Lecture 2: Number Representation

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Decimal Numbers: Base 10

Digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

Example:

3271 =

$$(3 \times 10^{3}) + (2 \times 10^{2}) + (7 \times 10^{1}) + (1 \times 10^{0})$$

Numbers: Positional Notation

- Number Base B ⇒ B symbols per digit:
  - Base 10 (Decimal): 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
    Base 2 (Binary): 0, 1
- Number representation:
  - $d_{31}d_{30} \ldots d_1d_0$ is a 32 digit number
  - value = $d_{31} \times B^{31} + d_{30} \times B^{30} + \ldots + d_1 \times B^1 + d_0 \times B^0$
- Binary: 0,1 (In binary digits called "bits")
  - 0b11010 $= 1\times2^4 + 1\times2^3 + 0\times2^2 + 1\times2^1 + 0\times2^0$
    $= 16 + 8 + 2$
    $= 26$
  - Here 5 digit binary # turns into a 2 digit decimal #
  - Can we find a base that converts to binary easily?

#s often written 0b…

Hexadecimal Numbers: Base 16

- Hexadecimal:
  0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F
  - Normal digits + 6 more from the alphabet
  - In C, written as 0x… (e.g., 0xFAB5)
- Conversion: Binary⇔Hex
  - 1 hex digit represents 16 decimal values
  - 4 binary digits represent 16 decimal values
  - ⇒ 1 hex digit replaces 4 binary digits
- One hex digit is a "nibble". Two is a "byte"
  - 2 bits is a "half-nibble". Shave and a haircut…
- Example:
  - 1010 1100 0011 (binary) = 0x_____ ?

Decimal vs. Hexadecimal vs. Binary

Examples:

1010 1100 0011 (binary)
= 0xAC3

10111 (binary)
= 0001 0111 (binary)
= 0x17

0x3F9
= 11 1111 1001 (binary)

How do we convert between hex and Decimal?
and Decimal?

00	0	0000
01	1	0001
02	2	0010
03	3	0011
04	4	0100
05	5	0101
06	6	0110
07	7	0111
08	8	1000
09	9	1001
10	A	1010
11	B	1011
12	C	1100
13	D	1101
14	E	1110
15	F	1111

MEMORIZE!

Which base do we use?

- Decimal: great for humans, especially when doing arithmetic
- Hex: if human looking at long strings of binary numbers, its much easier to convert to hex and look 4 bits/symbol
  - Terrible for arithmetic on paper
- Binary: what computers use;
  you will learn how computers do +, -, *, /
  - To a computer, numbers always binary
  - Regardless of how number is written:
  - $32_{ten} == 32_{10} == 0x20 == 100000_2 == 0b100000$
  - Use subscripts "ten", "hex", "two" in book, slides when might be confusing

What to do with representations of numbers?

- Just what we do with numbers!
  - Add them
  - Subtract them
  - Multiply them
  - Divide them
  - Compare them
- Example: 10 + 7 = 17
  - …so simple to add in binary that we can build circuits to do it!
  - subtraction just as you would in decimal
  - Comparison: How do you tell if X > Y ?

	1	1			
	1	0	1	0	
+	0	1	1	1	
1	0	0	0	1	

BIG IDEA: Bits can represent anything!!

- Characters?
  - 26 letters ⇒ 5 bits ($2^5 = 32$)
  - upper/lower case + punctuation ⇒ 7 bits (in 8) ("ASCII")
  - standard code to cover all the world's languages ⇒ 8,16,32 bits ("Unicode") `www.unicode.com`
- Logical values?
  - 0 ⇒ False, 1 ⇒ True
- colors ? Ex:
- locations / addresses? commands?
- MEMORIZE: N bits ⇔ at most 2^N things

Red (00) *Green (01)* *Blue (11)*

How to Represent Negative Numbers?

- So far, unsigned numbers
- Obvious solution: define leftmost bit to be sign!
  - $0 \Rightarrow +$, $1 \Rightarrow -$
  - Rest of bits can be numerical value of number
- Representation called sign and magnitude
- MIPS uses 32-bit integers. $+1_{ten}$ would be:

0000 0000 0000 0000 0000 0000 0000 0001

- And -1_{ten} in sign and magnitude would be:

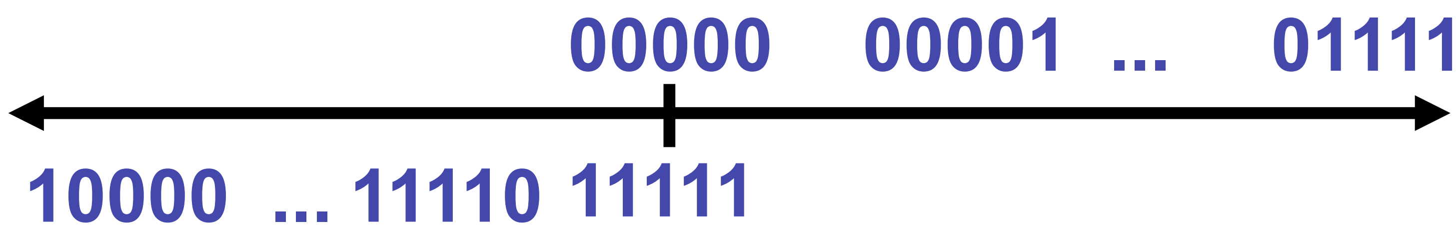
1000 0000 0000 0000 0000 0000 0000 0001

Shortcomings of sign and magnitude?

- Arithmetic circuit complicated
  - Special steps depending whether signs are the same or not
- Also, two zeros
  - 0x00000000 = $+0_{ten}$
  - 0x80000000 = -0_{ten}
  - What would two 0s mean for programming?
- Therefore sign and magnitude abandoned

Another try: complement the bits

- Example: $7_{10} = 00111_2$ $\quad -7_{10} = 11000_2$
- Called One' s Complement
- Note: positive numbers have leading 0s, negative numbers have leadings 1s.



- **What is -00000 ? Answer: 11111**
- **How many positive numbers in N bits?**
- **How many negative numbers?**

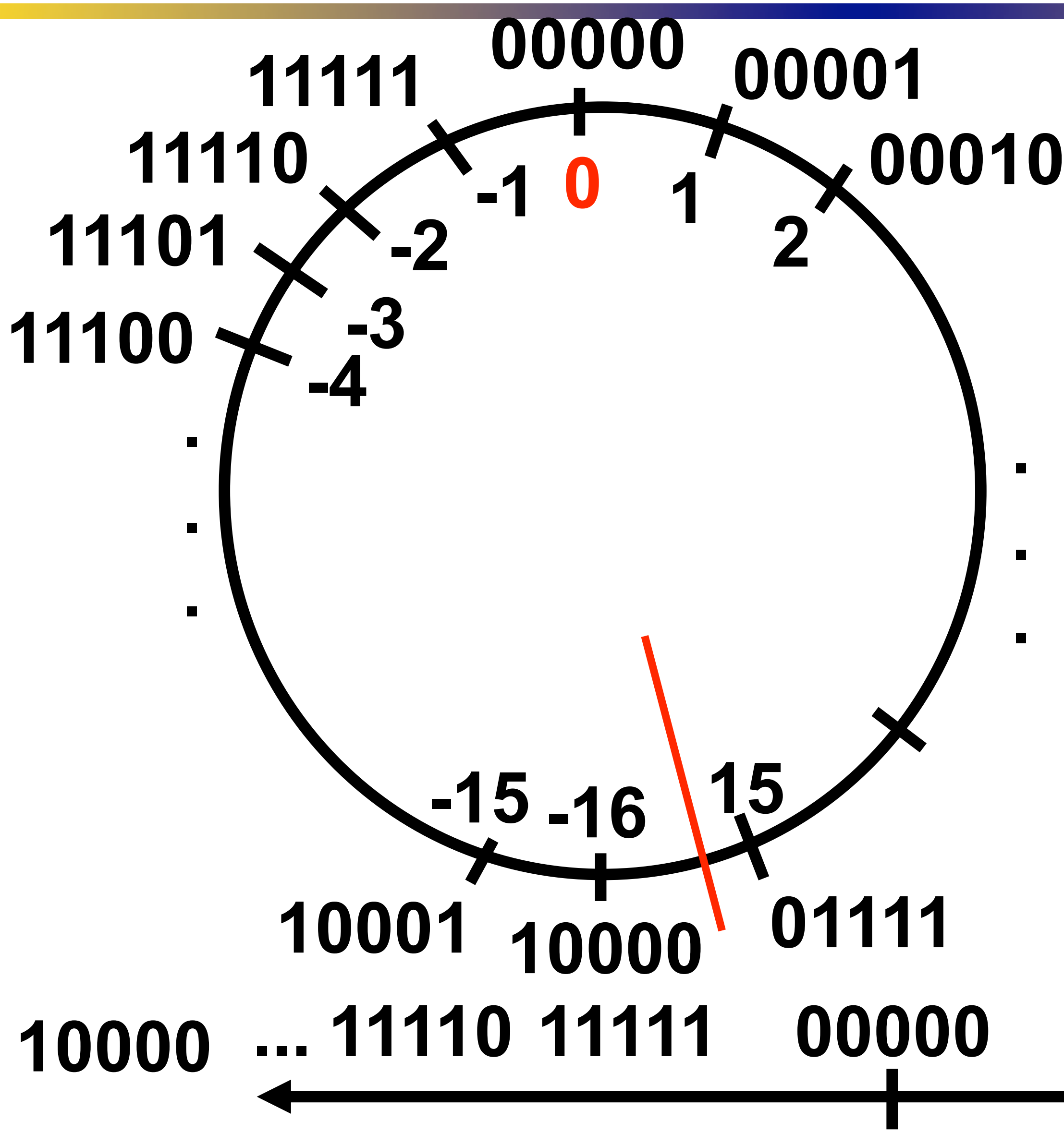
Shortcomings of One's complement?

- Arithmetic still somewhat complicated.
- Still two zeros
  - `0x00000000` = $+0_{ten}$
  - `0xFFFFFFFF` = -0_{ten}
- Although used for awhile on some computer products, one's complement was eventually abandoned because another solution was better.

Standard Negative Number Representation

- What is result for unsigned numbers if tried to subtract large number from a small one?
  - Would try to borrow from string of leading 0s, so result would have a string of leading 1s
    - 3 - 4 ⇒ 00…0011 – 00…0100 = 11…1111
  - With no obvious better alternative, pick representation that made the hardware simple
  - As with sign and magnitude, leading 0s ⇒ positive, leading 1s ⇒ negative
    - 000000...xxx is ≥ 0, 111111...xxx is < 0
    - except 1…1111 is -1, not -0 (as in sign & mag.)
- This representation is Two's Complement

2’s Complement Number “line”: N = 5



- 2^{N-1} non-negatives
- 2^{N-1} negatives
- one zero
- how many positives?

Two's Complement Formula

- Can represent positive and negative numbers in terms of the bit value times a power of 2:

  $d_{31} \text{ x } -(2^{31}) + d_{30} \text{ x } 2^{30} + \ldots + d_2 \text{ x } 2^2 + d_1 \text{ x } 2^1 + d_0 \text{ x } 2^0$

- Example: 1101_{two}

  $= 1\text{x}-(2^3) + 1\text{x}2^2 + 0\text{x}2^1 + 1\text{x}2^0$

  $= -2^3 + 2^2 + 0 + 2^0$

  $= -8 + 4 + 0 + 1$

  $= -8 + 5$

  $= -3_{ten}$

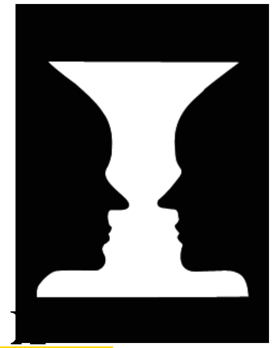
Two's Complement shortcut: Negation

- Change every 0 to 1 and 1 to 0 (invert or complement), then add 1 to the result
- Proof*: Sum of number and its (one's) complement must be $111...111_{two}$

  However, $111...111_{two} = -1_{ten}$

  Let $x' \Rightarrow$ one's complement representation of x

  Then $x + x' = -1 \Rightarrow x + x' + 1 = 0 \Rightarrow$ $-x = x' + 1$



- Example: -3 to +3 to -3

x : $1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1101_{two}$

x' : $0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0010_{two}$

+1: $0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0011_{two}$

()' : $1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1100_{two}$

+1: $1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1101_{two}$

You should be able to do this in your head…

Two's Complement for N=32

$0000 \ldots 0000\ 0000\ 0000\ 0000_{two}$ =	0_{ten}
$0000 \ldots 0000\ 0000\ 0000\ 0001_{two}$ =	1_{ten}
$0000 \ldots 0000\ 0000\ 0000\ 0010_{two}$ =	2_{ten}
. . .	
$0111 \ldots 1111\ 1111\ 1111\ 1101_{two}$ =	$2{,}147{,}483{,}645_{ten}$
$0111 \ldots 1111\ 1111\ 1111\ 1110_{two}$ =	$2{,}147{,}483{,}646_{ten}$
$0111 \ldots 1111\ 1111\ 1111\ 1111_{two}$ =	$2{,}147{,}483{,}647_{ten}$
$1000 \ldots 0000\ 0000\ 0000\ 0000_{two}$ =	$-2{,}147{,}483{,}648_{ten}$
$1000 \ldots 0000\ 0000\ 0000\ 0001_{two}$ =	$-2{,}147{,}483{,}647_{ten}$
$1000 \ldots 0000\ 0000\ 0000\ 0010_{two}$ =	$-2{,}147{,}483{,}646_{ten}$
. . .	
$1111 \ldots 1111\ 1111\ 1111\ 1101_{two}$ =	-3_{ten}
$1111 \ldots 1111\ 1111\ 1111\ 1110_{two}$ =	-2_{ten}
$1111 \ldots 1111\ 1111\ 1111\ 1111_{two}$ =	-1_{ten}

- One zero; 1st bit called sign bit
- 1 "extra" negative:no positive $2{,}147{,}483{,}648_{ten}$

Two's comp. shortcut: Sign extension

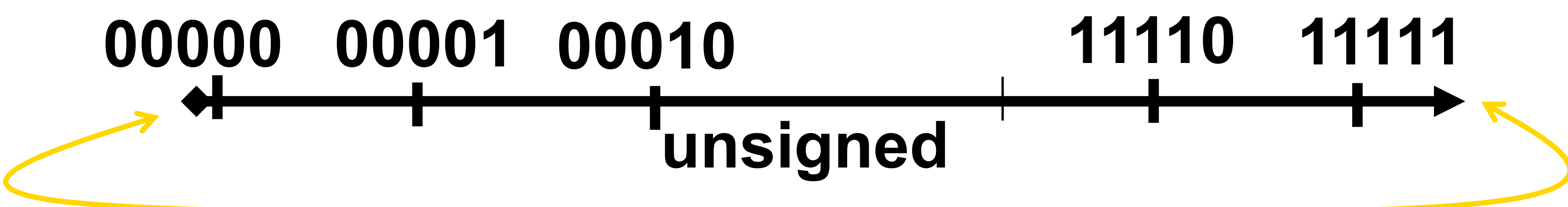
- Convert 2's complement number rep. using n bits to more than n bits
- Simply replicate the most significant bit (sign bit) of smaller to fill new bits
  - 2's comp. positive number has infinite 0s
  - 2's comp. negative number has infinite 1s
  - Binary representation hides leading bits; sign extension restores some of them
  - 16-bit -4_{ten} to 32-bit:

$$1111\ 1111\ 1111\ 1100_{two}$$

$$1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1100_{two}$$

What if too big?

- Binary bit patterns above are simply representatives of numbers. Strictly speaking they are called "numerals".
- Numbers really have an ∞ number of digits
  - with almost all being same (00…0 or 11…1) except for a few of the rightmost digits
  - Just don't normally show leading digits
- If result of add (or -, *, /) cannot be represented by these rightmost HW bits, overflow is said to have occurred.



Question

$X = 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1100_{two}$

$Y = 0011\ 1011\ 1001\ 1010\ 1000\ 1010\ 0000\ 0000_{two}$

A. X > Y (if signed)

B. X > Y (if unsigned)

C. Babylonians could represent ALL their integers from $[-2^{N-1}$ to $2^{N-1}]$ with N bits!

	ABC
0:	FFF
1:	FFT
2:	FTF
3:	FTT
4:	TFF
5:	TFT
6:	TTF
7:	TTT

Signed vs. Unsigned Variables

- Java and C declare integers `int`
  - Use two' s complement (signed integer)
- Also, C declaration `unsigned int`
  - Declares a unsigned integer
  - Treats 32-bit number as unsigned integer, so most significant bit is part of the number, not a sign bit

Kilo, Mega, Giga, Tera, Peta, Exa, Zetta, Yotta

physics.nist.gov/cuu/Units/binary.html

- Common use prefixes (all SI, except K [= k in SI])

Name	Abbr	Factor	SI size
Kilo	K	2^{10} = 1,024	10^{3} = 1,000
Mega	M	2^{20} = 1,048,576	10^{6} = 1,000,000
Giga	G	2^{30} = 1,073,741,824	10^{9} = 1,000,000,000
Tera	T	2^{40} = 1,099,511,627,776	10^{12} = 1,000,000,000,000
Peta	P	2^{50} = 1,125,899,906,842,624	10^{15} = 1,000,000,000,000,000
Exa	E	2^{60} = 1,152,921,504,606,846,976	10^{18} = 1,000,000,000,000,000,000
Zetta	Z	2^{70} = 1,180,591,620,717,411,303,424	10^{21} = 1,000,000,000,000,000,000,000
Yotta	Y	2^{80} = 1,208,925,819,614,629,174,706,176	10^{24} = 1,000,000,000,000,000,000,000,000

- Confusing! Common usage of "kilobyte" means 1024 bytes, but the "correct" SI value is 1000 bytes
- Hard Disk manufacturers & Telecommunications are the only computing groups that use SI factors, so what is advertised as a 30 GB drive will actually only hold about 28 x 2^{30} bytes, and a 1 Mbit/s connection transfers 10^{6} bps.

kibi, mebi, gibi, tebi, pebi, exbi, zebi, yobi

en.wikipedia.org/wiki/Binary_prefix

- New IEC Standard Prefixes [only to exbi officially]

Name	Abbr	Factor
kibi	Ki	2^{10} = 1,024
mebi	Mi	2^{20} = 1,048,576
gibi	Gi	2^{30} = 1,073,741,824
tebi	Ti	2^{40} = 1,099,511,627,776
pebi	Pi	2^{50} = 1,125,899,906,842,624
exbi	Ei	2^{60} = 1,152,921,504,606,846,976
zebi	Zi	2^{70} = 1,180,591,620,717,411,303,424
yobi	Yi	2^{80} = 1,208,925,819,614,629,174,706,176

As of this writing, this proposal has yet to gain widespread use…

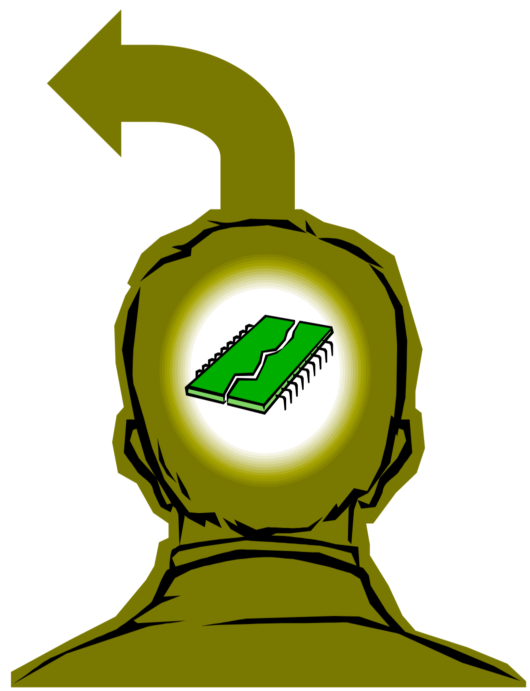
- International Electrotechnical Commission (IEC) in 1999 introduced these to specify binary quantities.
  - Names come from shortened versions of the original SI prefixes (same pronunciation) and *bi* is short for "binary", but pronounced "bee" :-(
  - Now SI prefixes only have their base-10 meaning and never have a base-2 meaning.

The way to remember #s

- What is 2^{34}? How many bits addresses (i.e., whats `ceil log`$_2$ = `lg` of) 2.5 TiB?
- Answer! 2^{XY} means…

X=0 ⇒ ---
X=1 ⇒ kibi ~10^3
X=2 ⇒ mebi ~10^6
X=3 ⇒ gibi ~10^9
X=4 ⇒ tebi ~10^{12}
X=5 ⇒ pebi ~10^{15}
X=6 ⇒ exbi ~10^{18}
X=7 ⇒ zebi ~10^{21}
X=8 ⇒ yobi ~10^{24}

Y=0 ⇒ 1
Y=1 ⇒ 2
Y=2 ⇒ 4
Y=3 ⇒ 8
Y=4 ⇒ 16
Y=5 ⇒ 32
Y=6 ⇒ 64
Y=7 ⇒ 128
Y=8 ⇒ 256
Y=9 ⇒ 512



MEMORIZE!

Summary

- We represent "things" in computers as particular bit patterns: N bits ⇒ 2^N things
- Decimal for human calculations, binary for computers, hex to write binary more easily
- 1's complement - mostly abandoned

00000 00001 ... 01111

10000 ... 11110 11111

- 2's complement universal in computing: cannot avoid, so learn

00000 00001 ... 01111

10000 ... 11110 11111

- Overflow: numbers ∞; computers finite,errors!