Closest Point Exterior Calculus

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Problem

Our research aims to develop a novel approach that combines the strengths of **Discrete Exterior Calculus** and the **Closest Point Method**, mitigating the issues caused by poor mesh quality, and offering a more **accurate** and **robust solution** for approximating PDEs on surfaces.

Background

DISCRETE EXTERIOR CALCULUS (DEC)

DEC works by creating a scheme to pose differential operators on a mesh. This means DEC is ultimately dependent on the mesh quality [3].

HIGH QUALITY MESH	LOW QUALITY MESH	NON-ORIENTABLE SURFACE

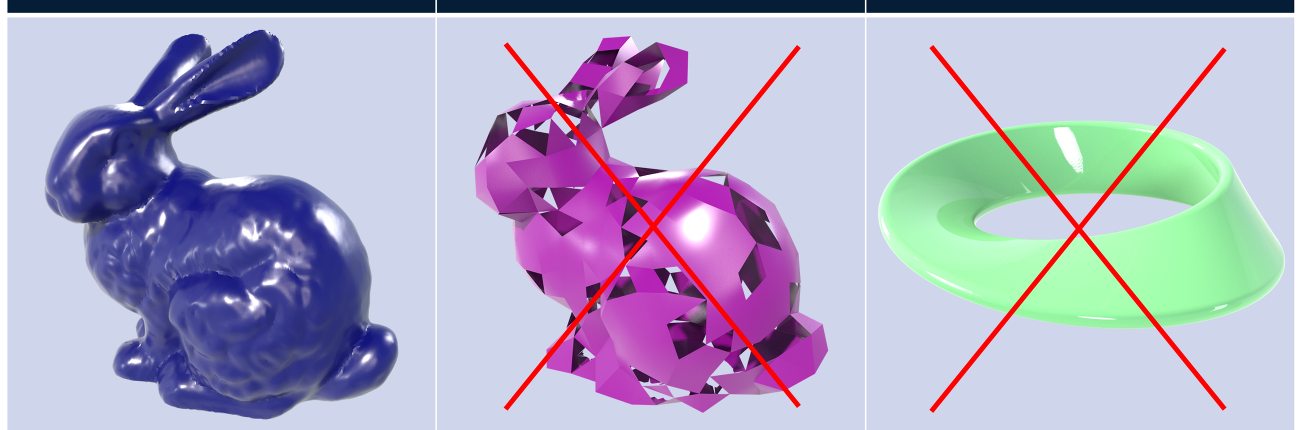

Figure 1: DEC works on high-quality meshes, but fails on low-quality meshes or complex surfaces

CLOSEST POINT METHOD (CPM)

CPM captures the shape of a surface and is **less dependent on mesh quality**. [4,2]

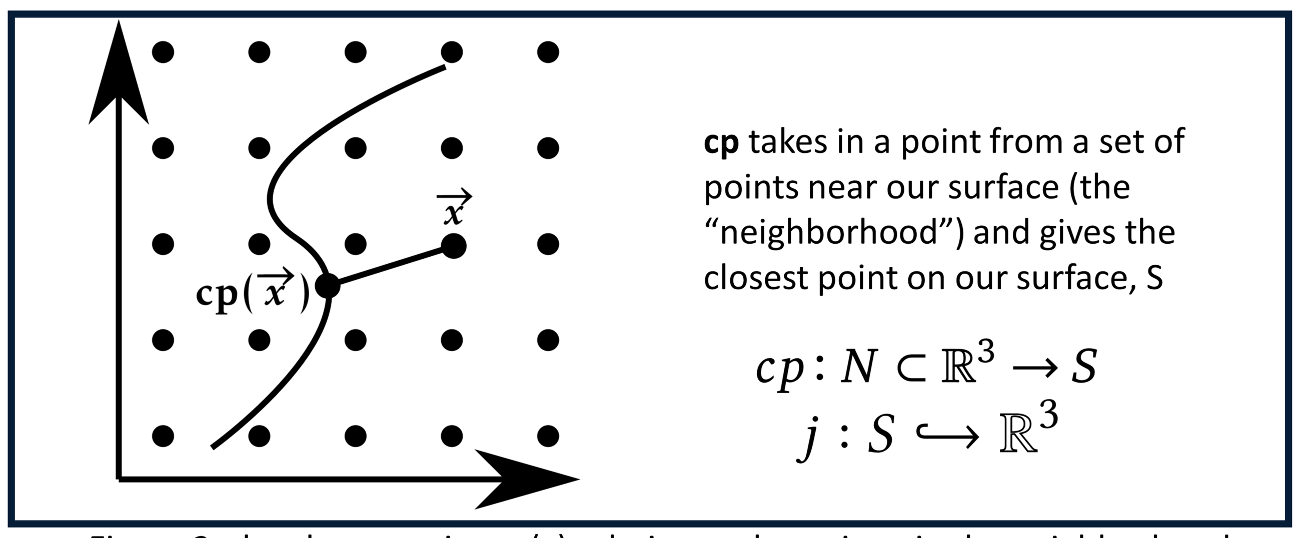

Figure 2: the closest point cp(x) relative to the point x in the neighborhood

One limitation is that there is no unified framework for solving arbitrary PDEs. For every new PDE, **a new solver must be written!**

References

[1] Hang Yin, Mohammad Sina Nabizadeh, Baichuan Wu, Stephanie Wang, and Albert Chern.
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[2] Nathan King, Haozhe Su, Mridul Aanjaneya, Steven Ruuth, Christopher Batty.
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[3] Anil Hirani. 2003. Discrete Exterior Calculus. CalTech PhD Thesis.
https://thesis.library.caltech.edu/1885/3/thesis_hirani.pdf

[4] Colin B. Macdonald, Jeremy Brandman, Steven J. Ruuth. 2011. Solving eigenvalue problems on curved surfaces using the Closest Point Method. Journal of Computational Physics,Volume 230, Issue 22,Pages 7944-7956,
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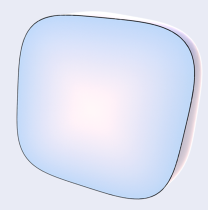
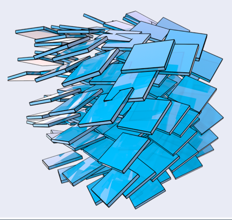
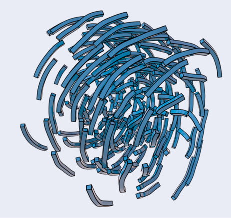
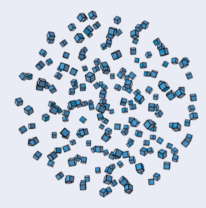
Results

DIFFERENTIAL FORMS IN $\mathbb{R}^3$

Name	0-form	1-form	2-form	3-form
Symbol	$f_{(0)} = f$	$\mathbf{u}_{(1)} [\![\mathbf{v}_{\text{vec}}]\!] = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$	$\mathbf{u}_{(2)} [\![\mathbf{v}_{\text{vec}}, \mathbf{w}_{\text{vec}}]\!] = \det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$	$f_{(3)} [\![\mathbf{u}_{\text{vec}}, \mathbf{v}_{\text{vec}}, \mathbf{w}_{\text{vec}}]\!] = \mathrm{f} \det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$
Meaning	Scalar field	Field of linear functions on vectors	Skew-symmetric bilinear form of vectors	Scalar times the volume form
Visual				 [1]

EXTERIOR CALCULUS OPERATORS IN $\mathbb{R}^3$

Output Degree	Wedge Product ($\wedge$)	Interior Product (i)	Exterior Derivative (d)	Hodge Star ($\star$)
0-form	$f_{(0)} \wedge g_{(0)} = (fg)_{(0)}$	$i_{\vec{u}} \mathbf{w}_{(1)} = (\mathbf{u} \cdot \mathbf{w})_{(0)}$	N/A	$\star f_{(3)} = f_{(0)}$
1-form	$f_{(0)} \wedge \mathbf{u}_{(1)} = f \mathbf{u}_{(1)}$	$i_{\vec{u}} \mathbf{w}_{(2)} = (\mathbf{w} \times \mathbf{u})_{(1)}$	$df_{(0)} = (\nabla f)_{(1)}$	$\star \mathbf{u}_{(2)} = \mathbf{u}_{(1)}$
2-form	$\mathbf{u}_{(1)} \wedge \mathbf{v}_{(1)} = (\mathbf{u} \times \mathbf{v})_{(2)}$	$i_{\vec{u}} f_{(3)} = f \mathbf{u}_{(2)}$	$d\mathbf{u}_{(1)} = (\nabla \times \mathbf{u})_{(2)}$	$\star \mathbf{u}_{(1)} = \mathbf{u}_{(2)}$
3-form	$\mathbf{u}_{(1)} \wedge \mathbf{v}_{(2)} = (\mathbf{u} \cdot \mathbf{v})_{(3)}$	N/A	$d\mathbf{u}_{(2)} = (\nabla \cdot \mathbf{u})_{(3)}$	$\star f_{(0)} = f_{(3)}$

We express the standard exterior calculus in $\mathbb{R}^3$ as above. To do exterior calculus on curved surfaces, utilize the closest point pullback given as follows:
(Note: $\mathbf{F} = d(j \circ cp)$, the Jacobian of cp)

CP PULLBACK IN $\mathbb{R}^3$

0- form	1-form	2-form	3-form
$cp^* j^* f_{(0)} = (f \circ cp \circ j)_{(0)}$	$cp^* j^* \mathbf{u}_{(1)} = (\mathbf{F}^\mathsf{T} \mathbf{u} \circ cp \circ j)_{(1)}$	$cp^* j^* \mathbf{u}_{(2)} = (\operatorname{cof}(\mathbf{F})^\mathsf{T} \mathbf{u} \circ cp \circ j)_{(2)}$	$cp^* j^* f_{(3)} = (\det(\mathbf{F}) f \circ cp \circ j)_{(3)} = 0$

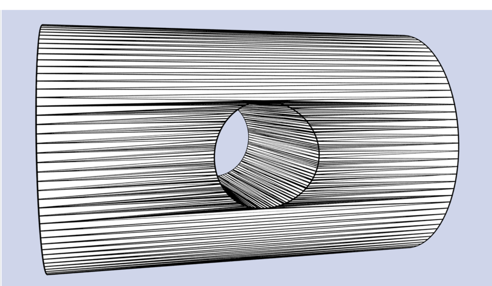
CP-EC OPERATORS IN $\mathbb{R}^3$

CP-Wedge Product	CP-Interior Product	CP-Exterior Derivative	CP-Hodge Star
$cp^*(\alpha \wedge^S \beta) = (cp^*\alpha) \wedge^{\mathbb{R}^3} (cp^*\beta)$	$cp^*(i^S_{\mathbf{Fu}} \alpha) = i^{\mathbb{R}^3}_{\mathbf{u}} cp^* \alpha$	$cp^*(d^S \alpha) = d^{\mathbb{R}^3} cp^* \alpha$	$cp^*(\star^S \alpha)\big\|_{j(S)} = i_{\vec{n}} \star^{\mathbb{R}^3} (cp^* \alpha)\big\|_{j(S)}$

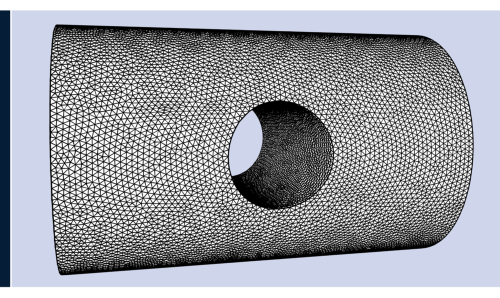
In order to express exterior calculus on surfaces (and by extension PDEs) in a way that is easier to discretize, we can formulate the surface exterior calculus in $\mathbb{R}^3$ using the closest point pullback defined by the above formulas. These operators form the basis of **Closest Point Exterior Calculus (CP-EC).**

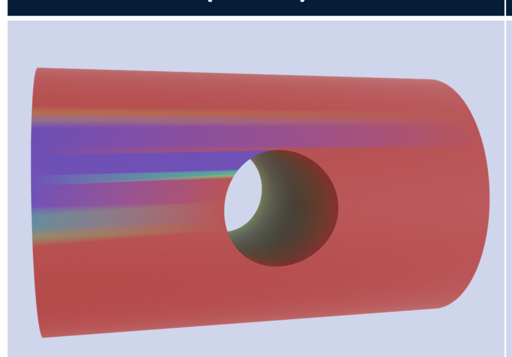
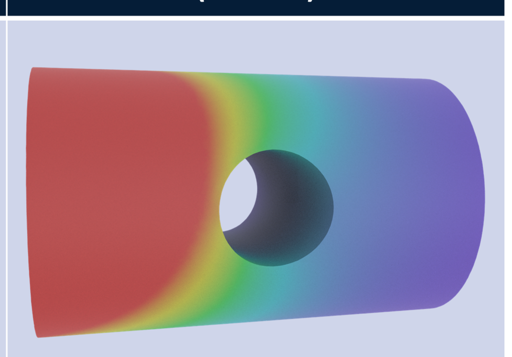
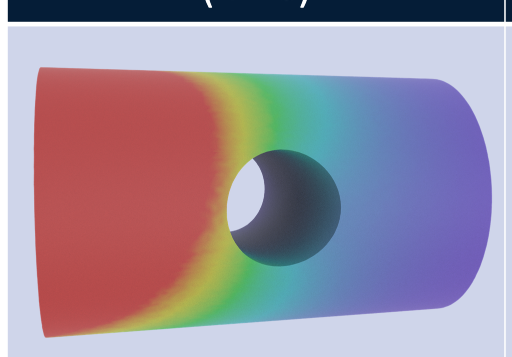
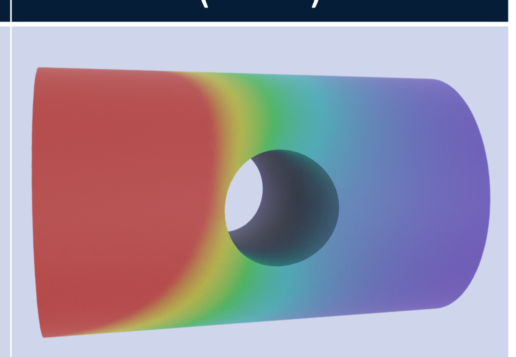
NUMERICAL TESTING

LOW QUALITY MESH



HIGH QUALITY MESH



HEAT FLOW SIMULATION ON A LOW QUALITY MESH (DEC)	HEAT FLOW SIMULATION ON A LOW QUALITY MESH (CPM)	HEAT FLOW SIMULATION ON A HIGH QUALITY MESH (DEC)	HEAT FLOW SIMULATION ON A HIGH QUALITY MESH (CPM)
			

Computed using **DEC**, while high quality mesh yields accurate representation of heat flow, **a poor quality mesh can lead to inaccurate simulation**. **CPM generates similar results** for both mesh qualities, demonstrating **CPM**'s resilience to the irregularities in the mesh, crucial for **robust and flexible physical simulation**.

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