Department of Computer Science and Engineering
University of California at San Diego

CSE 230
Winter 1998

Problem Set 9

DUE THURSDAY MARCH 12 IN CLASS.

1. Consider the following theorem which says that substitution preserves the well-formedness of λ-expressions, and its purported proof.

   Theorem: If S and T are λ-expressions, then the result of performing the substitution $S[x \mapsto T]$ is also a λ-expression.
   Proof: We will prove the claim by structural induction on S. There are six cases to consider.

   (i) If $S \equiv x$ then $S[x \mapsto T] \equiv T$ which is given to be a λ-expression.

   (ii) If $S \equiv y \not\equiv x$ then $S[x \mapsto T] \equiv y$ which is given to be a λ-expression.

   (iii) Suppose $S \equiv MN$. Now M and N are sub-expressions of S, so we can assume inductively that $M[x \mapsto T]$ and $N[x \mapsto T]$ are well-formed λ-expressions. Then $S[x \mapsto T] \equiv (M[x \mapsto T])(N[x \mapsto T])$ is a well-formed λ-expression.

   (iv) If $S \equiv \lambda x.W$ then $S[x \mapsto T] \equiv S$ which is given to be a λ-expression.

   (v) Suppose $S \equiv \lambda y.W$ where $y \not\equiv x$ and y is not free in T. Now W is a sub-expression of S, so we can assume inductively that $W[x \mapsto T]$ is well-formed. Then $S[x \mapsto T] \equiv \lambda y.(W[x \mapsto T])$ is well-formed.

   (vi) Suppose $S \equiv \lambda y.W$ where $y \not\equiv x$ and y actually is free in T. Then (because of capture) $S[x \mapsto T] \equiv \lambda z.((W[y \mapsto z])[x \mapsto T])$ where z is a fresh identifier that is not free in W or T. Now W is a sub-expression of S so the induction hypothesis says that $(W[y \mapsto z])[x \mapsto T]$ is well-formed, and therefore $S[x \mapsto T]$ is well-formed. ■

   This theorem is true but the proof is incorrect. Find and fix the error(s).

2. (i) Find a λ-expression S that is self-reproducing: for any list of λ-expressions $T_1, T_2, \ldots T_n$ it should be the case that $S\ T_1\ T_2 \ldots\ T_n$ β-reduces to S. Prove that your S works.

   (ii) [Klop] Let $B \equiv AAAAAAAAAAAAAAAAAAAAAAAAAA$ where

   $$A \equiv \lambda abcdefghijklmnopqstuvwxyzr.r(thisisafixedpointcombinator)$$

   Show that B is a fixed point combinator—prove that $BF = F(BF)$ for any F.

   (iii) [Barendregt] Find a λ-expression under 100mm long with a normal form of length over $10^{10^{10}}$ light years. Assume that symbols of the λ-calculus are 2mm wide, and that the speed of light is $3 \cdot 10^8$m/s.

   (iv) Remember that equality is the reflexive symmetric transitive closure of β-reduction and α-conversion. Show that if we assume that $\mathbf{S} \equiv \lambda xyz.(xz)(yz)$ and $\mathbf{I} \equiv \lambda x.x$ are equal, then we can conclude that any two λ-expressions are equal.