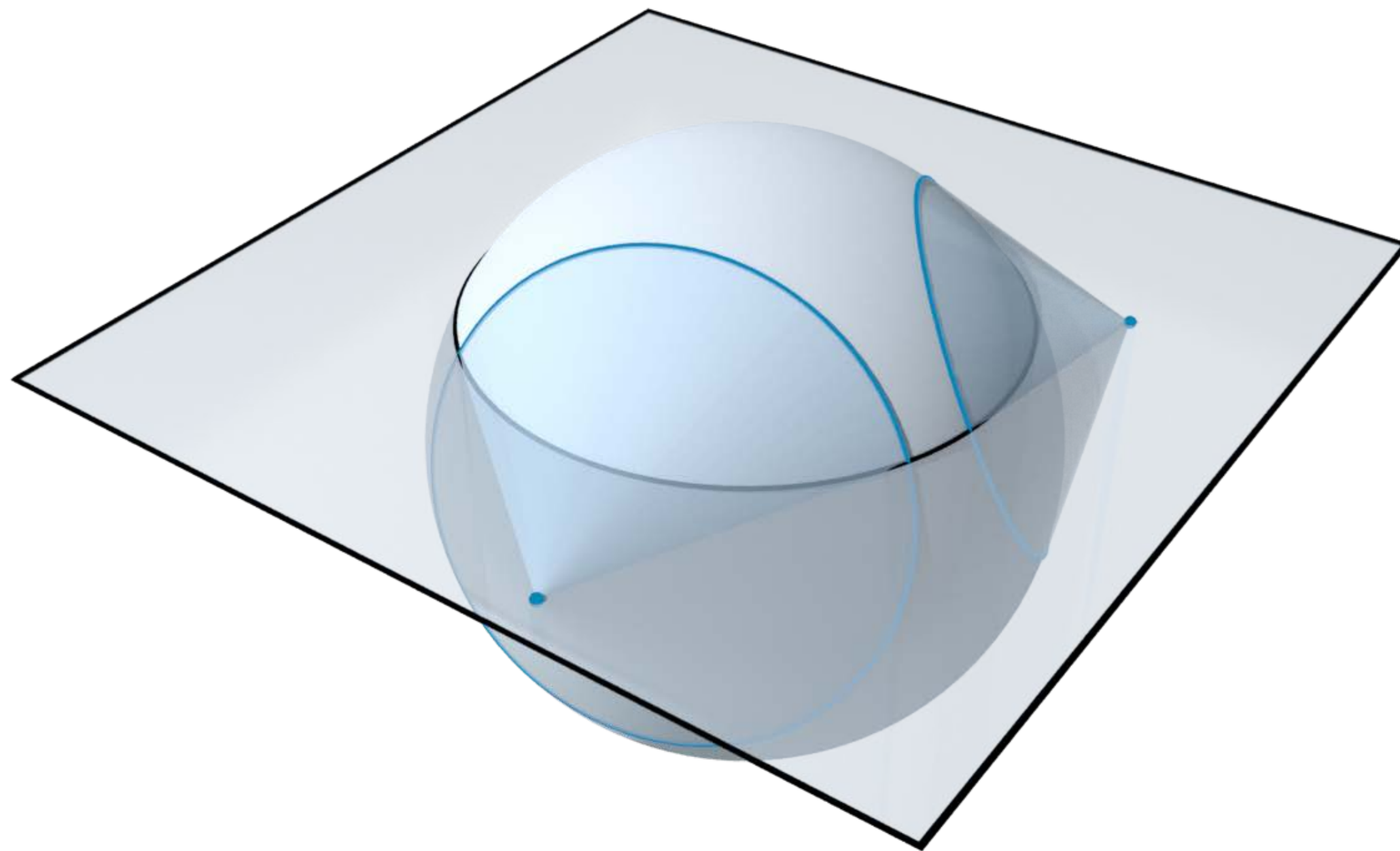
- ✓ Two points connect into a line
- ✓ Center and distance gives a circle
- ✓ 90° is well-defined
- ✗ Two lines meet unless parallel

Lobachevsky, Möbius, Grassmann,
Beltrami, Lie, Plücker, Cayley, Klein,...

What is Geometry?

- Relationships between shapes, order of spaces (18th, 19th century)

Non-Euclidean projective geometry



Lobachevsky, Möbius, Grassmann,
Beltrami, Lie, Plücker, Cayley, Klein,...

- Linear algebra
- Quantum mechanics
- Algebraic geometry

What is Geometry?

- Relationships between shapes, order of spaces
- Symmetries and invariants

Klein's Erlangen program 1872:

Ingredients of geometry:

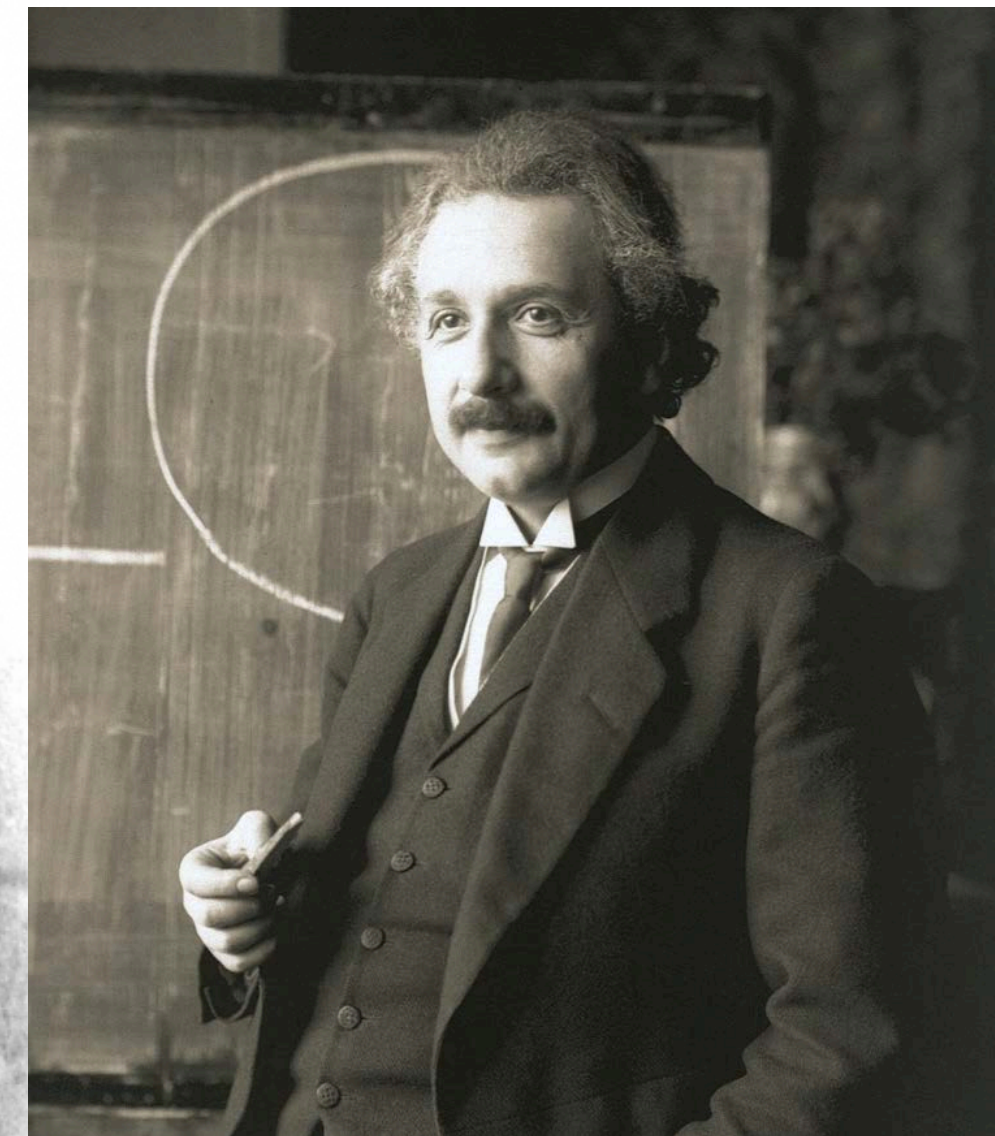
- Collection of objects
- Group of transformations
- Invariant notions



Felix Klein 1872



Emmy Noether 1915



Albert Eistein 1916

What is Geometry?

Ingredients of geometry:

- Collection of objects
- Group of transformations
- Invariant notions

$$\left\{ \mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \right\}$$

$$\tilde{\mathbf{p}} = \underset{\text{rotation}}{R} \mathbf{p} + \underset{\text{translation}}{\mathbf{t}}$$

- ▶ straight lines, planes
- ▶ distance
- ▶ circles, spheres
- ▶ angles
- ▶ area
- ▶ parallelism

- Non-invariant notions

- ▶ additions $\mathbf{p}_1 + \mathbf{p}_2$
- ▶ dot product $\mathbf{p}_1 \cdot \mathbf{p}_2$

Geometry in Euclidean space

What is Geometry?

Ingredients of geometry:

- Collection of objects
- Group of transformations
- Invariant notions

$$\left\{ \mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \right\} \quad \tilde{\mathbf{p}} = R\mathbf{p}_{\text{rotation}}$$

- ▶ straight lines, planes
- ▶ distance
- ▶ circles, spheres
- ▶ angles
- ▶ area
- ▶ parallelism
- ▶ additions $\mathbf{p}_1 + \mathbf{p}_2$
- ▶ dot product $\mathbf{p}_1 \cdot \mathbf{p}_2$

- Non-invariant notions

Geometry in Euclidean vector space

What is Geometry?

Ingredients of geometry:

- Collection of objects
- Group of transformations
- Invariant notions

$$\left\{ \mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \right\} \quad \tilde{\mathbf{p}} = A\mathbf{p}$$

invertible matrix

- ▶ straight lines, planes
- ▶ parallelism
- ▶ additions $\mathbf{p}_1 + \mathbf{p}_2$

- Non-invariant notions

- ▶ distance
- ▶ circles, spheres
- ▶ angles
- ▶ area
- ▶ dot product $\mathbf{p}_1 \cdot \mathbf{p}_2$

Geometry in vector space

What is Geometry?

Ingredients of geometry:

- Collection of objects
- Group of transformations
- Invariant notions

$$\left\{ \mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \right\}$$

$$\tilde{\mathbf{p}} = A\mathbf{p} + \mathbf{b}$$

translation
invertible matrix

- straight lines, planes
- parallelism
- midpoint $\frac{\mathbf{p}_1 + \mathbf{p}_2}{2}$

- Non-invariant notions

- distance
- circles, spheres
- angles
- area
- additions $\mathbf{p}_1 + \mathbf{p}_2$
- dot product $\mathbf{p}_1 \cdot \mathbf{p}_2$

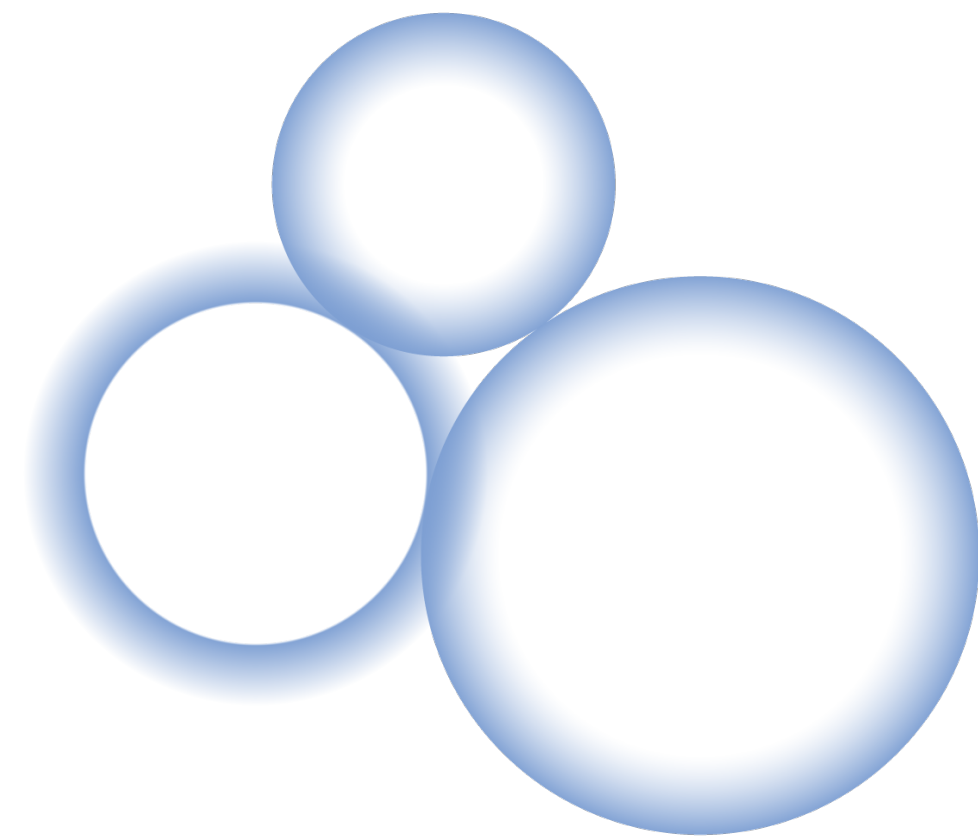
Geometry in affine space

What is Geometry?

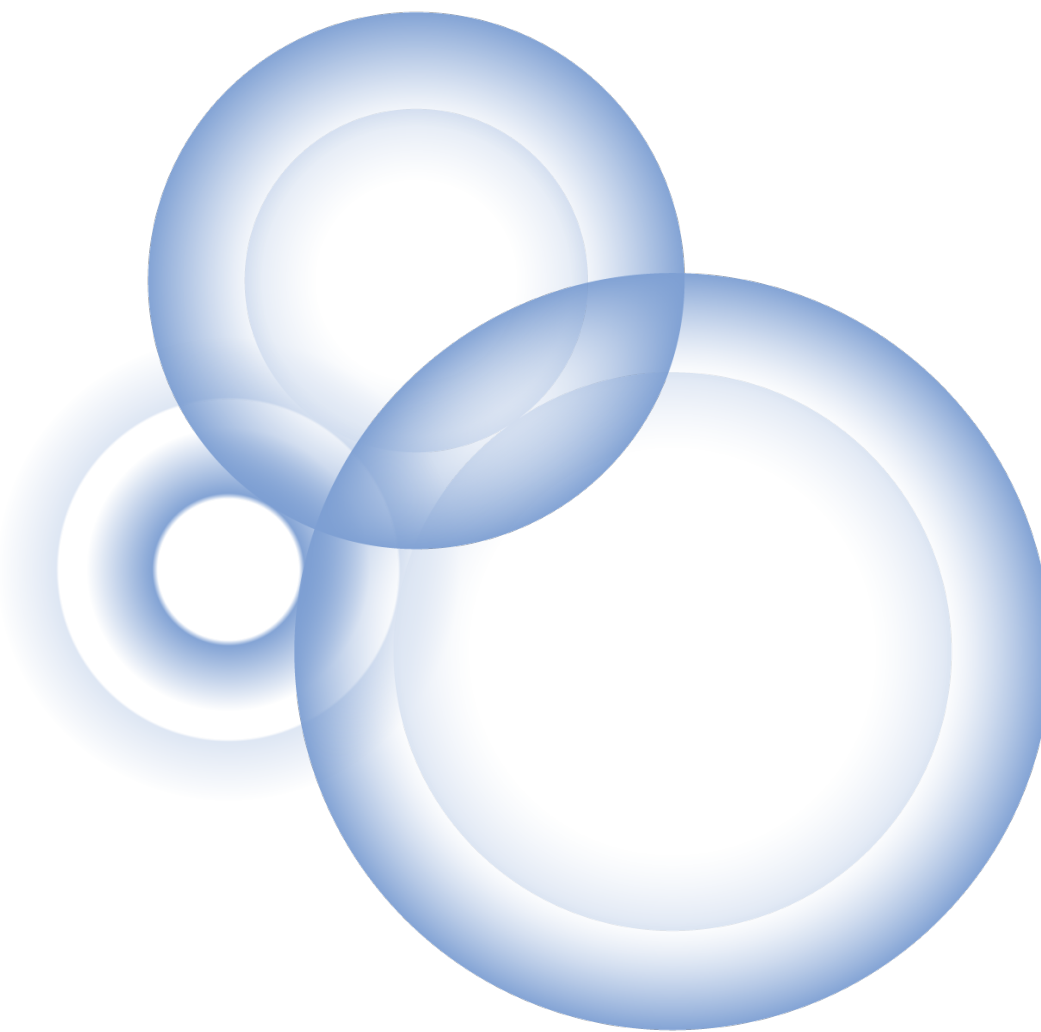
Ingredients of geometry:

- Collection of objects
- Group of transformations

oriented circles in $\mathbb{R}^2$

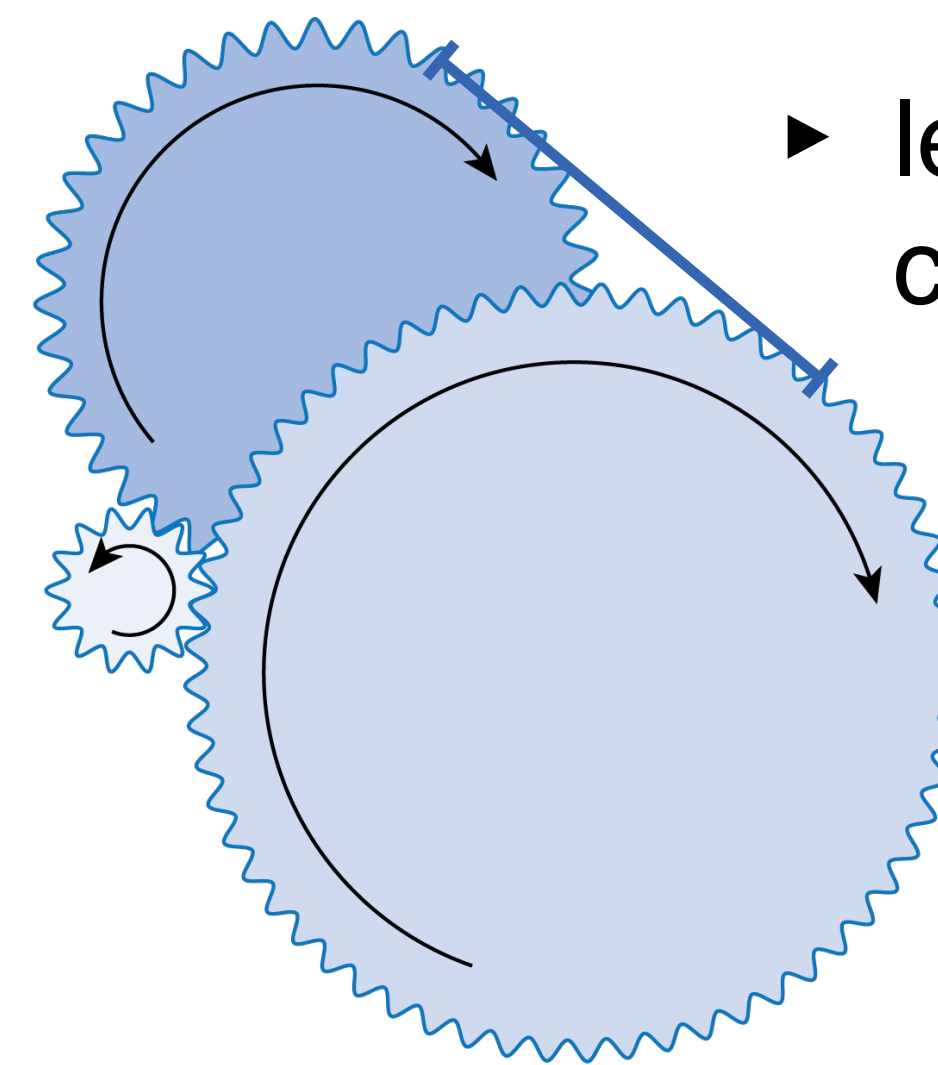


- ▶ rotate, translate
- ▶ normal offset (propagate wave)



- Invariant notions

- ▶ straight lines
- ▶ oriented contact



- ▶ lengths of common tangents

- Non-invariant notions

- ▶ intersection
- ▶ point

Laguerre geometry of oriented circles

What is Geometry?

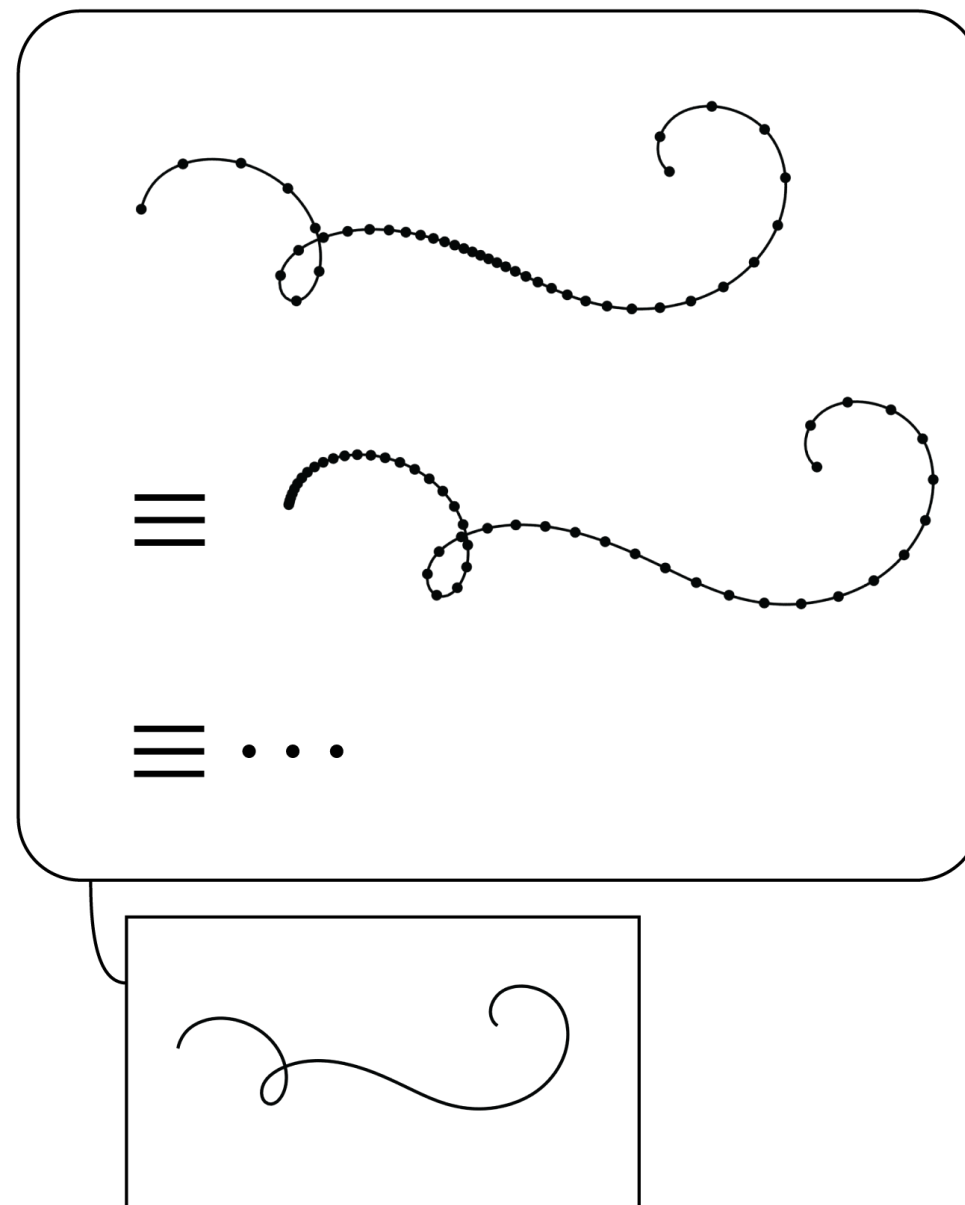
Ingredients of geometry:

- Collection of objects
- Group of transformations
- Invariant notions

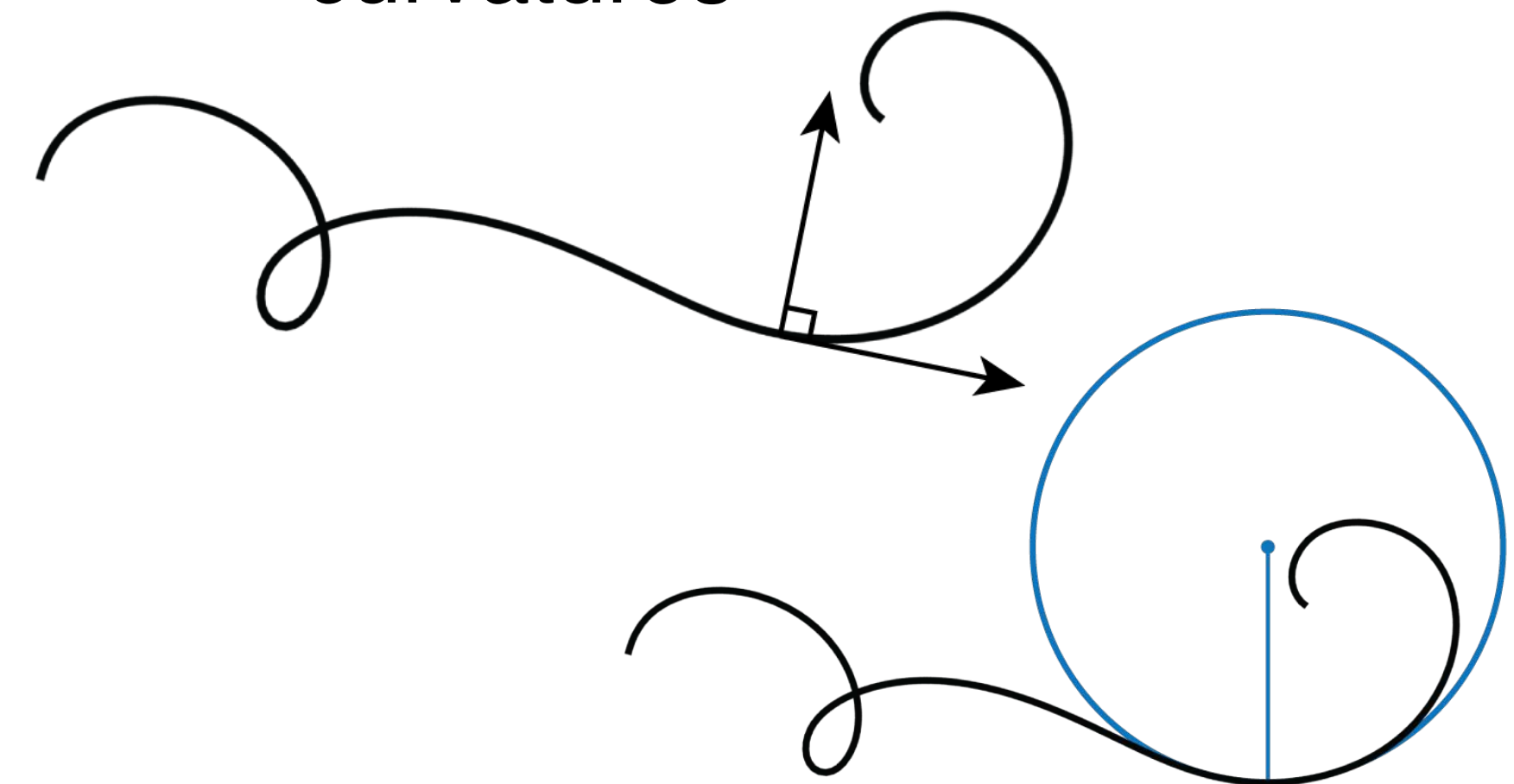
Particle moving
along a trajectory



Reparametrizations



- ▶ tangents, normals
- ▶ curvatures



- Non-invariant notions
 - ▶ particle speed
 - ▶ particle acceleration

Differential geometry of curves

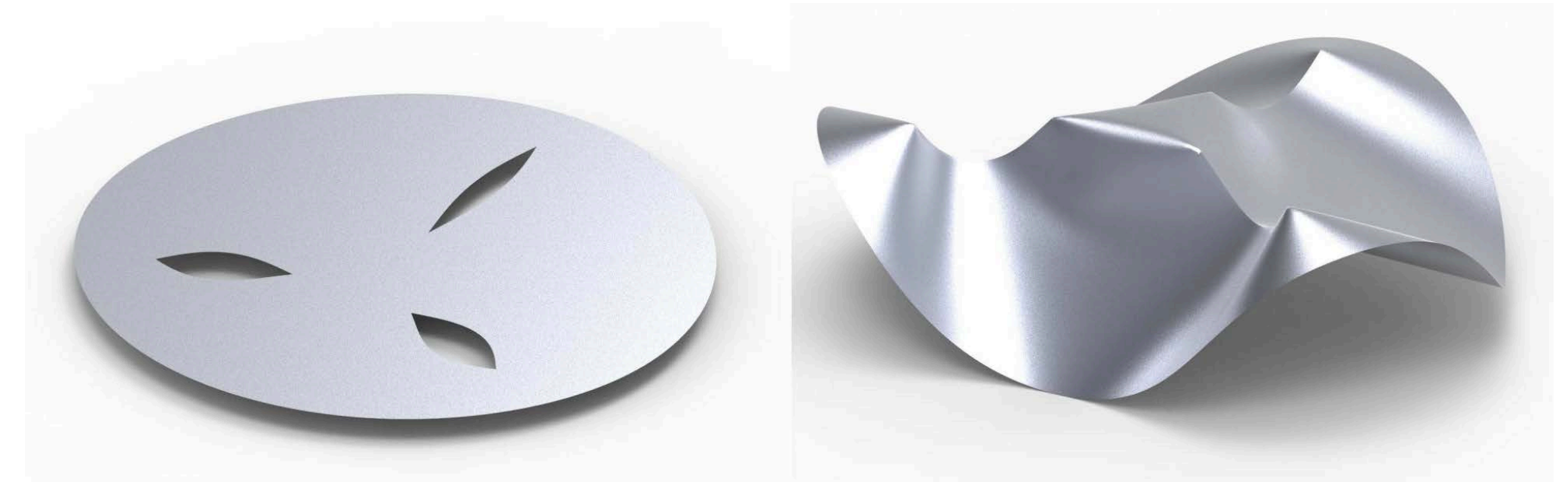
Differential Geometry

Why do we care?

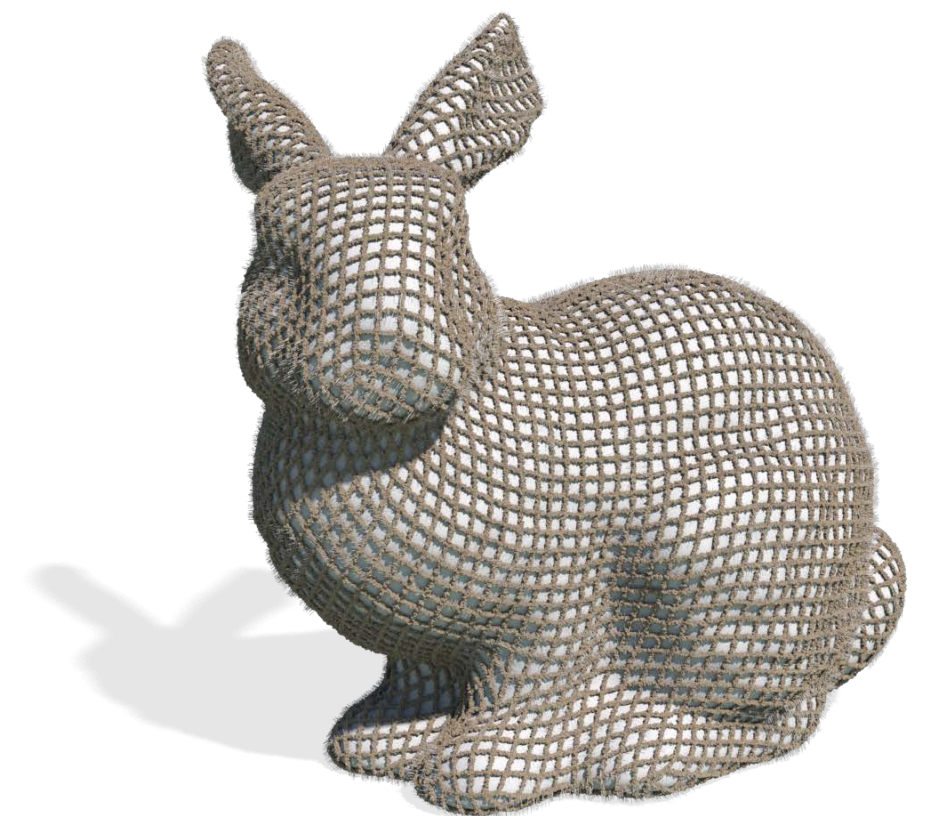
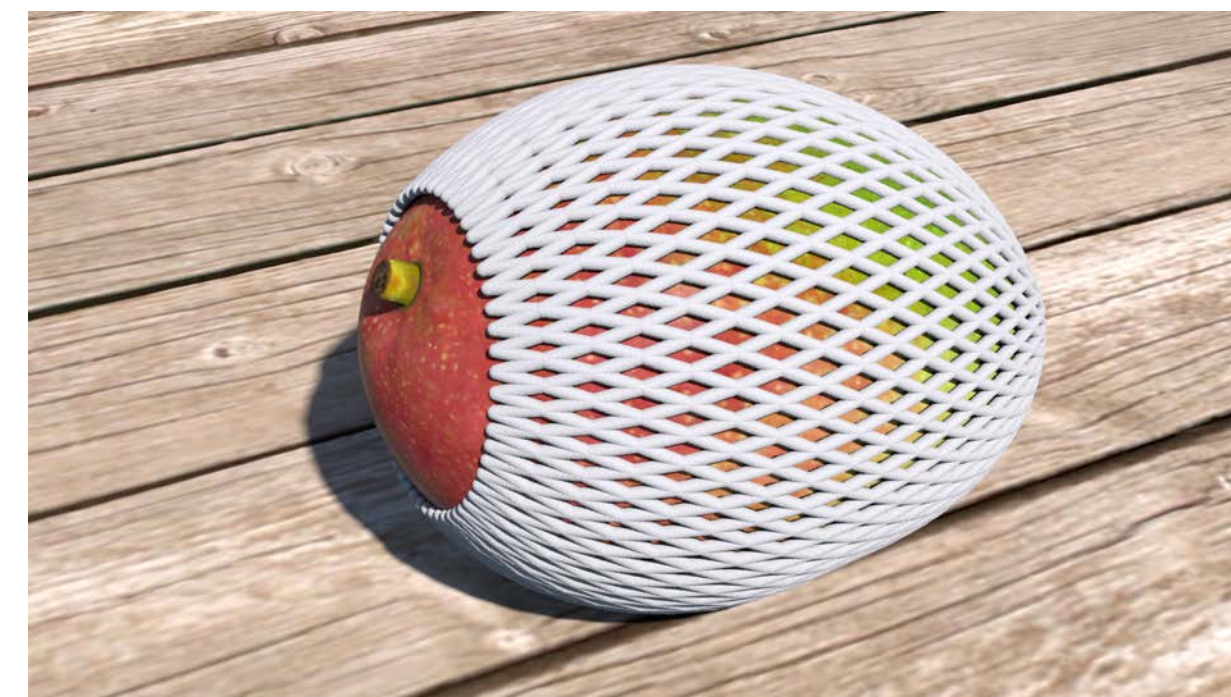
- geometry of surfaces



Konaković, Crane, Deng, Bouaziz, Piker & Pauly 2016



Jiang, Wang, Rist, Wallner & Pottmann 2020



Sageman-Furnas, Chern, Ben-Chen, Vaxman 2019

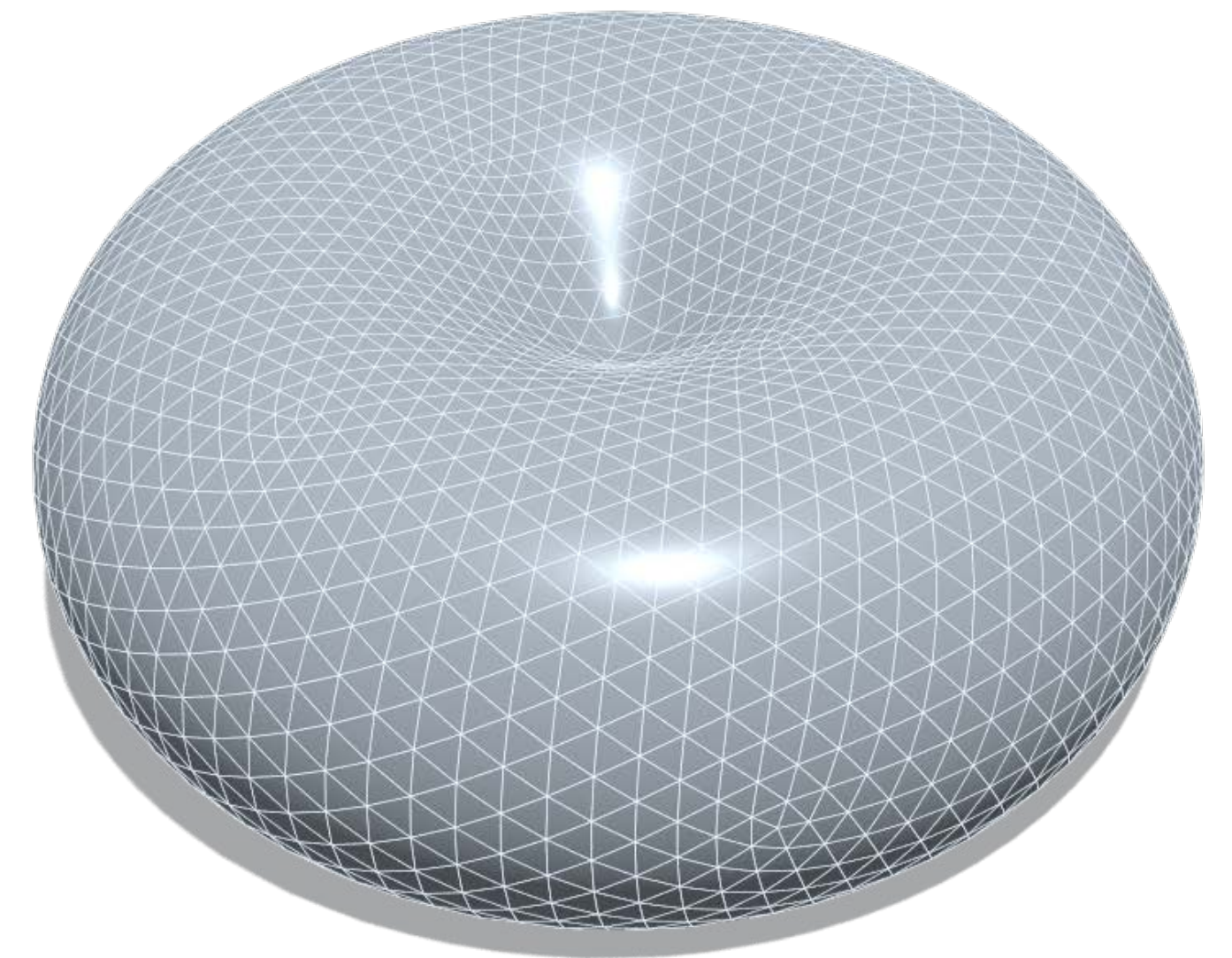
Differential Geometry

Why do we care?

- geometry of surfaces
- mothertongue of physical theories

Elasticity of lipid bilayer membrane (Helfrich 1973)

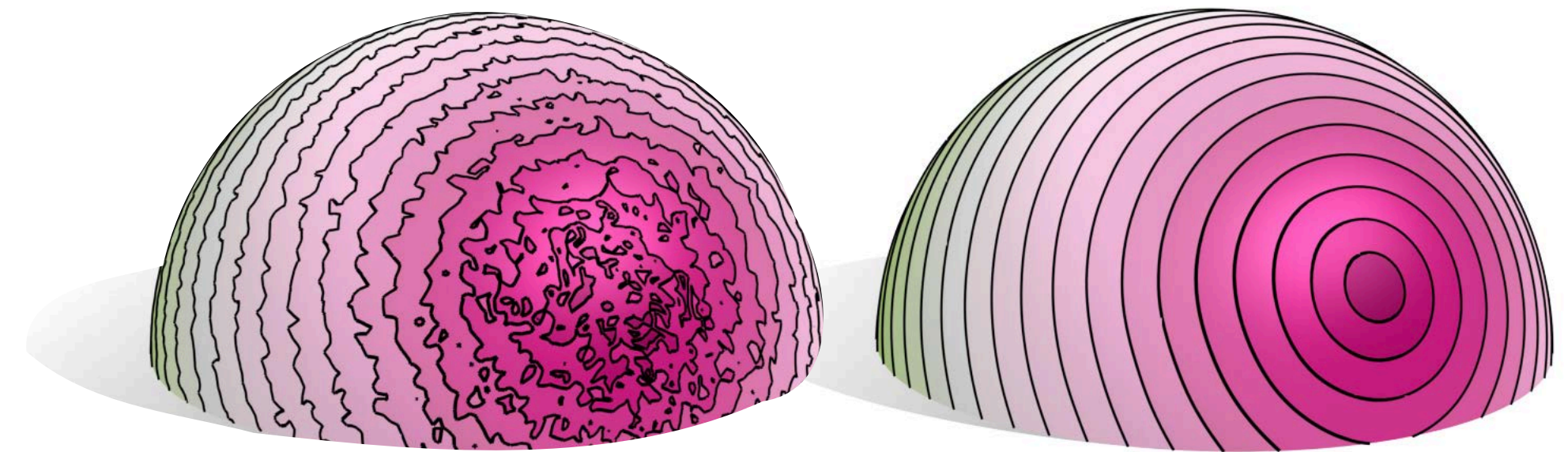
$$E = \int_S \alpha dA + \beta \underbrace{(H - H_0)^2}_{\text{mean curvature}} dA + \gamma \underbrace{K}_{\text{Gaussian curvature}} dA$$



Differential Geometry

Why do we care?

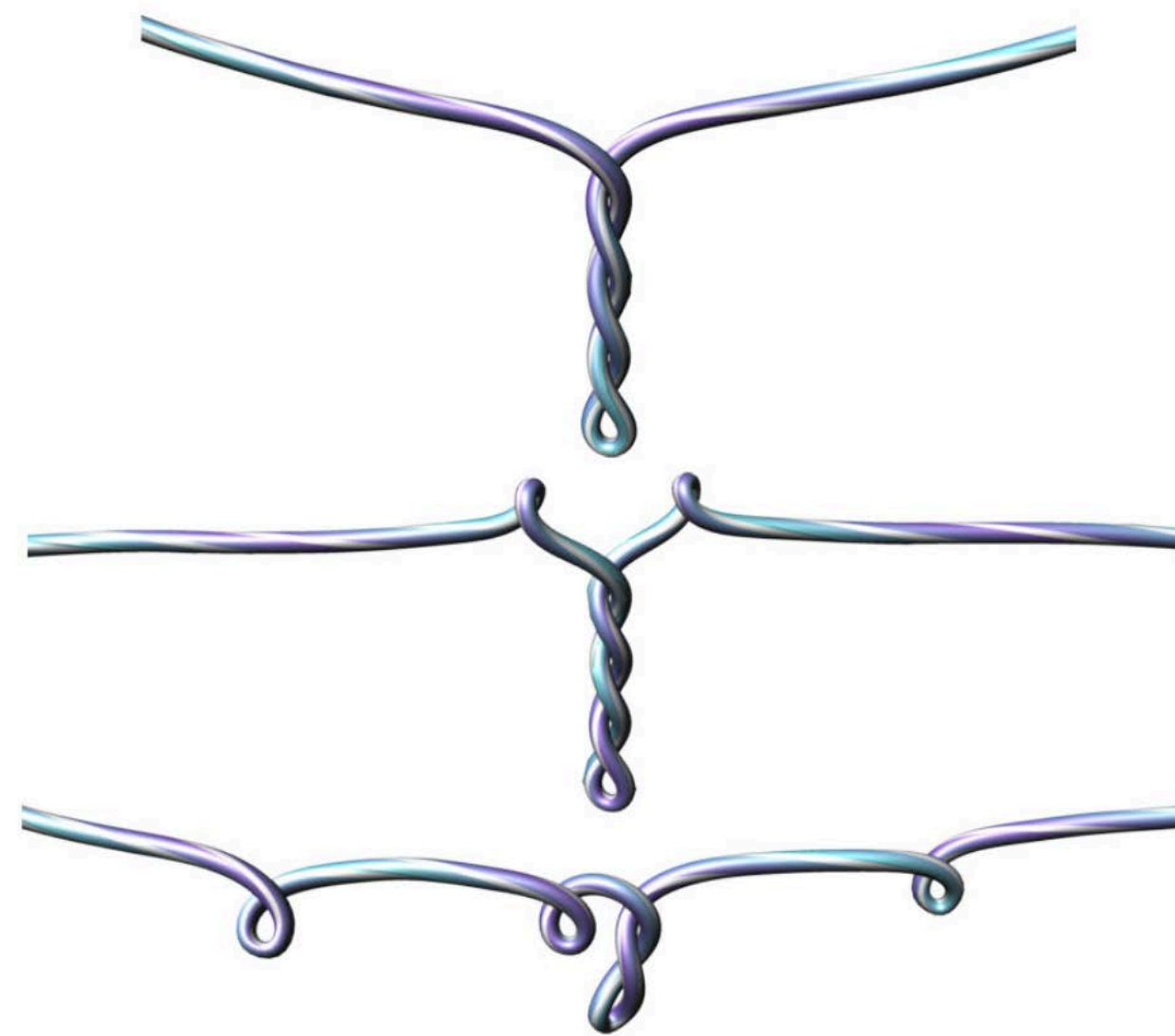
- geometry of surfaces
- mothertongue of physical theories
- computation: simulation, processing



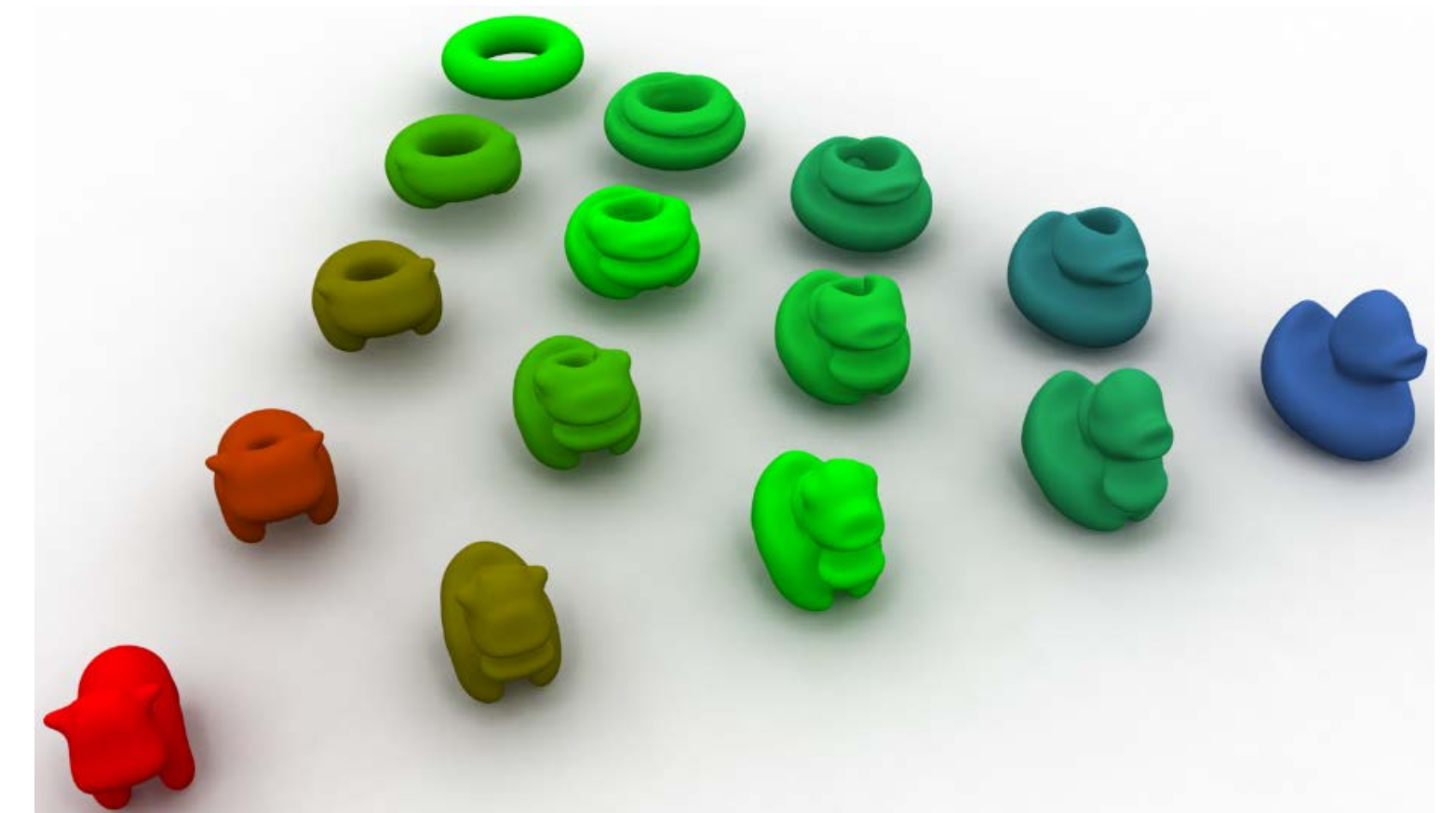
Stein, Grinspun, Wardetzky & Jacobson 2018



Chern, Knöppel, Pinkall & Schröder 2017



Bergou, Wardetzky, Robinson, Audoly & Grinspun 2008

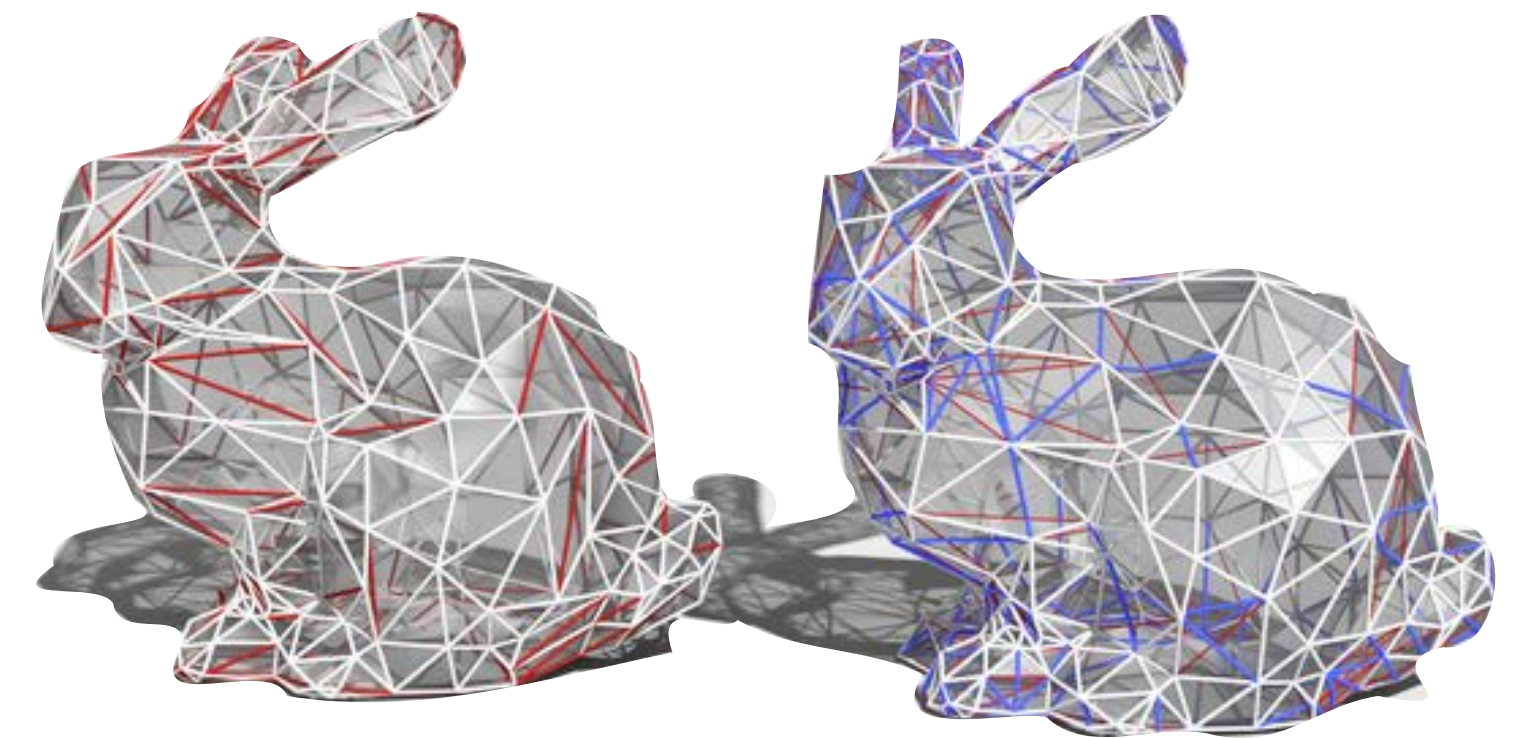


Solomon, de Goes, Peyré, Cuturi, Butscher, Nguyen, Du & Guibas 2015

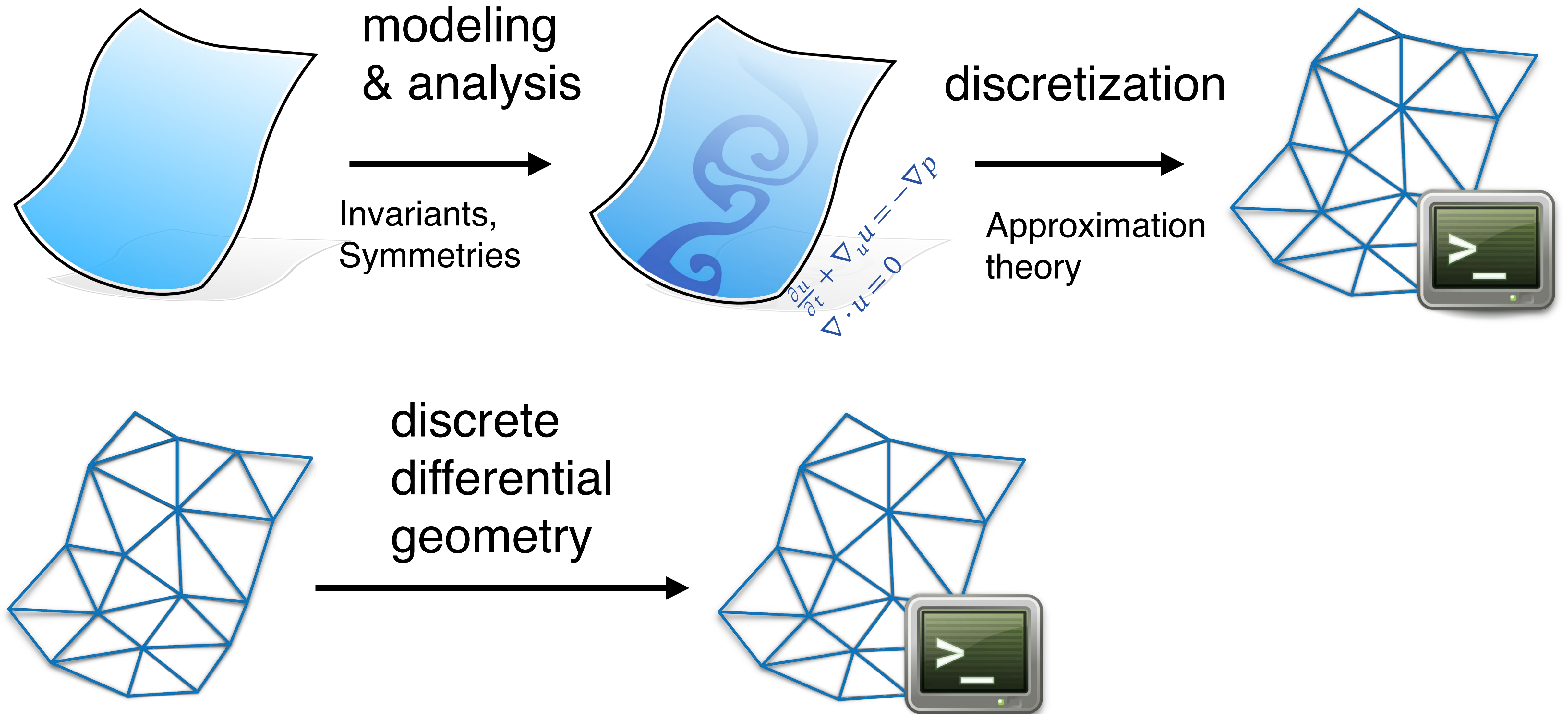
Differential Geometry

How do we apply differential geometry ideas?

- Surfaces are collections of samples and connectivity
- Apply continuous ideas
BUT: setting is discrete
- What is the right way?
 - ▶ Discrete v.s. Discretized



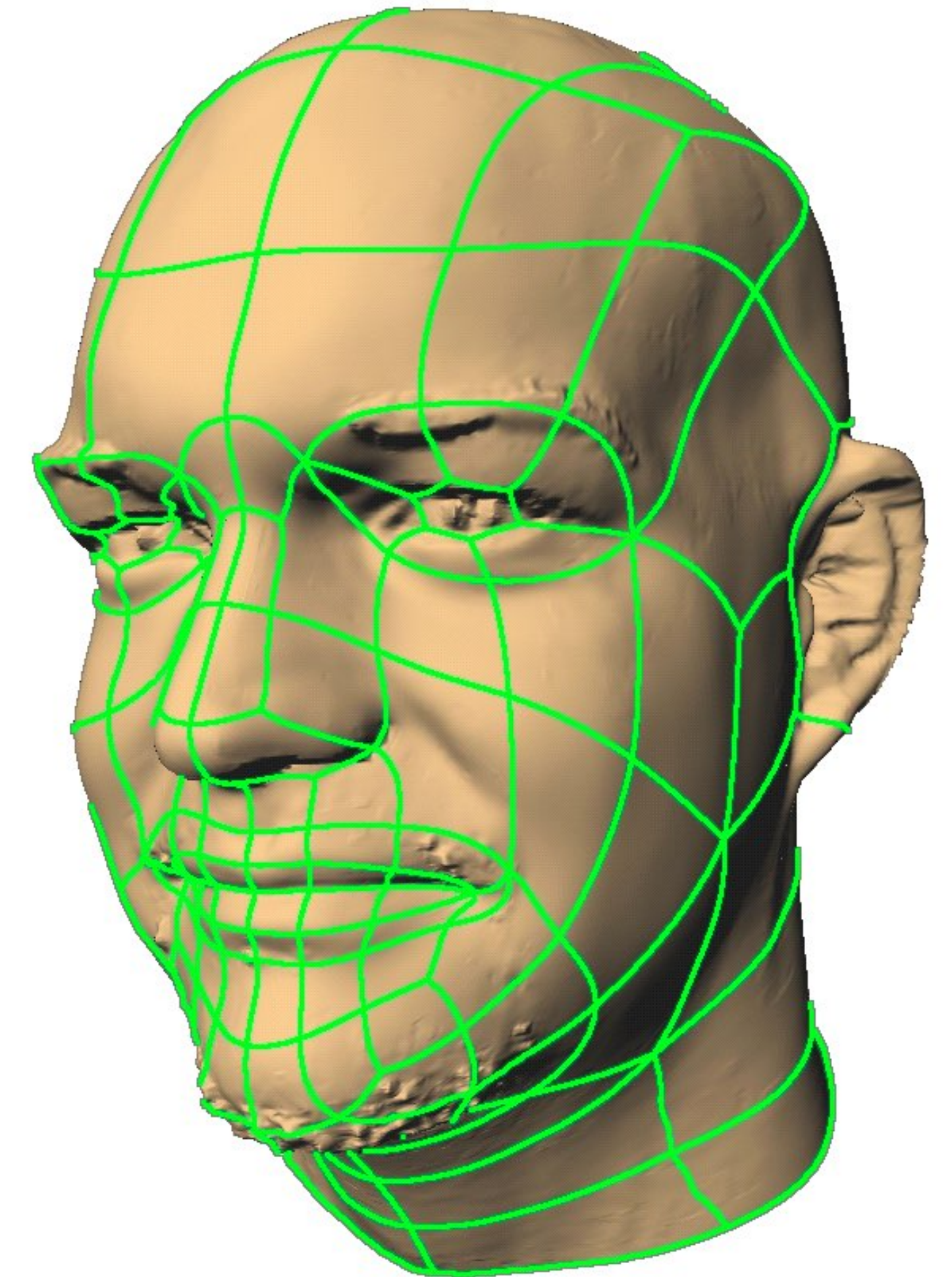
Discretized v.s. Discrete



Discretized

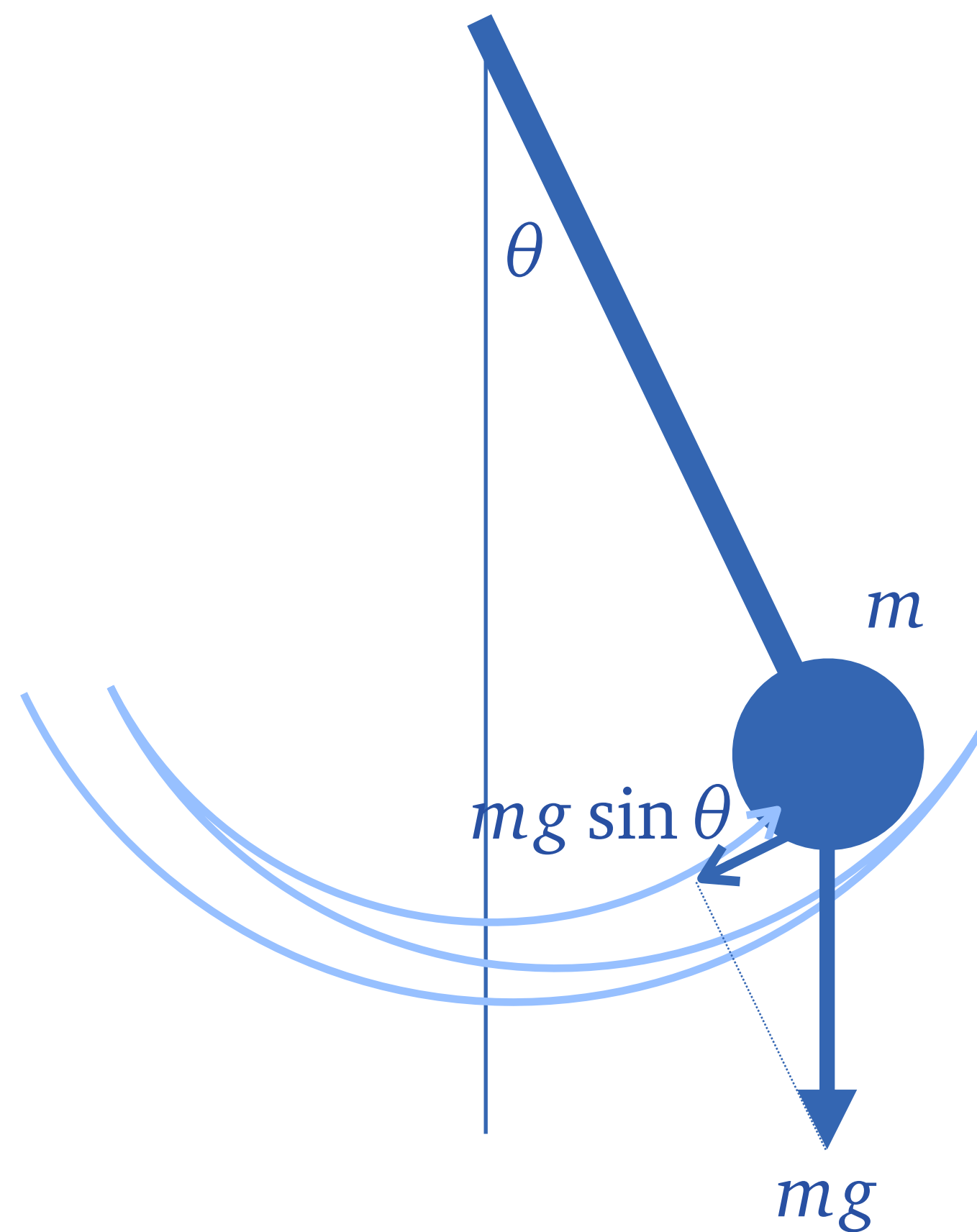
Build smooth manifold structure

- Collection of charts with compatibility on their overlaps
- Realize everything as smooth functions
 - ▶ Polynomial, rational functions
 - ▶ Spectral basis
- Numerical analysis paradigm
 - ▶ Truncation error
 - ▶ Stability



Discretized

Example: pendulum equation.

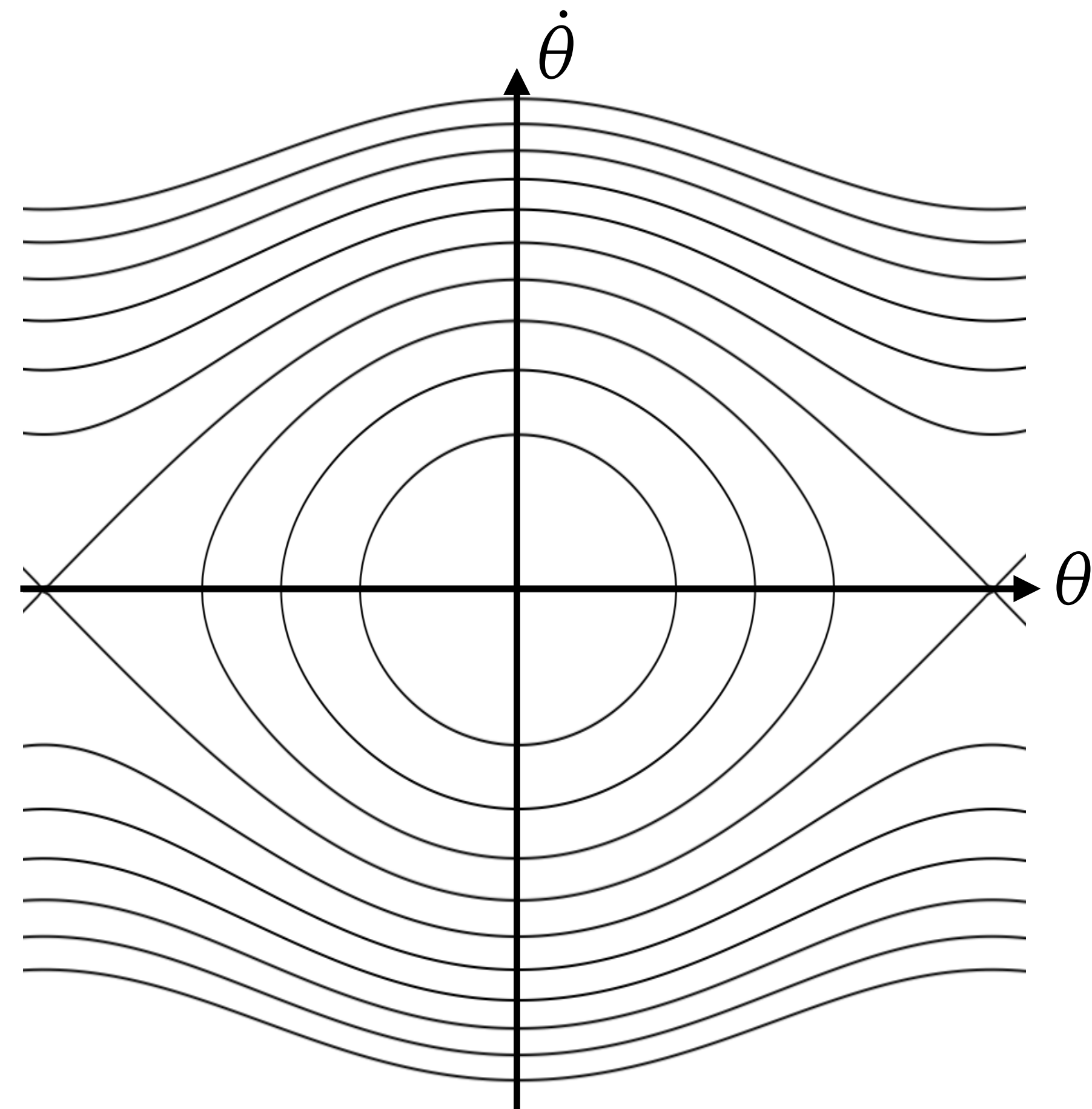


$$m\ddot{\theta} = -mg \sin \theta$$

✓ Energy conservation

$$\frac{1}{2}m\dot{\theta}^2 - mg \cos \theta = \text{const}$$

✓ Integrable system



Discretized

Example: pendulum equation.

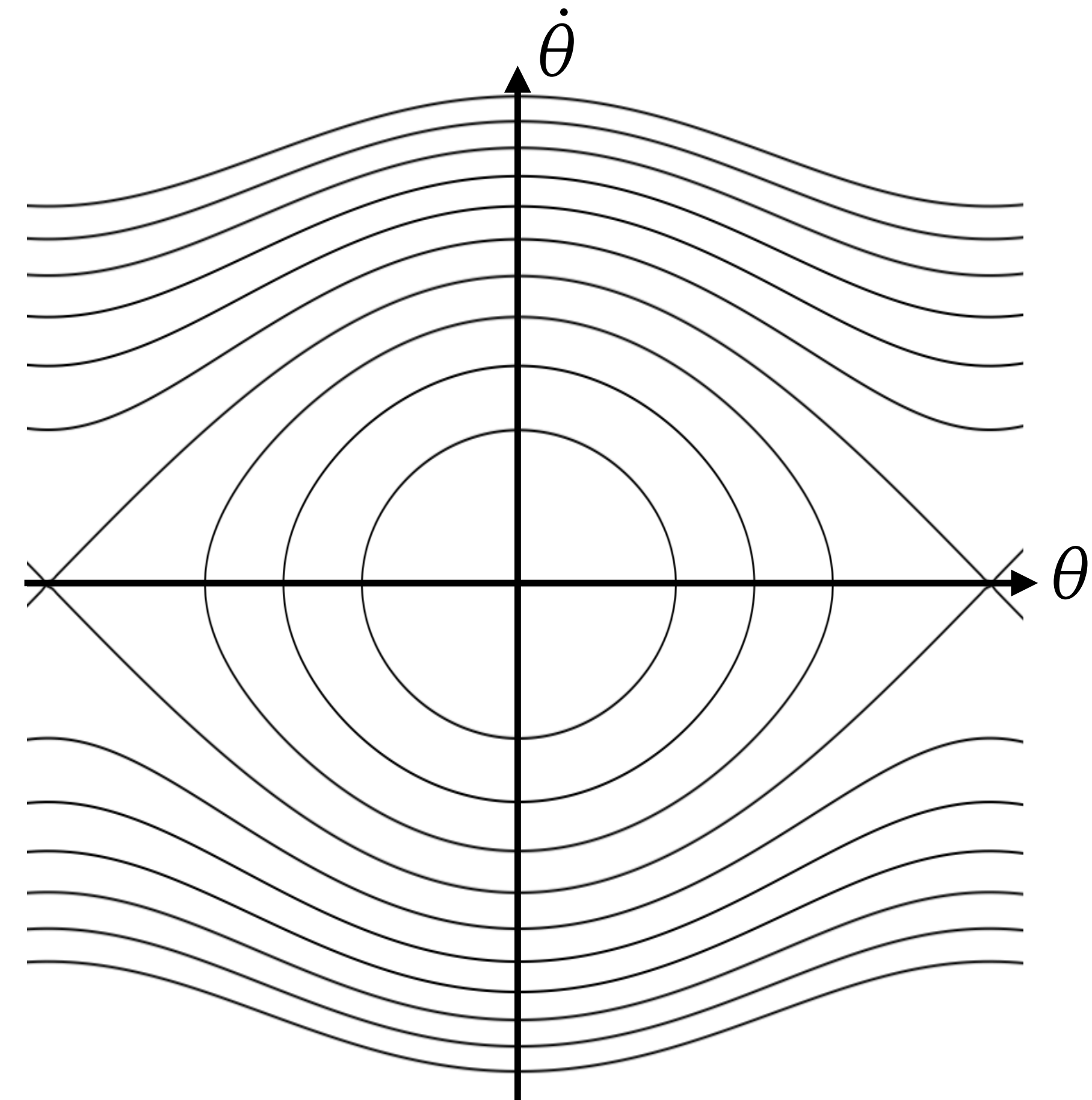
$$m\ddot{\theta} = -mg \sin \theta$$

- High order differential equation solver
(4th order Runge–Kutta method)

Given $(\theta_i, v_i = \dot{\theta}_i)$

$$\begin{aligned} \triangleright \quad \theta_{i+1/2}^* &= \theta_i + \frac{\Delta t}{2} v_i & v_{i+1/2}^* &= v_i - \frac{\Delta t}{2} \sin \theta_i \\ \triangleright \quad \theta_{i+1/2}^{**} &= \theta_i + \frac{\Delta t}{2} v_{i+1/2}^* & v_{i+1/2}^{**} &= v_i - \frac{\Delta t}{2} \sin \theta_{i+1/2}^* \\ \triangleright \quad \theta_{i+1}^{***} &= \theta_i + \Delta t v_{i+1/2}^{**} & v_{i+1}^{***} &= v_i - \Delta t \sin \theta_{i+1/2}^{**} \end{aligned}$$

$$\begin{aligned} \text{Output} \quad \theta_{i+1} &= \theta_i + \frac{\Delta t}{6} \left(v_i + 2v_{i+1/2}^* + 2v_{i+1/2}^{**} + v_{i+1}^{***} \right) \\ v_{i+1} &= v_i - \frac{\Delta t}{6} \left(\sin \theta_i + 2 \sin \theta_{i+1/2}^* + 2 \sin \theta_{i+1/2}^{**} + \sin \theta_{i+1}^{***} \right) \end{aligned}$$



Discretized

Example: pendulum equation.

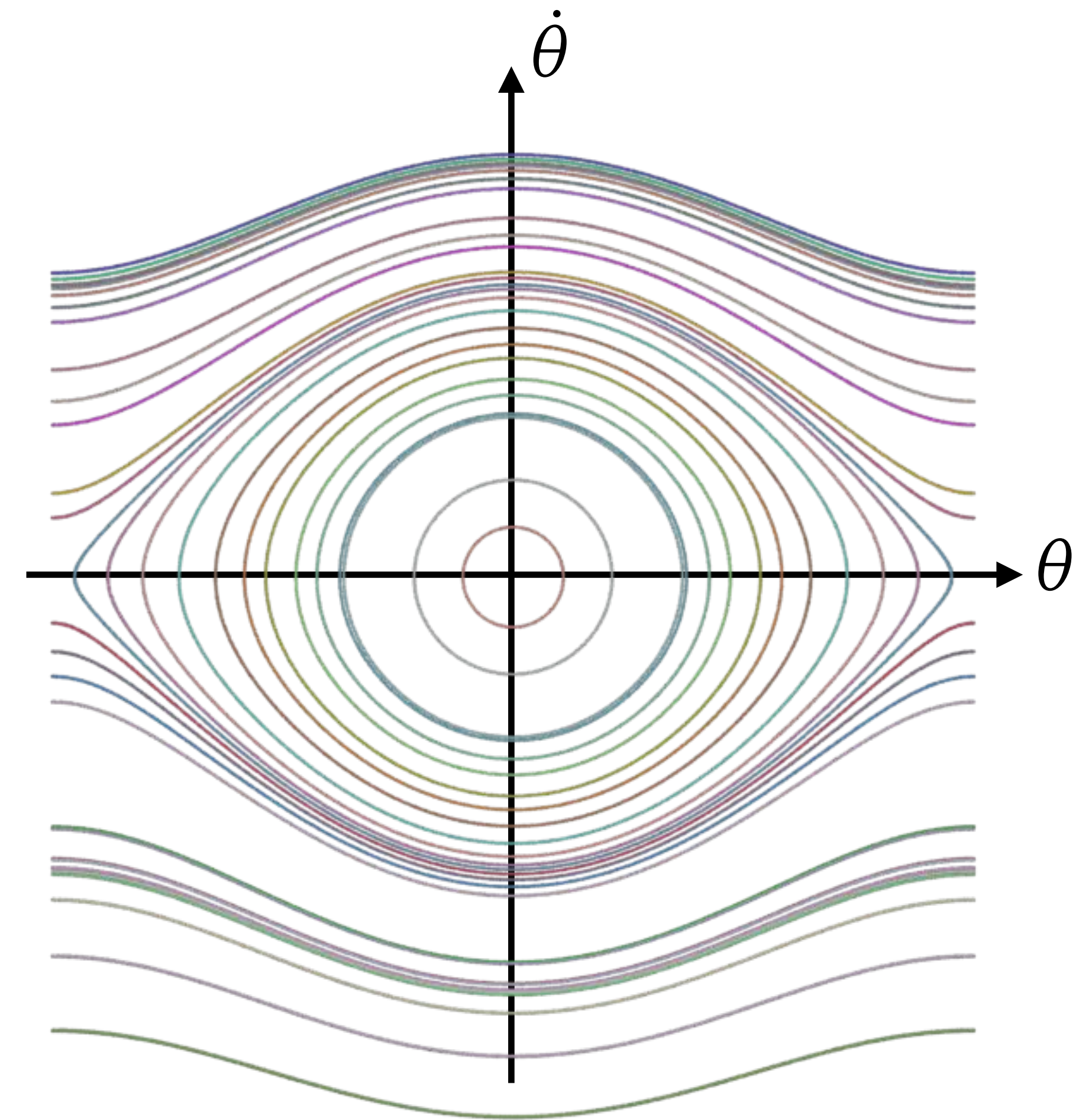
$$m\ddot{\theta} = -mg \sin \theta$$

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$\Delta t = 0.1$

Discretized

Example: pendulum equation.

$$m\ddot{\theta} = -mg \sin \theta$$

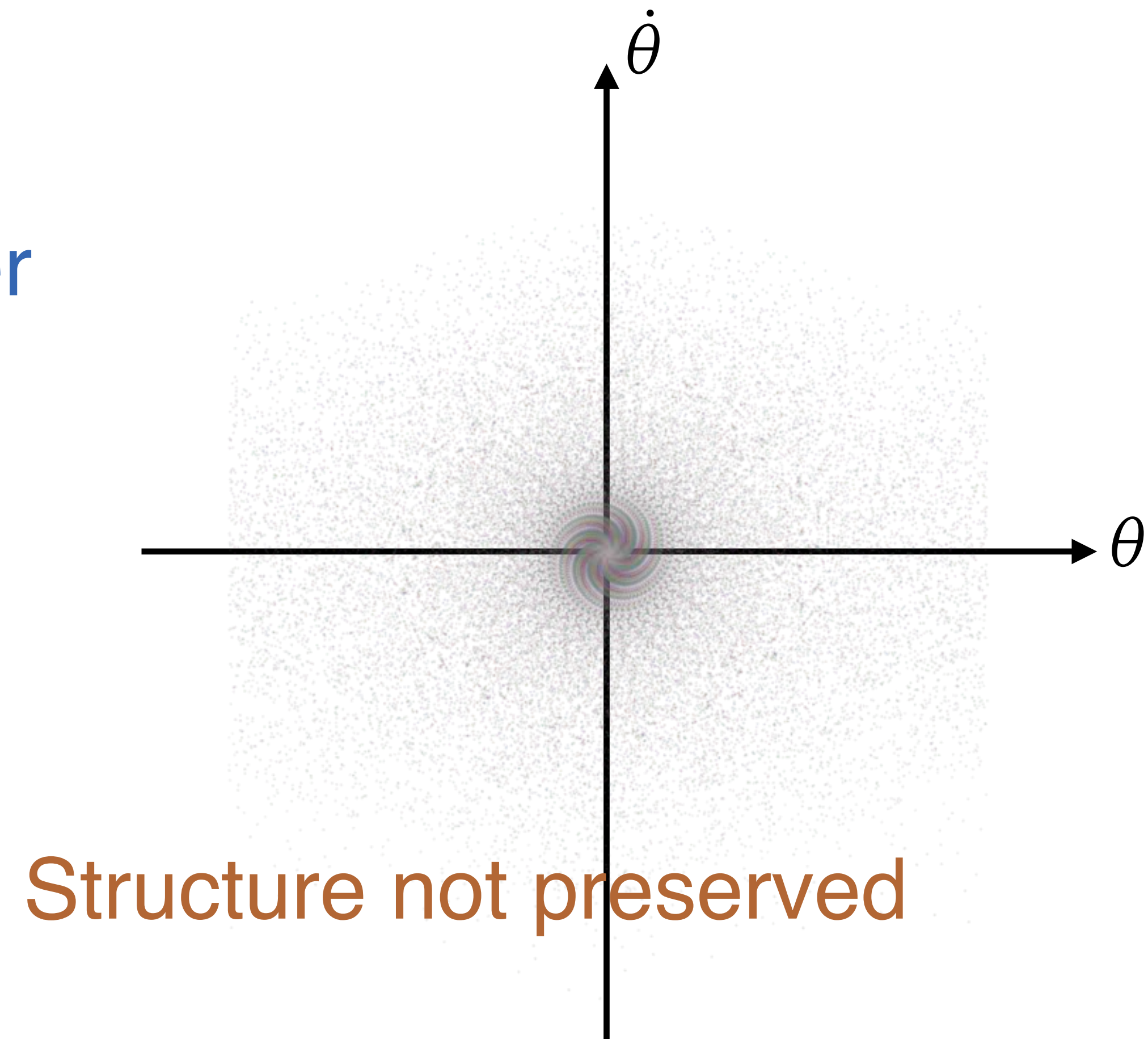
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$$\text{Output} \quad \theta_{i+1} = \theta_i + \frac{\Delta t}{6} \left(v_i + 2v_{i+1/2}^* + 2v_{i+1/2}^{**} + v_{i+1}^{***} \right)$$

$$v_{i+1} = v_i - \frac{\Delta t}{6} \left(\sin \theta_i + 2 \sin \theta_{i+1/2}^* + 2 \sin \theta_{i+1/2}^{**} + \sin \theta_{i+1}^{***} \right)$$



$$\Delta t = 0.9$$

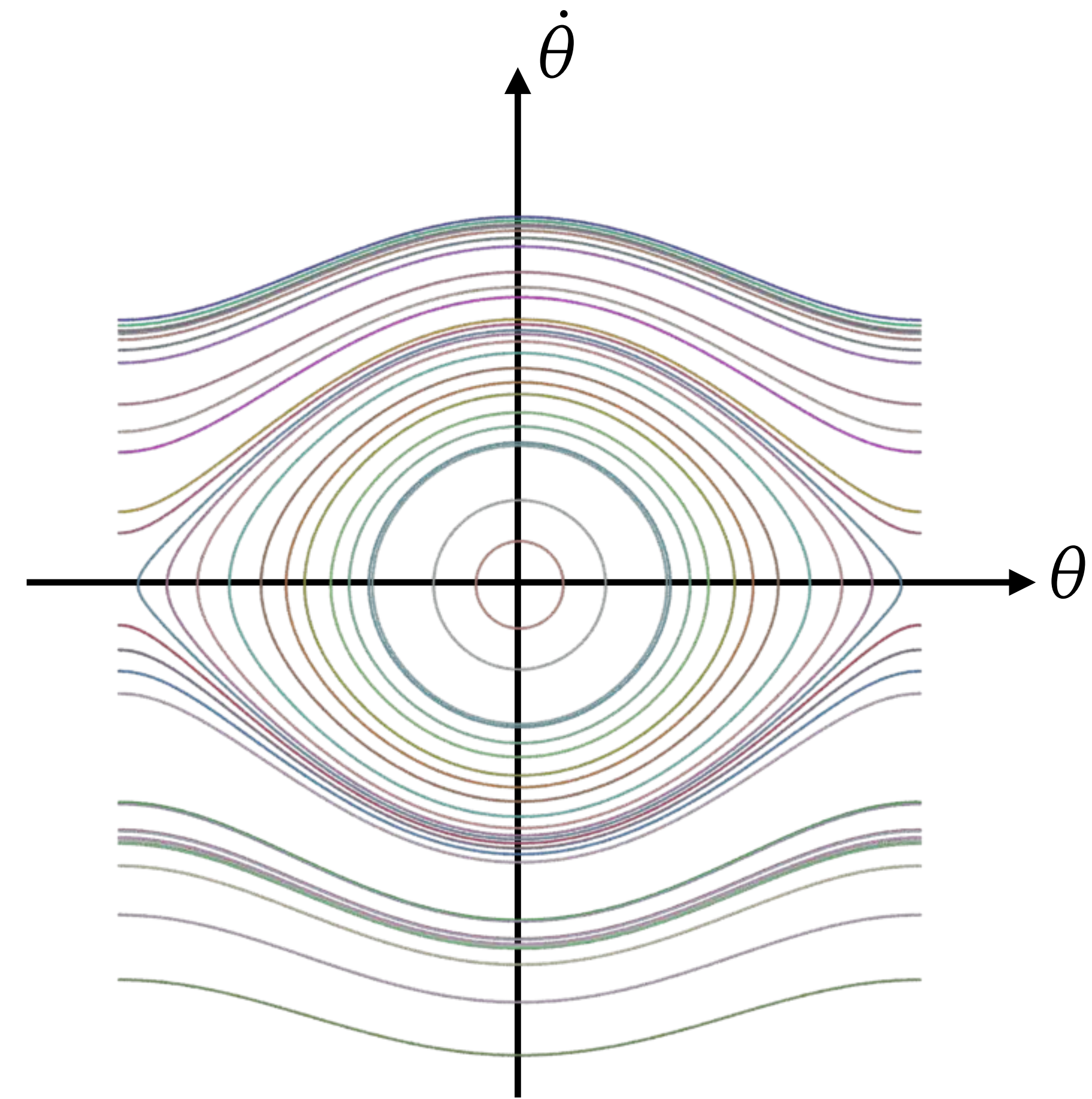
Discrete

Example: pendulum equation.

$$m\ddot{\theta} = -mg \sin \theta$$

- A 2nd order discretization

$$\frac{\theta_{i-1} - 2\theta_i + \theta_{i+1}}{\Delta t^2} = -\sin \theta_i$$



$$\Delta t = 0.1$$

Discrete

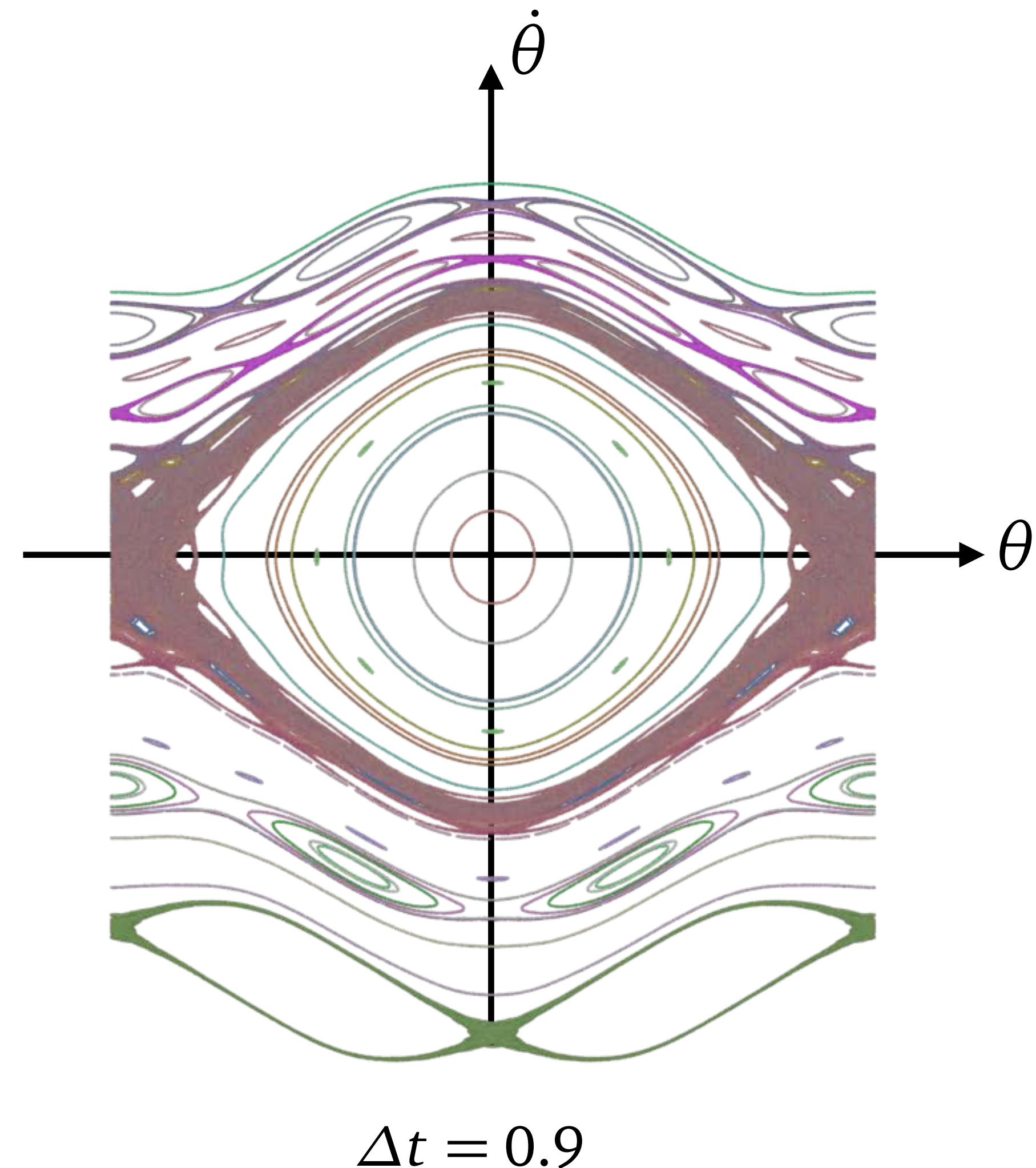
Example: pendulum equation.

$$m\ddot{\theta} = -mg \sin \theta$$

- A 2nd order discretization

$$\frac{\theta_{i-1} - 2\theta_i + \theta_{i+1}}{\Delta t^2} = -\sin \theta_i$$

✓ Energy conservation
in “asteroid belts”



Discrete

Example: pendulum equation.

$$m\ddot{\theta} = -mg \sin \theta$$

Least action principle

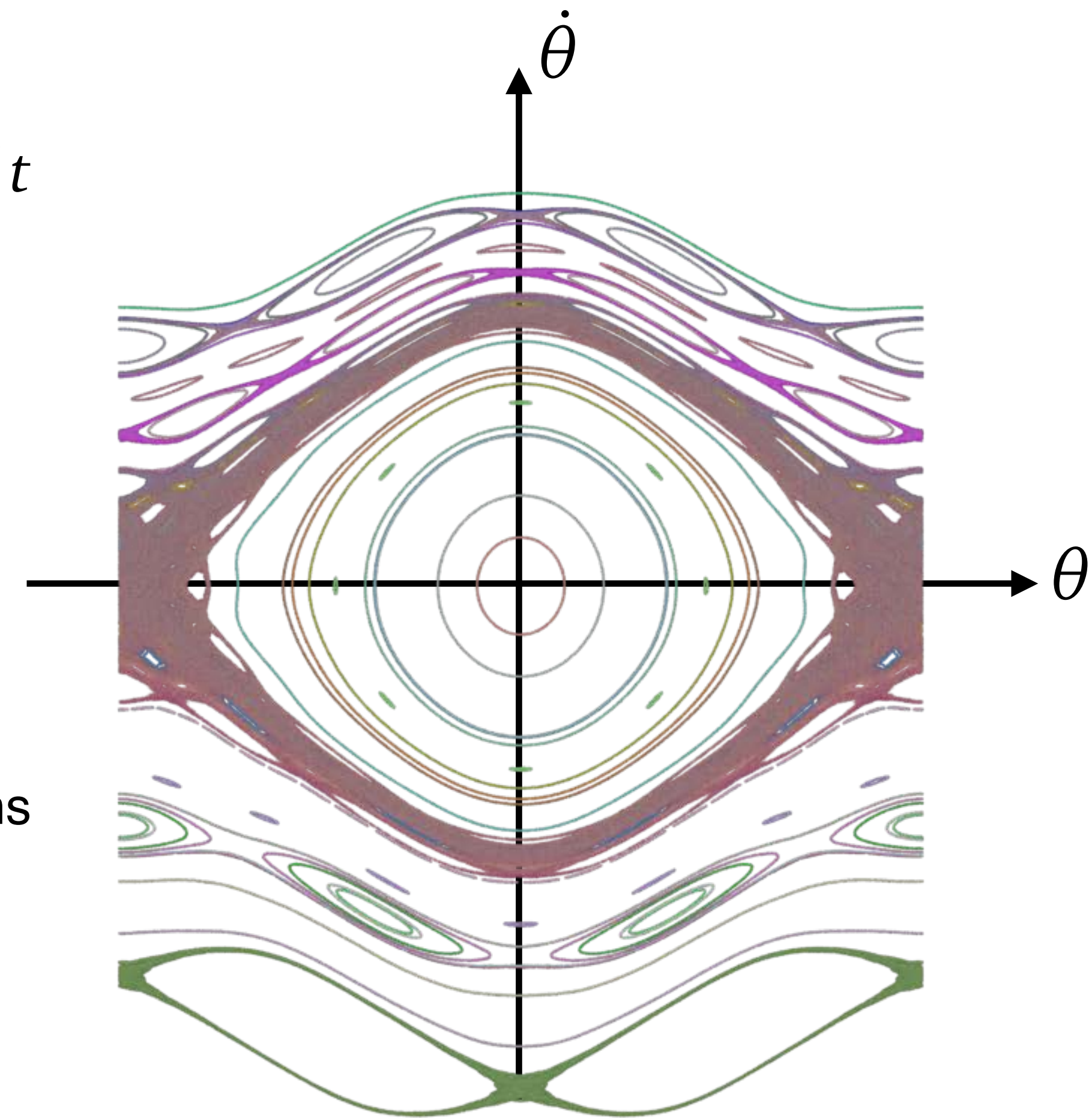
$$\int \left(\frac{m}{2} \dot{\theta}^2 + mg \cos \theta \right) dt$$

- A 2nd order discretization

$$\frac{\theta_{i-1} - 2\theta_i + \theta_{i+1}}{\Delta t^2} = -\sin \theta_i$$

First introduce discrete action,
then derive the least action paths

✓ Energy conservation
in “asteroid belts”



$\Delta t = 0.9$

Discrete

Example: pendulum equation.

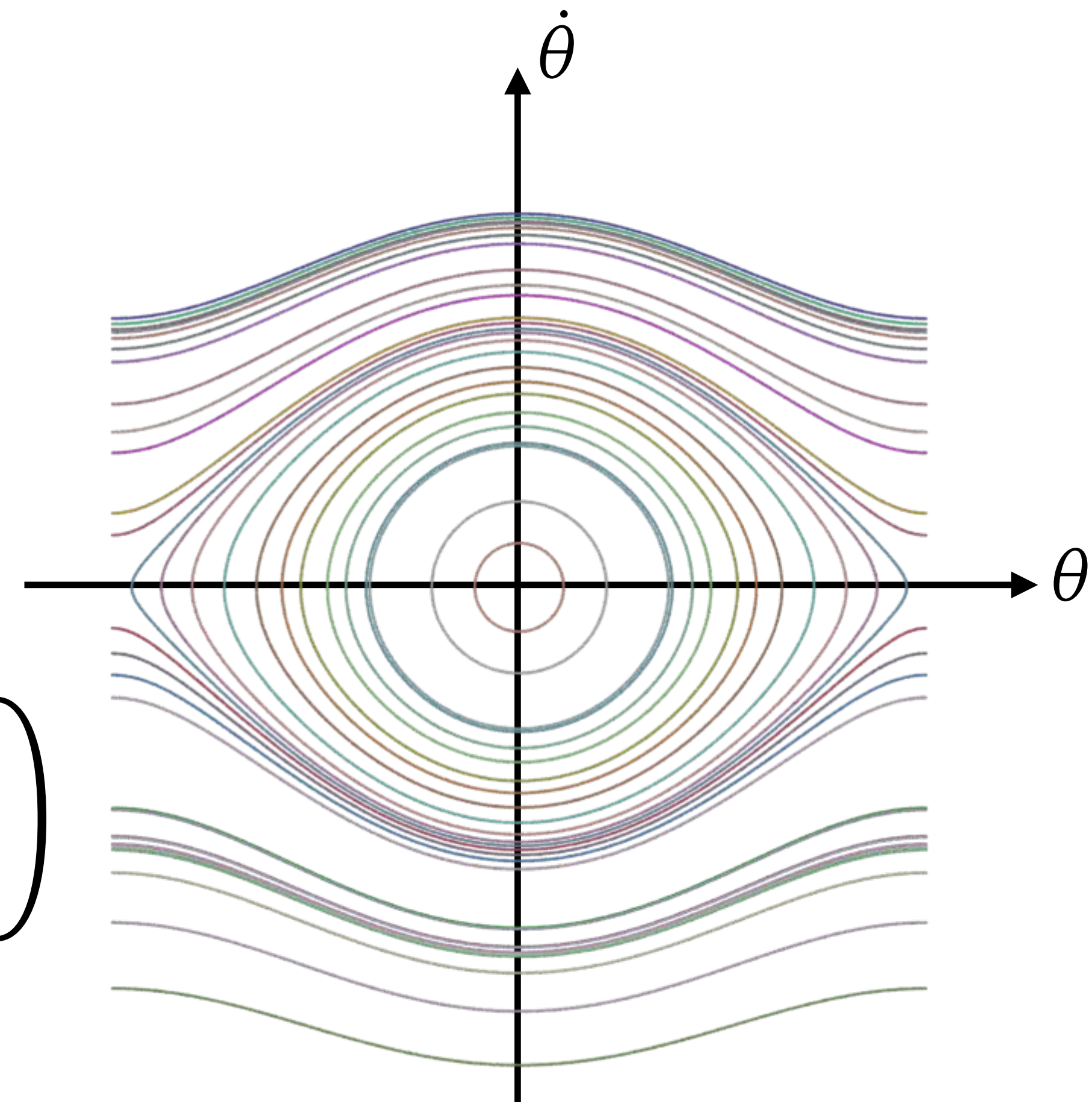
$$m\ddot{\theta} = -mg \sin \theta$$

- Another 2nd order discretization

$$\frac{\theta_{i-1} - 2\theta_i + \theta_{i+1}}{\Delta t^2} = -\sin \theta_i$$

$$\frac{\theta_{i-1} - 2\theta_i + \theta_{i+1}}{\Delta t^2} = 4 \arg \left(1 + \frac{\Delta t^2}{4} e^{-i\theta_i} \right)$$

✓ Integrable system



$\Delta t = 0.1$

Discrete

Example: pendulum equation.

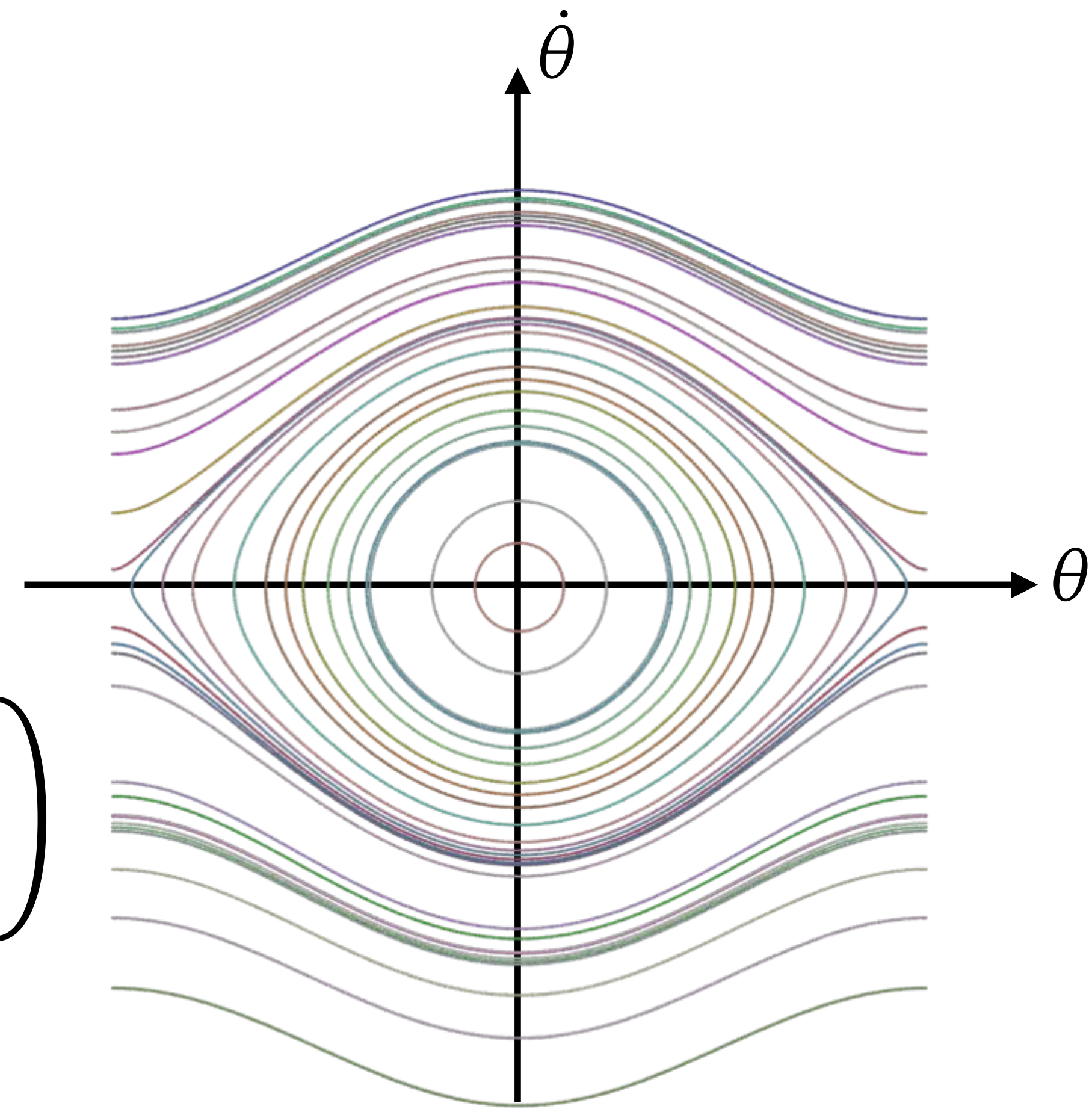
$$m\ddot{\theta} = -mg \sin \theta$$

- Another 2nd order discretization

$$\frac{\theta_{i-1} - 2\theta_i + \theta_{i+1}}{\Delta t^2} = -\sin \theta_i$$

$$\frac{\theta_{i-1} - 2\theta_i + \theta_{i+1}}{\Delta t^2} = 4 \arg \left(1 + \frac{\Delta t^2}{4} e^{-i\theta_i} \right)$$

✓ Integrable system



$\Delta t = 0.9$

Discretized v.s. Discrete

general

4th order Runge–Kutta

Variational integrator

Discrete integrable system **rare**

no-structure

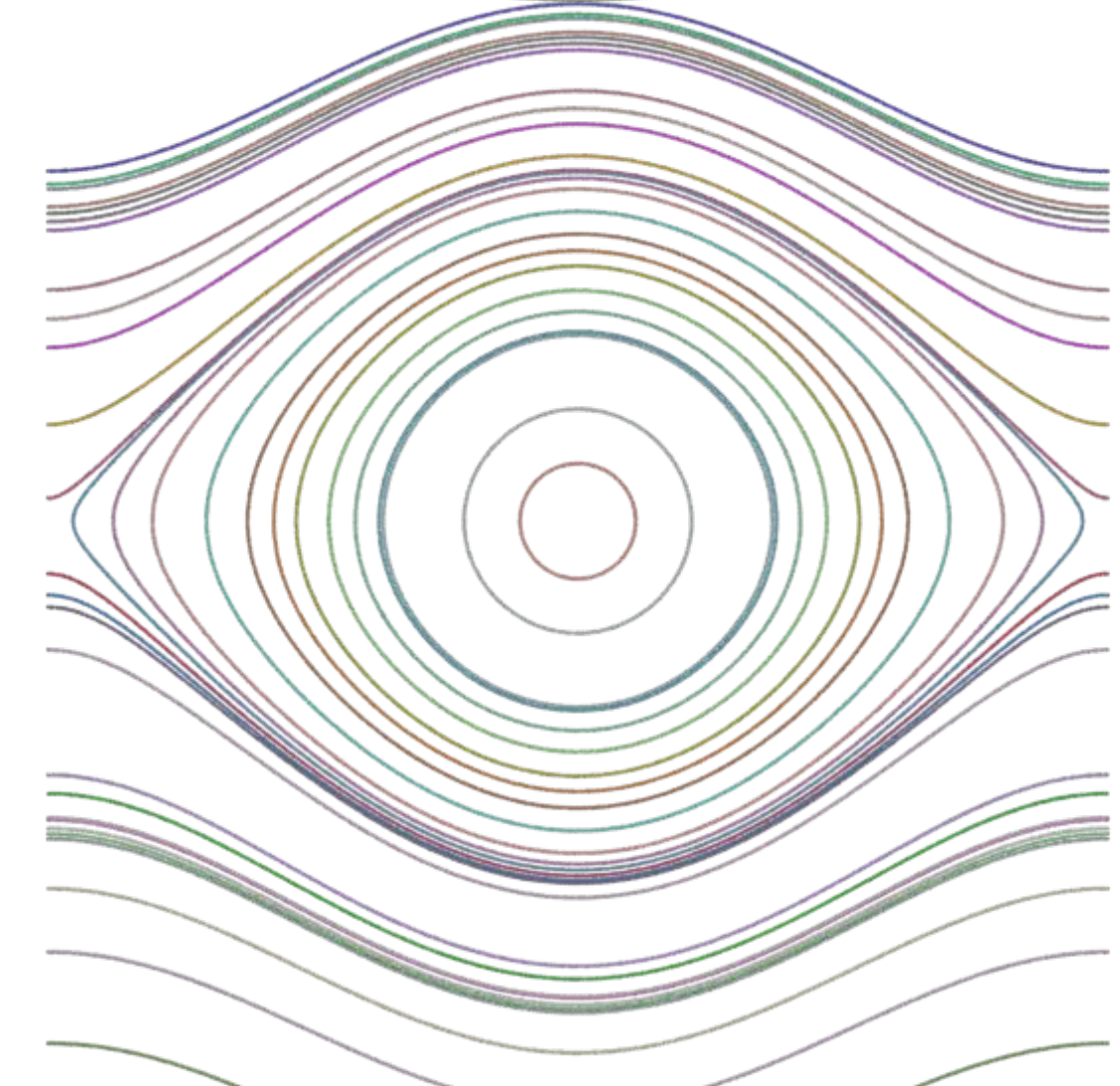
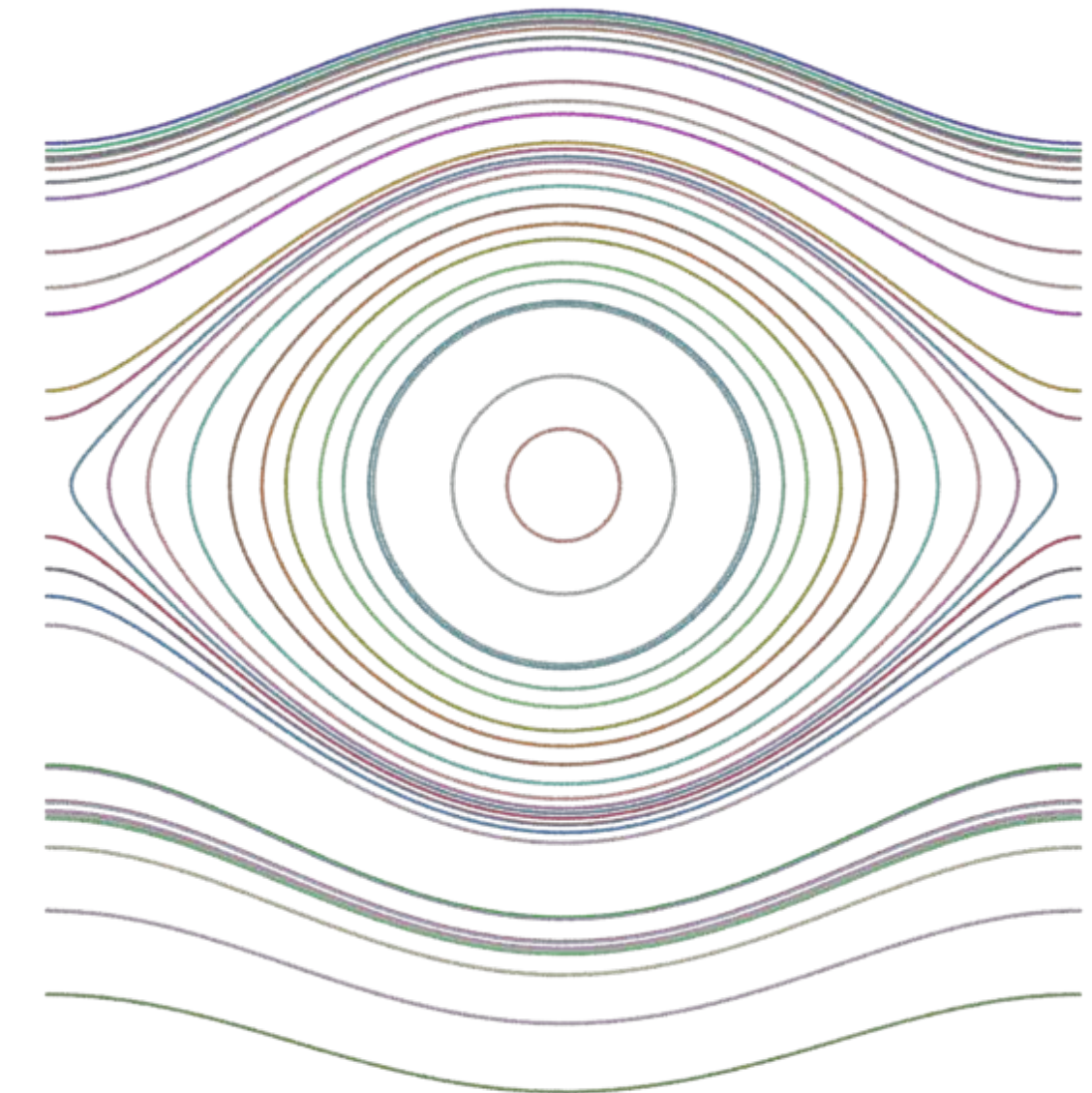
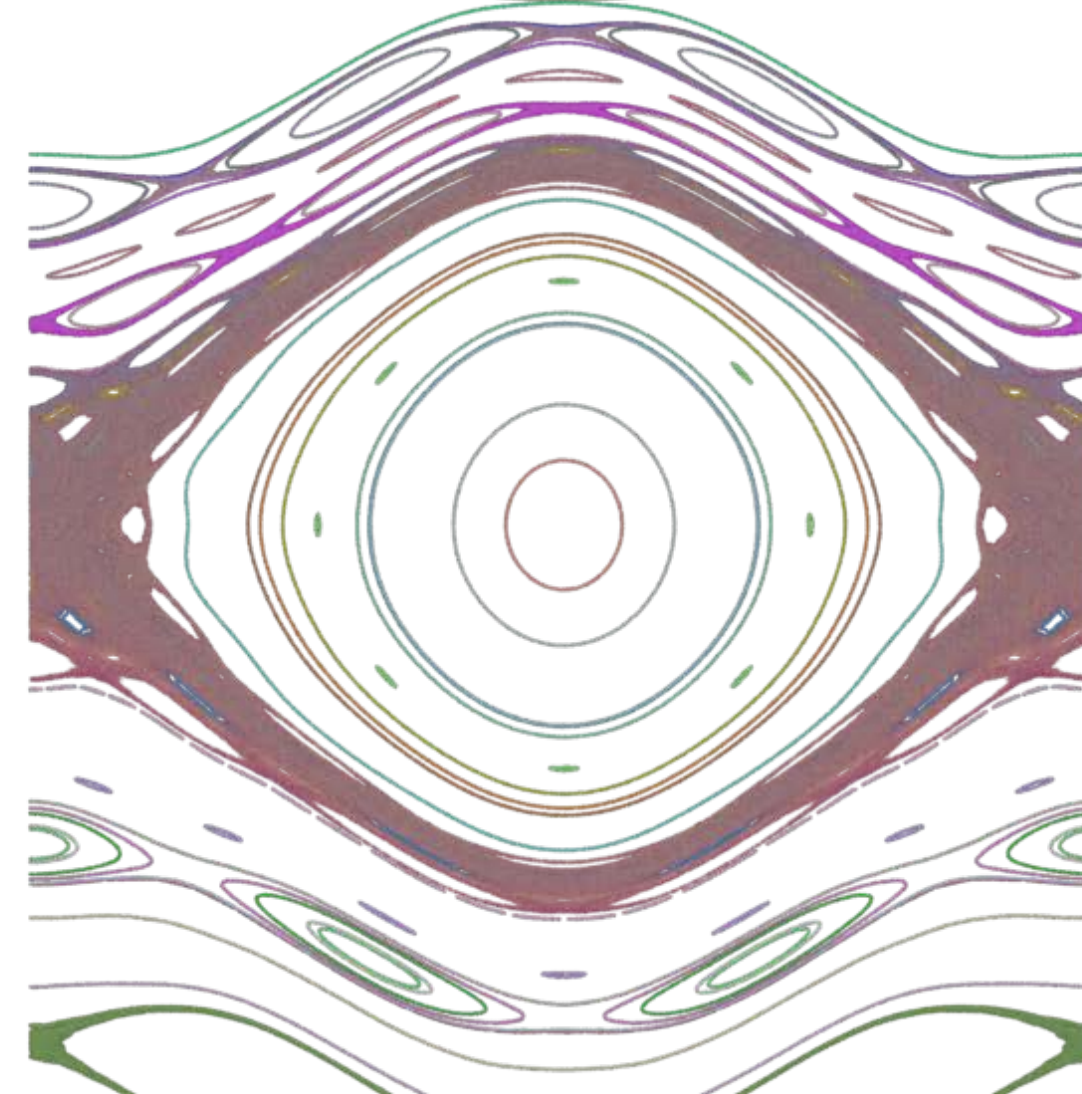
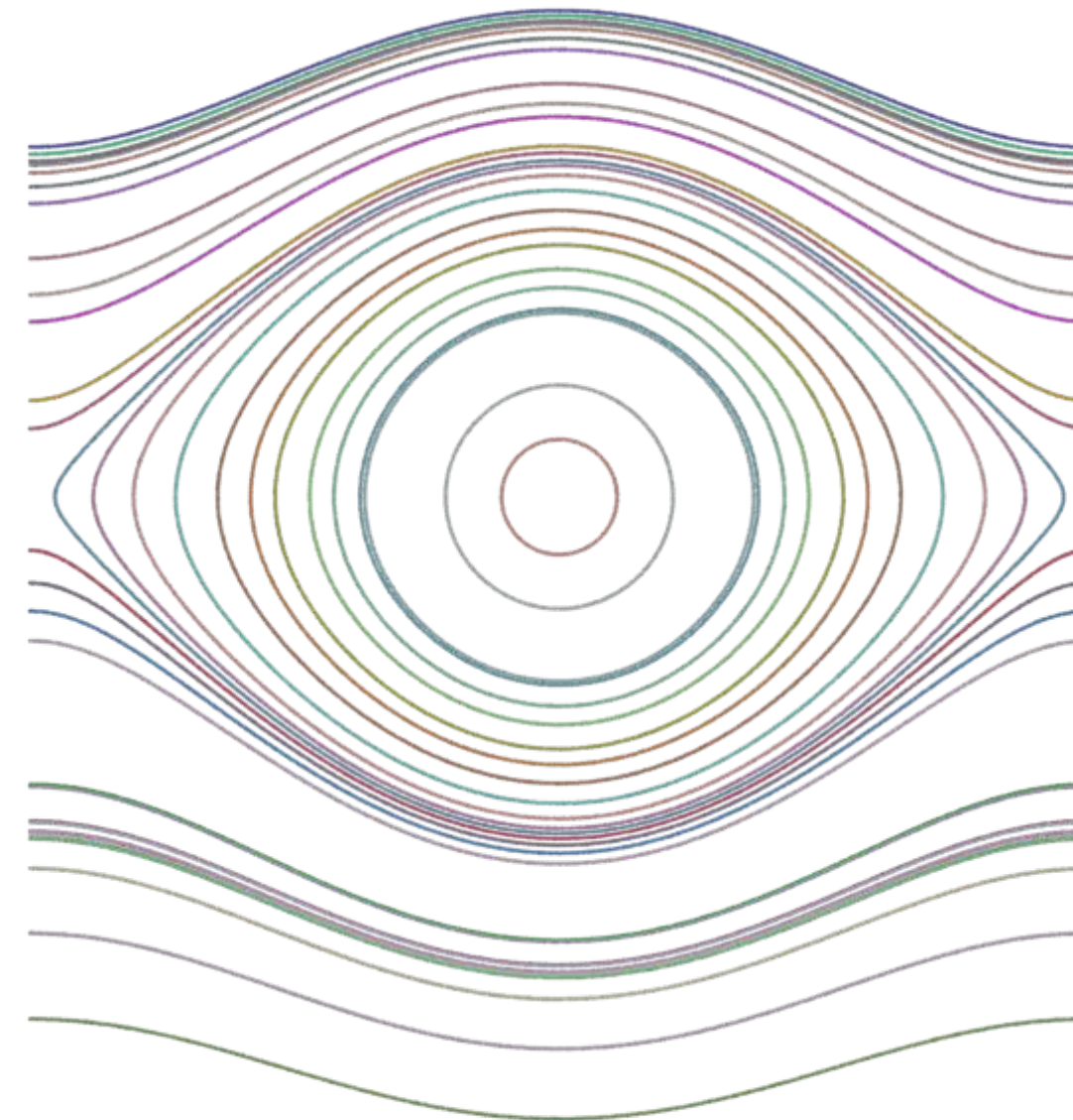
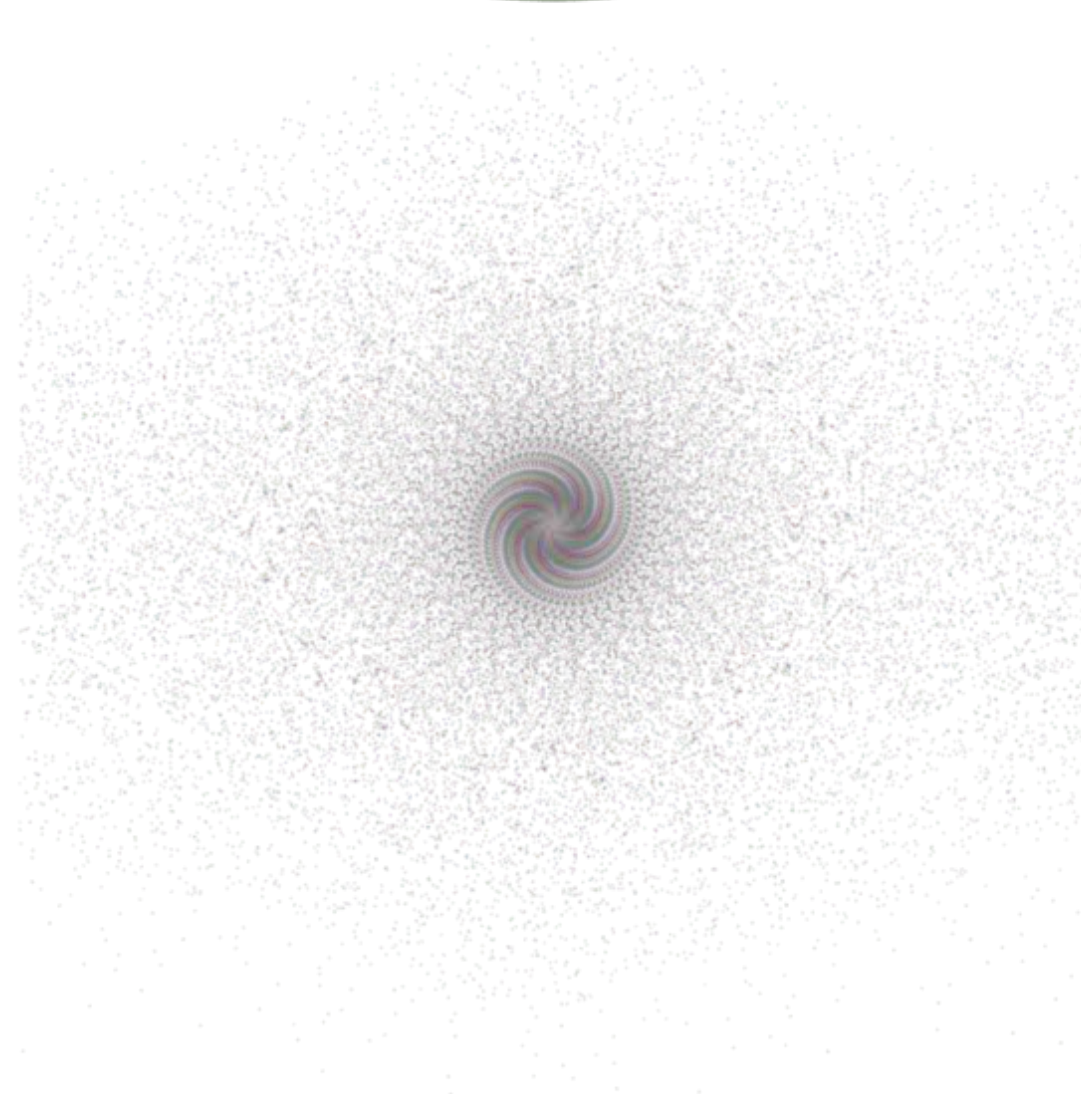
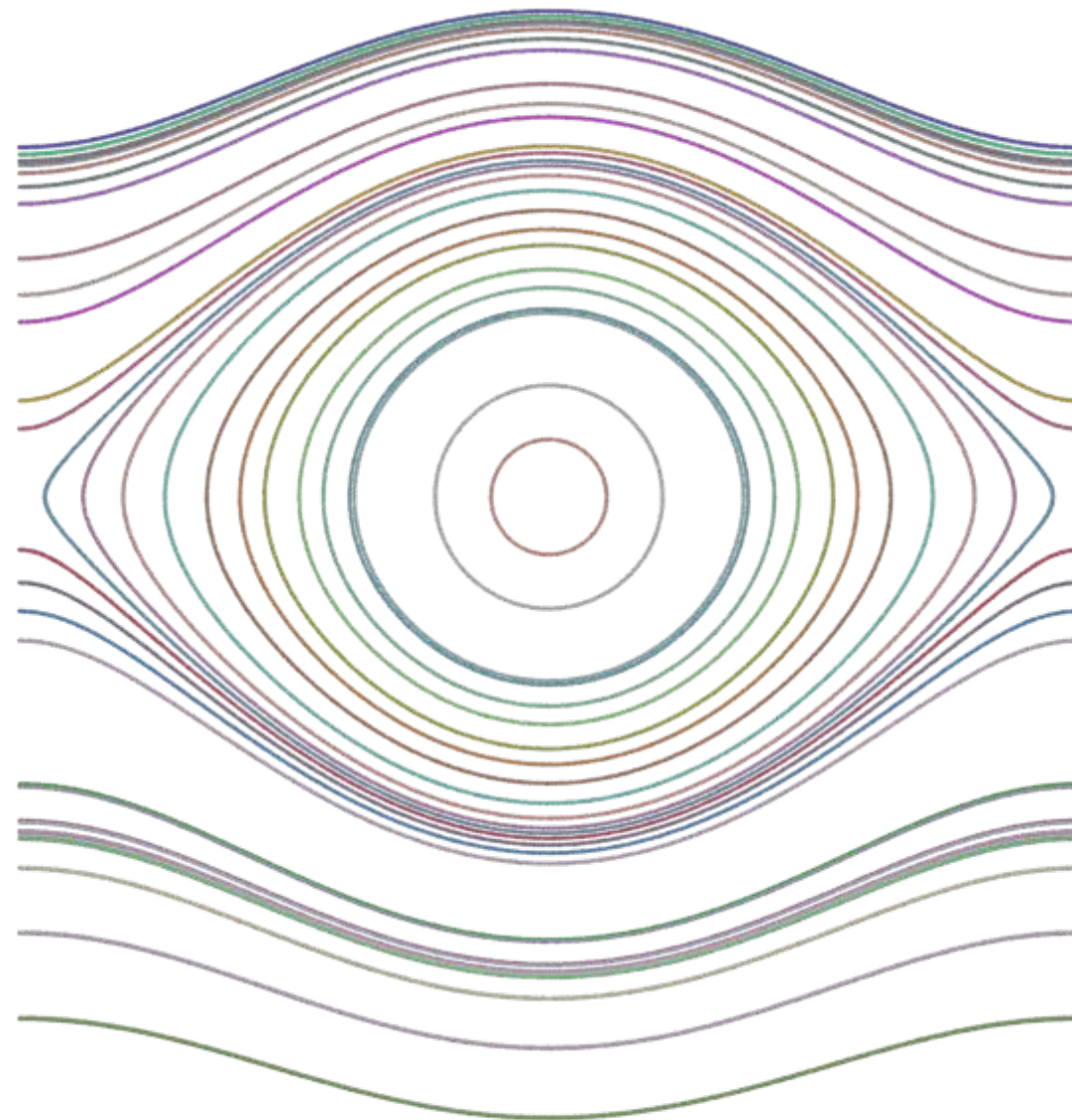
quantitative
high-precision

structure
preserving

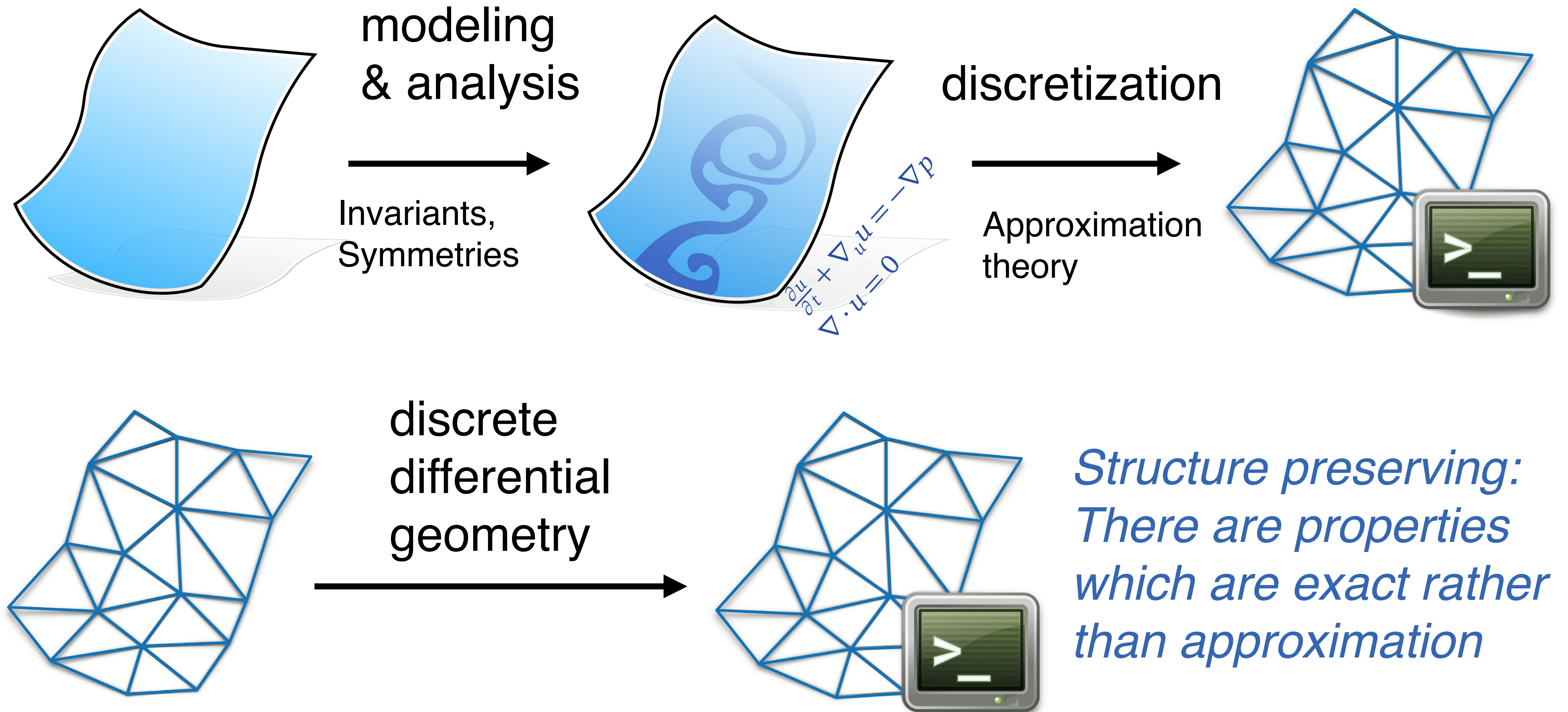
qualitative
exact

$\Delta t = 0.1$

$\Delta t = 0.9$



Discretized v.s. Discrete



Discretized v.s. Discrete



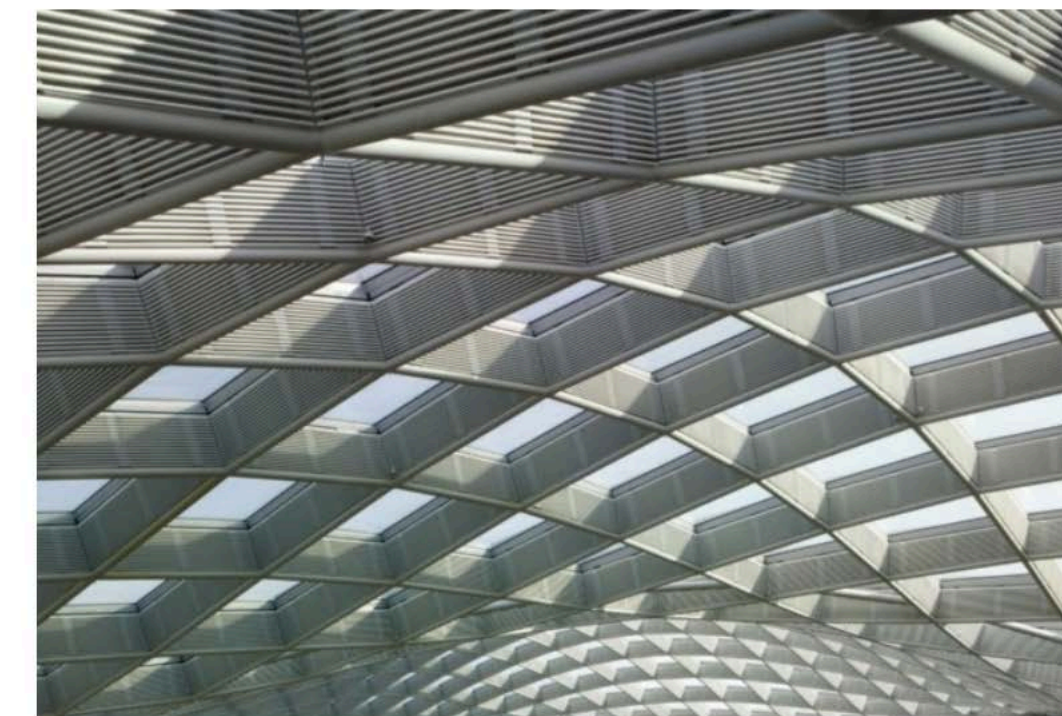
Pellis & Pottmann 2018



Wang, Pellis, Rist, Pottmann & Müller 2019



Yas Marina hotel in Abu Dhabi

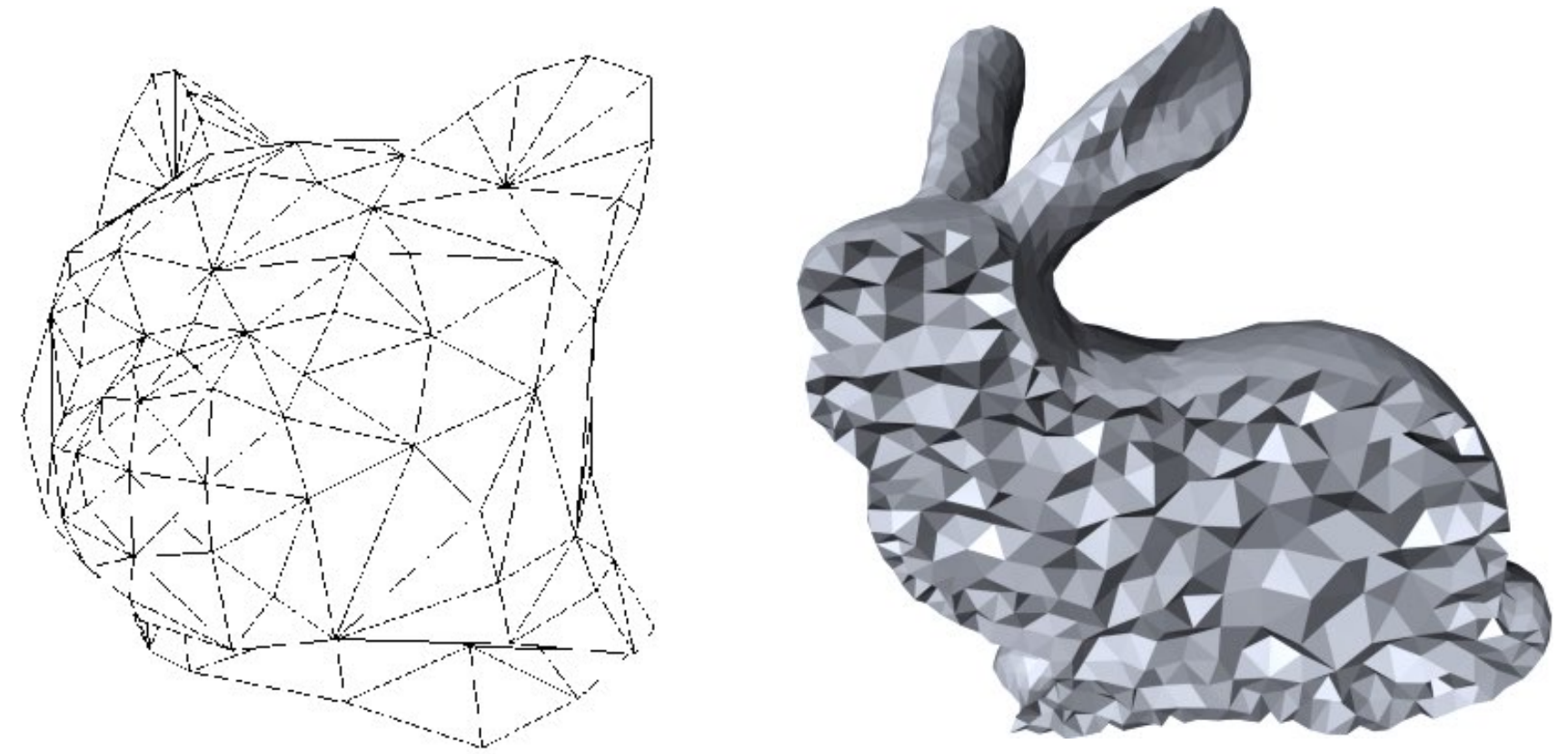


Smithsonian American Art Museum

Discrete Differential Geometry

Building from the ground up

- Discrete geometry is given
 - ▶ Triangle, tetrahedral mesh



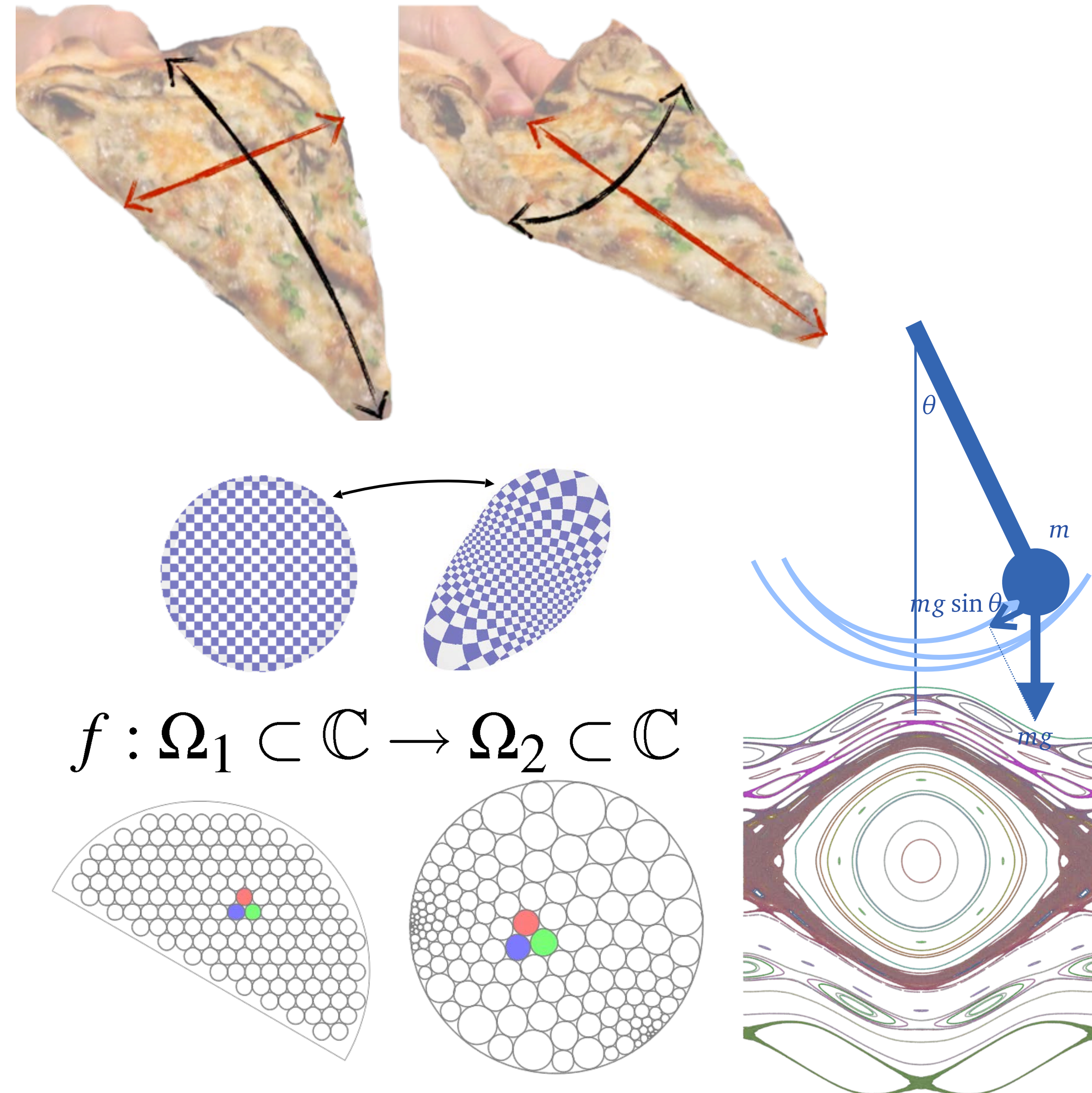
- How to do calculus?

- ▶ Preserve crucial properties, e.g. $\int_a^b f'(x) dx = f(b) - f(a)$

Discrete Differential Geometry

Building from the top down

- High level theorems
 - ▶ Gauss Theorema Egregium
 - ▶ Riemann mapping theorem
 - ▶ Least action principle

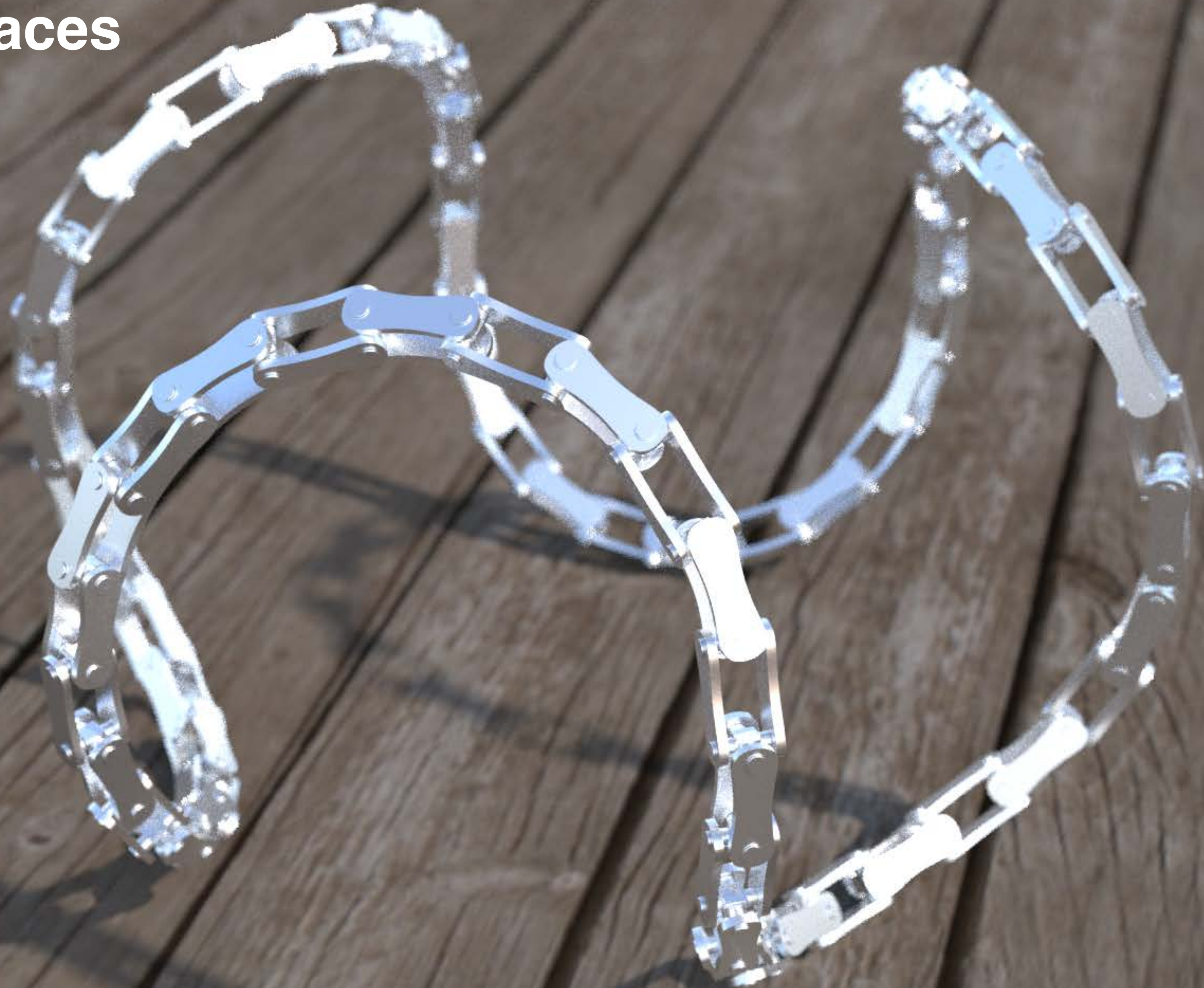


Curves and Surfaces

curvature

torsion

theorems of
Gauss



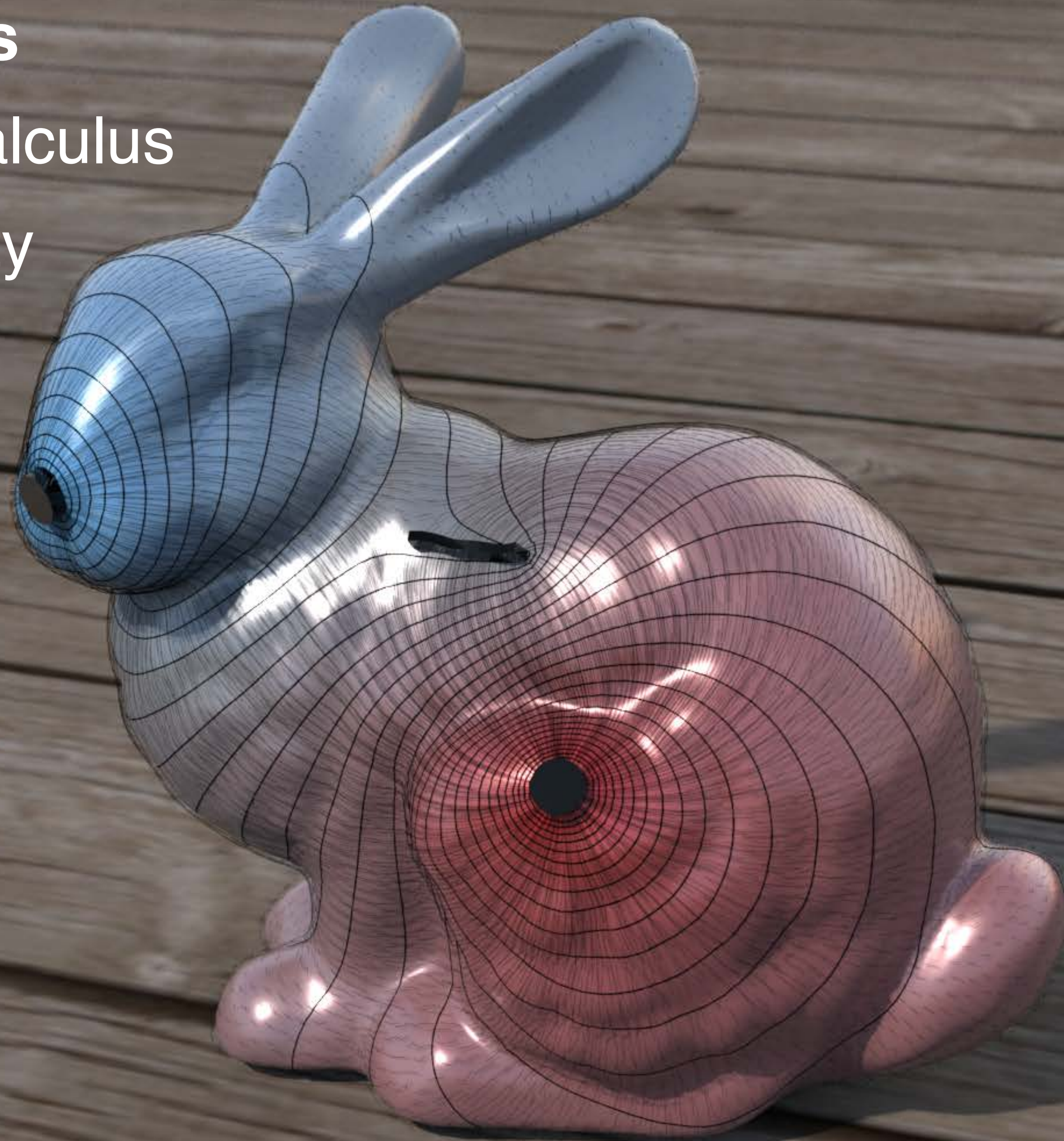
Exterior Calculus

coordinate-free calculus

scalar, flux, density

PDEs

Laplacian

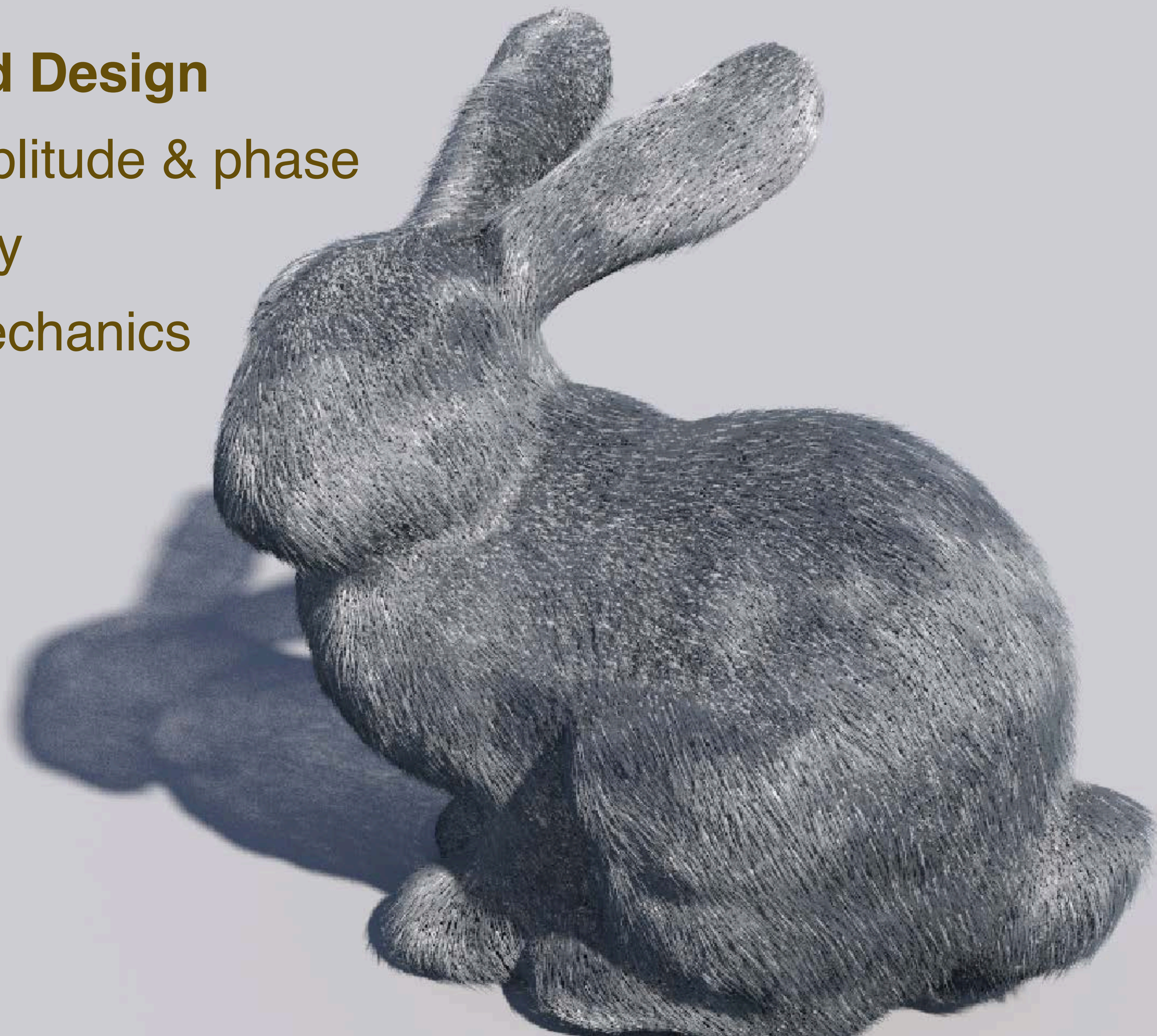


Vector Field Design

vectors: amplitude & phase

gauge theory

quantum mechanics



The End

- Course Evaluation

