A pot in Shahr-e Sukhteh, Iran, 3200 BCE
Animation

Phenakistoscope (19th century)
Disney’s Snow White and the Seven Dwarfs (1939) is the first hand-drawn featured film.
First 3D computer animation
Toy Story (1995)
Rigging and Key-frame animation

Animate through the control skeleton

Spline interpolation between keyframe
Rigging and Key-frame animation

Dinosaur input device (Jurassic Park 1993)
Motion capture (Lord of the Rings 2001–2003)
Flock simulation

Flock of birds

Flock simulation
Crowd simulation (World War Z 2013)
Physics simulation

Rigid body motions
Physics simulation

Fluid simulation (Good Dinosaur 2015)
Newton’s Law of Motion
Equations of motion

- List all the variables that are sufficient to determine the motion
  \[ x : \text{position of the object} \]
  \[ v : \text{velocity of the object} \]

- These variables describe the state. Let them be functions of time.
  \[ x(t) \quad v(t) \]

- Newton’s Law of motion
  \[ \frac{dx(t)}{dt} = \dot{x}(t) = v(t) \]
  \[ m \frac{dv(t)}{dt} = m\dot{v}(t) = f(x(t), v(t)) \]
Equations of motion

- Newton’s Law of motion
  \[ \frac{dx(t)}{dt} = \dot{x}(t) = v(t) \]
  \[ m \frac{dv(t)}{dt} = m \ddot{v}(t) = f(x(t), v(t)) \]
  more famously known as
  \[ f = ma \]

- These equations are called **Ordinary Differential Equations** (ODE)

- Any ODE involving high order derivatives (high order ODEs) can be rewritten as a 1st order ODE.
To complete the Newton’s equation of motion, we need to describe the force as a function of the state.

\[ f(x, v) = -k(x - x_0) - \mu v \]
To complete the Newton’s equation of motion, we need to describe the force as a function of the state.

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\(x_0\) is the spring stiffness.
To complete the Newton’s equation of motion, we need to describe the force as a function of the state.

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\[ f(x, v) = -k(x - x_0) - \mu v \]

\[
\begin{align*}
\dot{x} &= v \\
\dot{v} &= \frac{1}{m} \left(-k(x - x_0) - \mu v\right)
\end{align*}
\]
Multiple objects

- N-body problem

\[
\begin{align*}
\dot{x}_1 &= v_1 \\
\vdots & \\
\dot{x}_N &= v_N \\
\dot{v}_1 &= f_1(\mathbf{x}_1, \ldots, \mathbf{x}_N, v_1, \ldots, v_N) \\
\vdots & \\
\dot{v}_N &= f_N(\mathbf{x}_1, \ldots, \mathbf{x}_N, v_1, \ldots, v_N)
\end{align*}
\]
Multiple objects

• N-body problem

\[
\begin{align*}
\dot{x}_1 &= v_1 \\
\vdots \\
\dot{x}_N &= v_N \\
\dot{v}_1 &= f_1(x_1, \ldots, x_N, v_1, \ldots, v_N) \\
\vdots \\
\dot{v}_N &= f_N(x_1, \ldots, x_N, v_1, \ldots, v_N)
\end{align*}
\]
Numerical Integration
Numerical ODE

- Generic ODE \( \dot{y} = f(y) \)
- Discretize time into time-frames \( y^{(n)} = y(n \Delta t) \)
- Euler method \( \frac{y^{(n+1)} - y^{(n)}}{\Delta t} \approx f(y^{(n)}) \)

\[ y^{(n+1)} \approx y^{(n)} + \Delta t \cdot f(y^{(n)}) \]
Numerical ODE

- Euler method
  \[ y^{(n+1)} \approx y^{(n)} + \Delta t \cdot f(y^{(n)}) \]

- Runge–Kutta method (RK4)
  \[ k_1 = f(y^{(n)}) \]
  \[ k_2 = f(y^{(n)} + \frac{\Delta t}{2} k_1) \]
  \[ k_3 = f(y^{(n)} + \frac{\Delta t}{2} k_2) \]
  \[ k_4 = f(y^{(n)} + \Delta t k_3) \]
  \[ y^{(n+1)} = y^{(n)} + \frac{\Delta t}{6} (k_1 + 2k_2 + 2k_3 + k_4) \]
Equations of Motion from Energy
Problem with $F=ma$

- Newton’s formula $F=ma$ is useful only for very simple systems.
- Arguing what are the total force and torque on a moderately simple mechanical system can be very complicated.
Suppose the force is conservative \[ f(x) = -\frac{\partial}{\partial x} U(x) \]

The function \( U(x) \) is called the potential energy.

The kinetic energy is given by \( T(v) = \frac{1}{2} m |v|^2 \).

Newton’s equation \( f = ma \) can be reformulated as

\[
\frac{d}{dt} \left( \frac{\partial}{\partial v} T(x, v) \right) + \frac{\partial}{\partial x} U(x) = 0
\]

It is much easier to describe the energy than the force.
\[
\frac{d}{dt} \left( \frac{\partial}{\partial \mathbf{v}} T(\mathbf{x}, \mathbf{v}) \right) + \frac{\partial}{\partial \mathbf{x}} U(\mathbf{x}) = 0
\]

- **Example: compound pendulum**
  - State variables: \( \theta, \dot{\theta} \)
  - Kinetic energy
    \[
    T = \int_0^\ell \frac{1}{2} \frac{m}{\ell} (r \dot{\theta})^2 \, dr = \frac{1}{6} \frac{m}{\ell} \ell^3 \dot{\theta}^2 = \frac{1}{6} m \ell^2 \dot{\theta}^2
    \]
  - Potential energy
    \[
    U = \int_0^\ell -\frac{m}{\ell} g r \cos \theta \, dr = -\frac{1}{2} mg \ell \cos \theta
    \]
  - Derivatives:
    \[
    \frac{\partial T}{\partial \dot{\theta}} = \frac{1}{3} m \ell^2 \dot{\theta}, \quad \frac{\partial U}{\partial \theta} = \frac{1}{2} mg \ell \sin \theta
    \]
  - Equation of motion
    \[
    \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}} \right) + \frac{\partial U}{\partial \theta} = 0
    \]
    \[
    \implies \quad \ddot{\theta} = -\frac{3}{2} \frac{g}{\ell} \sin \theta
    \]
\[
\frac{d}{dt} \left( \frac{\partial}{\partial v} T(x, v) \right) + \frac{\partial}{\partial x} U(x) = 0
\]

- **Example: double compound pendulum**
  - State variables: \( \theta, \phi, \dot{\theta}, \dot{\phi} \)
  - Kinetic energy
    \[
    T = \int_0^\ell \frac{1}{2} \frac{m}{\ell} (r \dot{\theta})^2 + \int_0^\ell \frac{1}{2} \frac{m}{\ell} \left( (\ell \dot{\theta} + r \cos(\phi - \theta) \dot{\phi})^2 + (r \sin(\phi - \theta) \dot{\phi})^2 \right) dr
    \]
  - Potential energy
    \[
    U = -mg \frac{\ell}{2} \cos \theta - mg \left( \ell \cos \theta + \frac{\ell}{2} \cos \phi \right)
    \]
  - Equation of motion
    \[
    \begin{align*}
    \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}} \right) + \frac{\partial U}{\partial \theta} &= 0 \\
    \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\phi}} \right) + \frac{\partial U}{\partial \phi} &= 0
    \end{align*}
    \]
Rigid Body Motion
Rigid body motion

- The motion’s state variable is just the translation and rotation (and their time-derivatives)
  - Rotation matrix $R$ and the translation vector $c$
  - Denote the derivative of the translation
    \[ \dot{c} = v \]
  - The rotation matrix satisfies $R^T R = R R^T = \text{id}$
  - This implies that $\dot{R}$ always take the form
    \[ \dot{R} = W R = R A \]
    for some skew-symmetric matrices $W, A$.
  - Skew-symmetric matrices are cross product with vectors
    \[ \dot{R} = [\omega \times]R = R[\Omega \times] \]
Rigid body motion

• The motion’s state variable is just the translation and rotation (and their time-derivatives) $c, R, v, \omega$
  
  ▶ Derive the kinetic energy as a quadratic function of $v, \omega$
  
  ▶ Use the Lagrange’s principle to derive the equations of motion.

$$
\begin{aligned}
\dot{c} &= v \\
\dot{R} &= [\omega \times ]R \\
\frac{d}{dt} \left( \frac{\partial T}{\partial v} \right) &= 0 \\
\frac{d}{dt} \left( \frac{\partial T}{\partial \omega} \right) &= 0
\end{aligned}
$$

▶ If we take $c$ to be the center of mass (CM), then the 3rd and 4th equations decouple.

▶ We can focus on the rotation part.
Rigid body motion

- **State variables**: rotation $R$, angular velocity vector $\omega$
  - Angular velocities are defined so that $\dot{R} = [\omega \times] R = R[\Omega \times]$
  - World and body coordinate: $\omega \in \mathbb{R}^3_{\text{World}}$, $\Omega \in \mathbb{R}^3_{\text{Body}}$, $\omega = R\Omega$

- **Kinetic energy is a quadratic function**
  - $T = \frac{1}{2} \Omega^T I_{\text{Body}} \Omega = \frac{1}{2} \omega^T I_{\text{World}} \omega$
  - The 3x3 symmetric positive definite matrix $I$ is called the moment of inertia.
  - For rigid body, $I_{\text{body}}$ is time-independent;
    $I_{\text{World}}(t) = R(t)I_{\text{Body}}R(t)^T$ co-rotates.

- **Derivative** $\frac{\partial T}{\partial \omega} = I_{\text{World}} \omega$
Rigid body motion

• Derivative \( \frac{\partial T}{\partial \omega} = I_{\text{World}} \omega \)

• Equation of motion \( \frac{d}{dt} \left( \frac{\partial T}{\partial \omega} \right) = 0 \)

\[ \Rightarrow \frac{d}{dt} \left( I_{\text{World}}(t) \omega(t) \right) = 0 \]

\[ \Rightarrow \frac{d}{dt} \left( R(t) I_{\text{Body}} R(t)^T \omega(t) \right) = 0 \]

\[ \Rightarrow \dot{R} I_{\text{Body}} R^T \omega + I_{\text{Body}} \dot{R}^T \omega + R I_{\text{Body}} R^T \dot{\omega} = 0 \]

\[ \Rightarrow [\omega \times] R I_{\text{Body}} R^T \omega + I_{\text{Body}} R^T [\omega \times]^T \omega + R I_{\text{Body}} R^T \dot{\omega} = 0 \]

\[ \Rightarrow \omega \times (I_{\text{World}} \omega) + I_{\text{World}} \dot{\omega} = 0 \]

Euler equation of rigid body motion
Poinset’s ellipsoid (see whiteboard)

- blue vector: angular velocity $\omega$
- red vector: angular velocity $L$
- cyan trajectory of the angular velocity on the rolling ellipsoid: polhode
- green trajectory of the angular velocity in the world: herpolhode

https://vimeo.com/286156112
Tennis Racket Theorem

Dzhanibekov effect
Next time

- The Lagrangian mechanics formula works for fluid dynamics
- Continuum mechanics and geometric approaches