Estimation of Essential Matrix, Linear

Computer Vision II

CSE 252B

Lecture 14

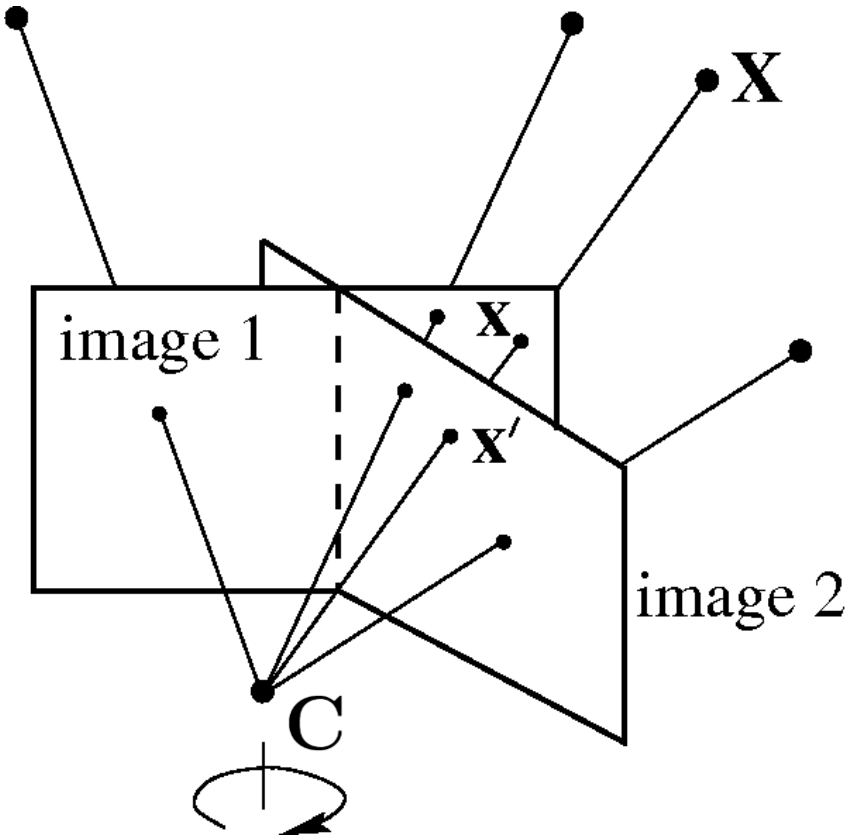
Announcements

- Assignment 4 is due Mar 4, 11:59 PM
- Reading
  - Section 9.6
  - Kukelova et al. “Polynomial Eigenvalue Solutions to the 5-pt and 6-pt Relative Pose Problems”

Two view geometry

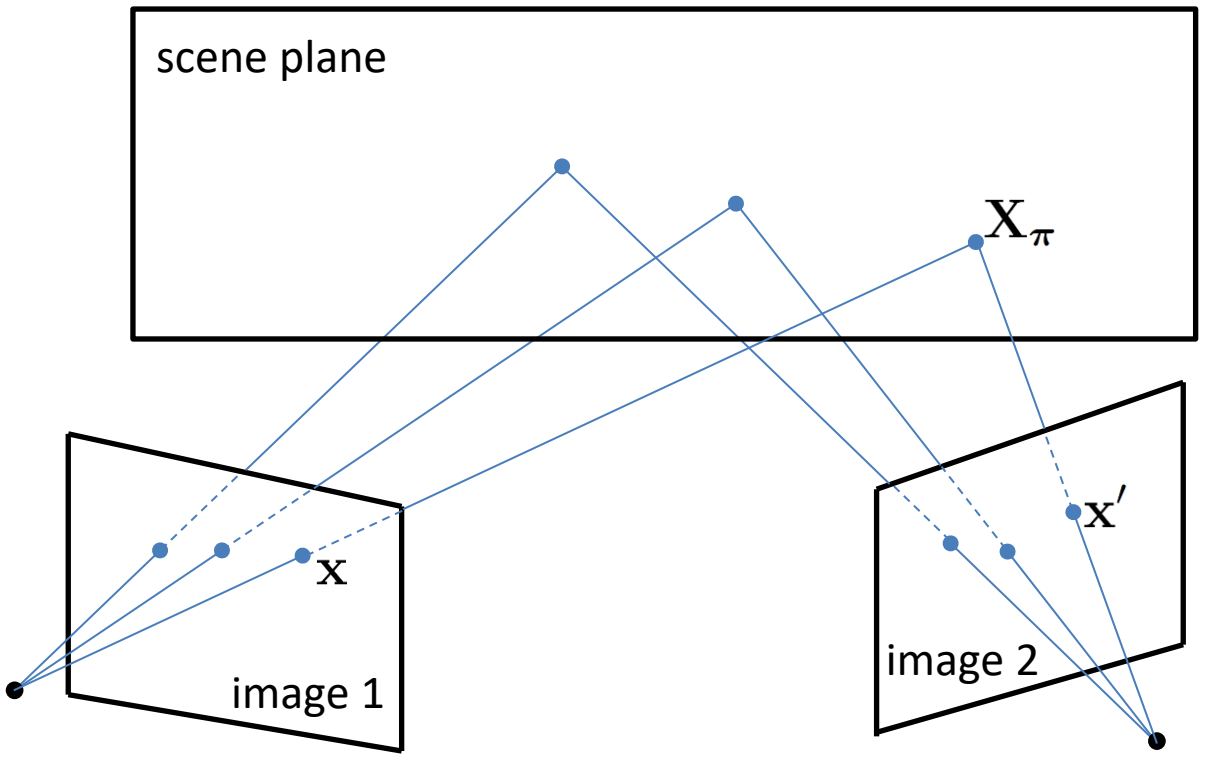
Two view geometry

- Rotation about the same camera center



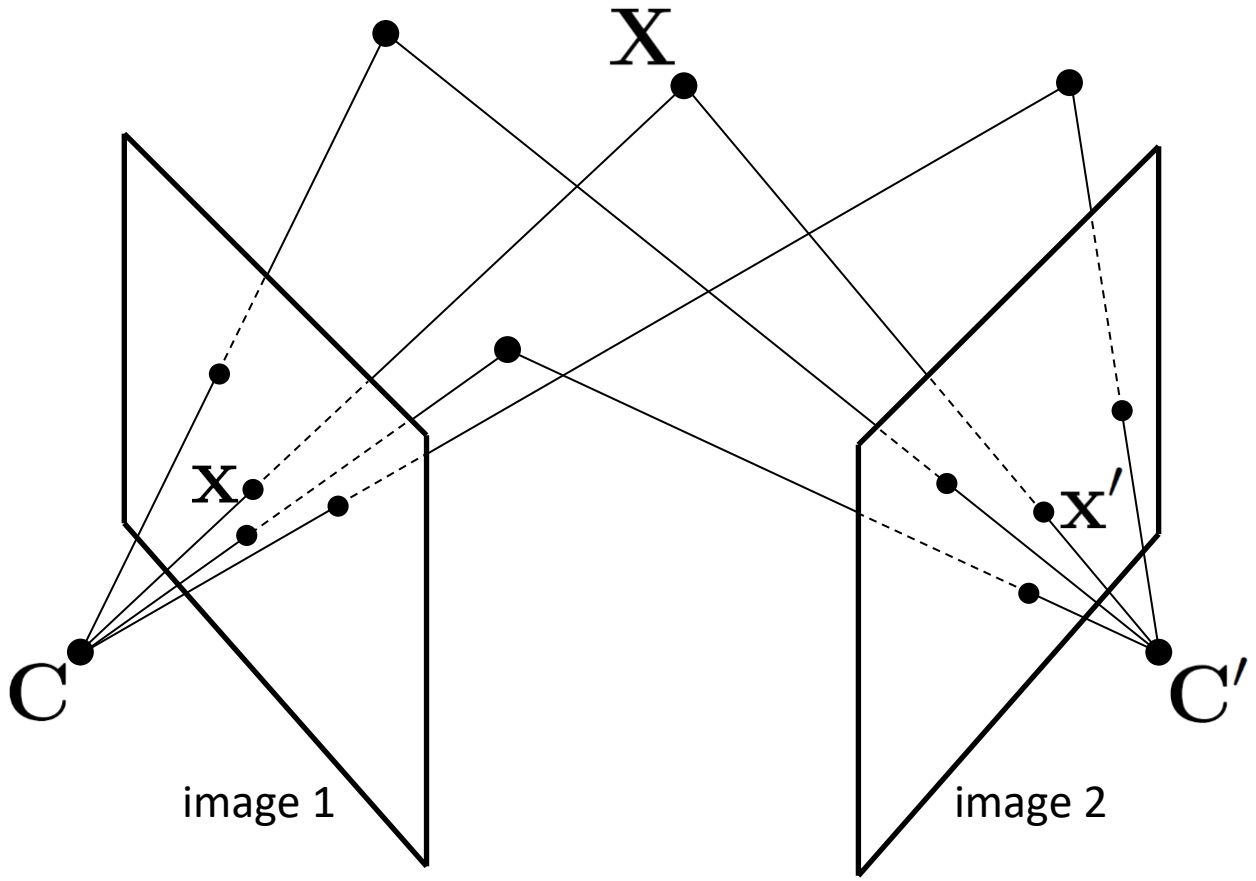
Two view geometry

- Imaging a plane



Two view geometry

- Imaging a 3D scene



Calibrated camera

$$\mathtt{K} = \begin{bmatrix} \alpha_x & s & x_0 \\ 0 & \alpha_y & y_0 \\ 0 & 0 & 1 \end{bmatrix}$$

where

$\alpha_x = f m_x$	Focal length in x direction in terms of pixel dimensions
$\alpha_y = f m_y$	Focal length in y direction in terms of pixel dimensions
s	Skew
$(x_0, y_0)^\top = (p_x m_x, p_y m_y)^\top$	Coordinates of principal point in terms of pixel dimensions

where m_x and m_y are number of pixels per unit disance in x and y directions, respectively

(and lens distortion parameters)

- If camera calibration parameters are known, then use normalized camera projection matrix and image points in normalized coordinates

Normalized camera projection matrix

$$\hat{\mathbf{x}} = \hat{\mathrm{P}}\mathbf{X}$$

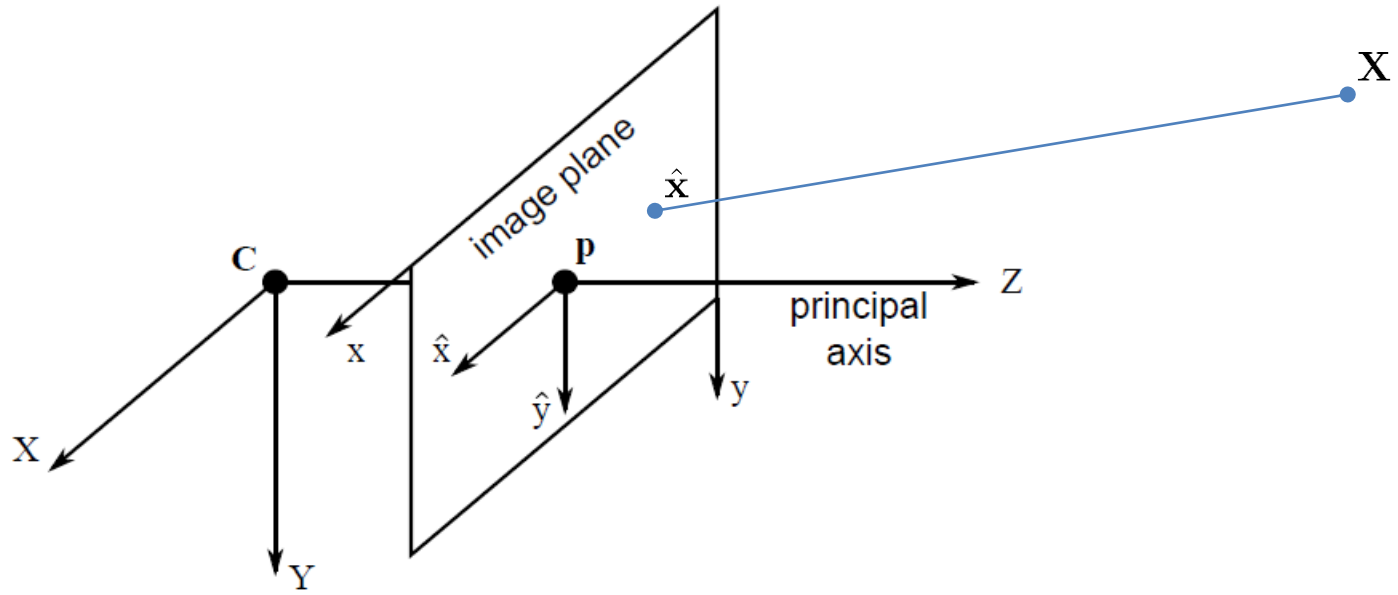
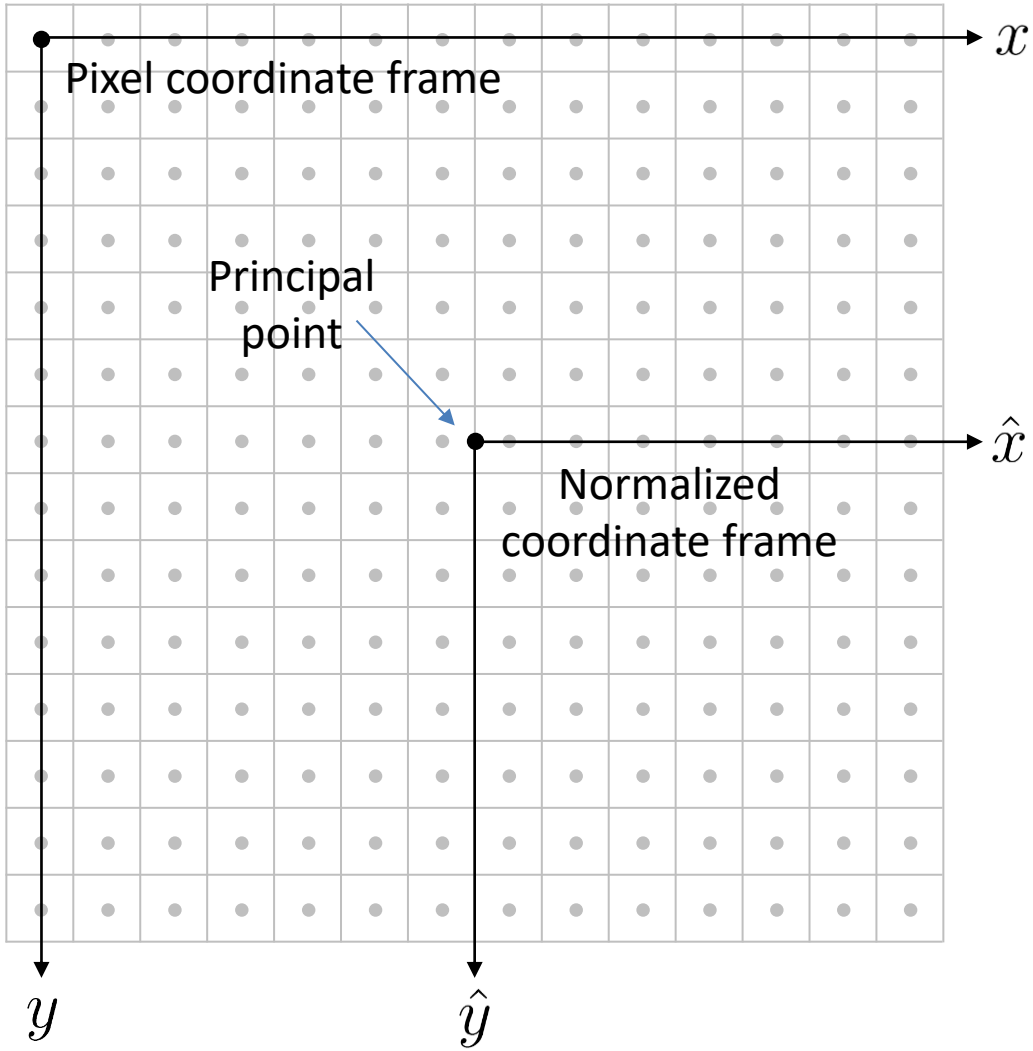


Image coordinate frames



Camera projection

Project 3D point in world coordinate frame $\mathbf{X}$ under camera projection matrix $\mathtt{P} = \mathtt{K}[\mathtt{R} \,|\, \mathbf{t}]$

$$\begin{aligned} \mathbf{x} &= \mathtt{P}\mathbf{X} \\ \mathbf{x} &= \mathtt{K}[\mathtt{R} \,|\, \mathbf{t}]\mathbf{X} \\ \mathtt{K}^{-1}\mathbf{x} &= [\mathtt{R} \,|\, \mathbf{t}]\mathbf{X} \\ \hat{\mathbf{x}} &= \hat{\mathtt{P}}\mathbf{X} \end{aligned}$$

Camera calibration matrix

where

Camera projection matrix	$\mathtt{P} = \mathtt{K}[\mathtt{R} \,|\, \mathbf{t}]$	$= \mathtt{K}\hat{\mathtt{P}}$
Normalized camera projection matrix	$\hat{\mathtt{P}} = [\mathtt{R} \,|\, \mathbf{t}]$	$= \mathtt{K}^{-1}\mathtt{P}$
Image point in pixel coordinates	$\mathbf{x} = \mathtt{K}[\mathtt{R} \,|\, \mathbf{t}]\mathbf{X}$	$= \mathtt{K}\hat{\mathbf{x}}$
Image point in normalized coordinates	$\hat{\mathbf{x}} = [\mathtt{R} \,|\, \mathbf{t}]\mathbf{X}$	$= \mathtt{K}^{-1}\mathbf{x}$

Imaging geometry models

- Single view
  - Uncalibrated: camera projection matrix
  - Calibrated: normalized camera projection matrix
- Two views
  - Rotation about the same camera center
    - Uncalibrated: 2D projective transformation matrix
    - Calibrated: 3D rotation matrix
  - Imaging a plane
    - Uncalibrated: 2D projective transformation matrix
    - Calibrated: 2D projective transformation matrix
  - Imaging a 3D scene
    - Uncalibrated: fundamental matrix
    - Calibrated: essential matrix

Essential matrix

Given the canonical normalized cameras $\hat{\mathtt{P}} = [\mathtt{I} \,|\, \mathbf{0}]$ and $\hat{\mathtt{P}}' = [\mathtt{R} \,|\, \mathbf{t}]$, a 3D point projects to $\hat{\mathbf{x}}$ and $\hat{\mathbf{x}}'$, respectively

The 3D point $\tilde{\mathbf{X}}_{\text{cam}} = \lambda \hat{\mathbf{x}}$ in the first camera coordinate frame.

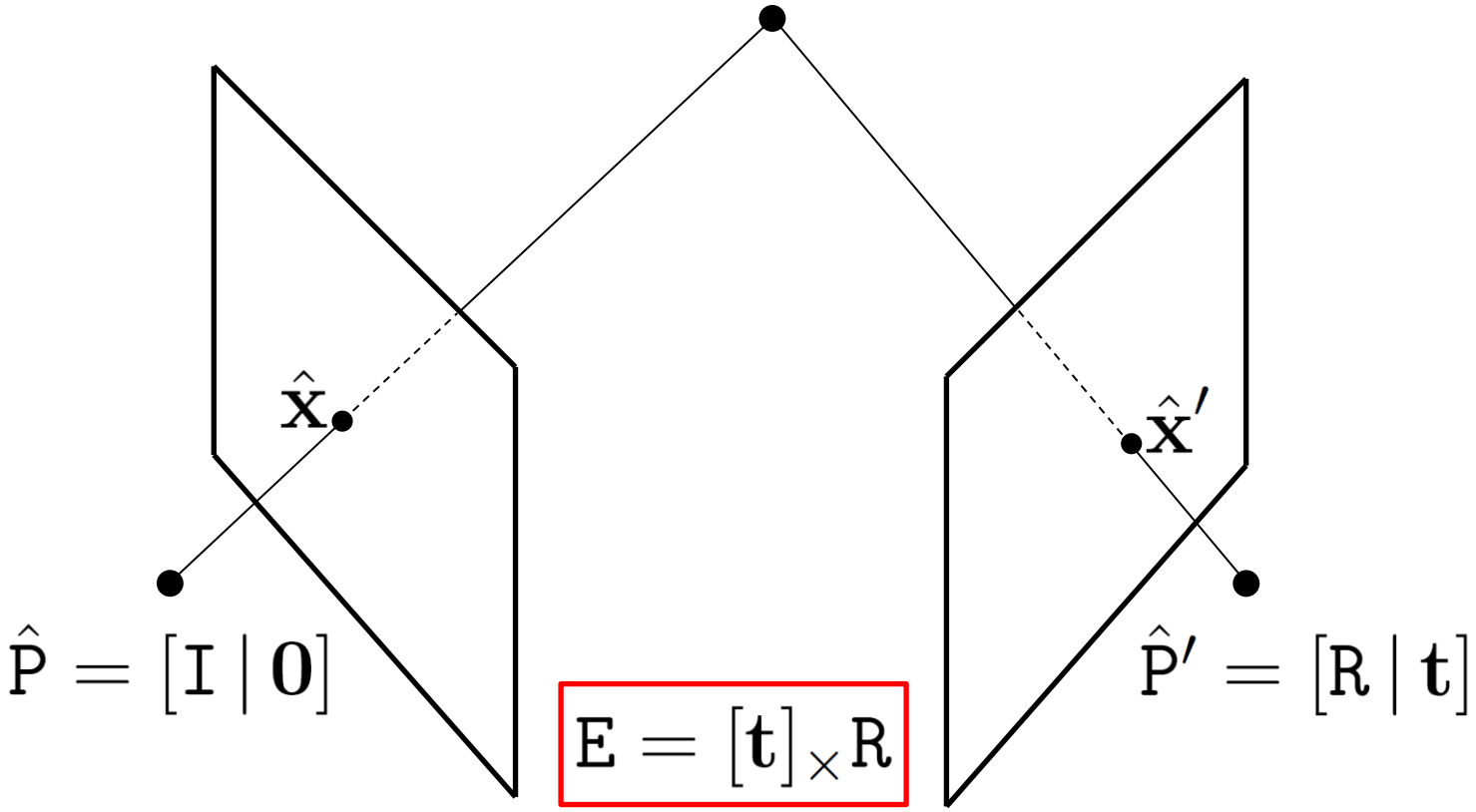
The 3D point $\tilde{\mathbf{X}}'_{\text{cam}} = \lambda' \hat{\mathbf{x}}'$ in the second camera coordinate frame.

Map $\tilde{\mathbf{X}}_{\text{cam}}$ from the first camera coordinate frame to the second camera coordinate frame.

$$
\begin{aligned}
\mathtt{R}\tilde{\mathbf{X}}_{\text{cam}} + \mathbf{t} &= \tilde{\mathbf{X}}'_{\text{cam}} \\
\mathtt{R}(\lambda \hat{\mathbf{x}}) + \mathbf{t} &= \lambda' \hat{\mathbf{x}}' \\
\lambda \mathtt{R} \hat{\mathbf{x}} + \mathbf{t} &= \lambda' \hat{\mathbf{x}}' \\
[\mathbf{t}]_\times (\lambda \mathtt{R} \hat{\mathbf{x}} + \mathbf{t}) &= [\mathbf{t}]_\times (\lambda' \hat{\mathbf{x}}') \\
\lambda [\mathbf{t}]_\times \mathtt{R} \hat{\mathbf{x}} + [\mathbf{t}]_\times \mathbf{t} &= \lambda' [\mathbf{t}]_\times \hat{\mathbf{x}}' \\
\lambda [\mathbf{t}]_\times \mathtt{R} \hat{\mathbf{x}} &= \lambda' [\mathbf{t}]_\times \hat{\mathbf{x}}' \\
\hat{\mathbf{x}}'^\top (\lambda [\mathbf{t}]_\times \mathtt{R} \hat{\mathbf{x}}) &= \hat{\mathbf{x}}'^\top (\lambda' [\mathbf{t}]_\times \hat{\mathbf{x}}') \\
\lambda \hat{\mathbf{x}}'^\top [\mathbf{t}]_\times \mathtt{R} \hat{\mathbf{x}} &= \lambda' \hat{\mathbf{x}}'^\top [\mathbf{t}]_\times \hat{\mathbf{x}}' \\
\lambda \hat{\mathbf{x}}'^\top [\mathbf{t}]_\times \mathtt{R} \hat{\mathbf{x}} &= 0 \\
\hat{\mathbf{x}}'^\top [\mathbf{t}]_\times \mathtt{R} \hat{\mathbf{x}} &= 0 \\
\hat{\mathbf{x}}'^\top \mathtt{E} \hat{\mathbf{x}} &= 0, \text{ where the essential matrix } \mathtt{E} = [\mathbf{t}]_\times \mathtt{R}
\end{aligned}
$$

$\mathtt{E}$ is a rank 2 homogeneous matrix with singular values $(\sigma, \sigma, 0)^\top$

Essential matrix



Essential matrix

$$\hat{\boldsymbol{\ell}}' = \mathtt{E}\hat{\mathbf{x}} \text{ and } \hat{\mathbf{x}}'^{\top}\hat{\boldsymbol{\ell}}' = 0$$

$$\hat{\mathbf{x}}'^{\top}\mathtt{E}\hat{\mathbf{x}} = 0 \qquad \text{Epipolar constraint}$$

$\hat{\boldsymbol{\ell}}' = \mathtt{E}\hat{\mathbf{x}}$	The epipolar line corresponding to the point $\hat{\mathbf{x}}$
$\hat{\boldsymbol{\ell}} = \mathtt{E}^{\top}\hat{\mathbf{x}}'$	The epipolar line corresponding to the point $\hat{\mathbf{x}}'$
$\mathtt{E}\hat{\mathbf{e}} = \mathbf{0}$	The epipole $\hat{\mathbf{e}}$ is the null space of $\mathtt{E}$
$\hat{\mathbf{e}}'^{\top}\mathtt{E} = \mathbf{0}^{\top}$	The epipole $\hat{\mathbf{e}}'$ is the left null space of $\mathtt{E}$
$\mathtt{E}^{\top}\hat{\mathbf{e}}' = \mathbf{0}$	The epipole $\hat{\mathbf{e}}'$ is the null space of $\mathtt{E}^{\top}$

The essential matrix $\mathtt{E}$ from general normalized cameras $\hat{\mathtt{P}} = [\mathtt{R} \,|\, \mathbf{t}]$ and $\hat{\mathtt{P}}' = [\mathtt{R}' \,|\, \mathbf{t}']$
Transform the cameras such that first normalized camera projection matrix is $[\mathtt{I} \,|\, \mathbf{0}]$ as follows

$$[\mathtt{R} \,|\, \mathbf{t}]\mathtt{H}_{\mathtt{E}}^{-1} = [\mathtt{I} \,|\, \mathbf{0}] \text{ and } [\mathtt{R}' \,|\, \mathbf{t}']\mathtt{H}_{\mathtt{E}}^{-1} = [\mathtt{R}'\mathtt{R}^{\top} \,|\, \mathbf{t}' - \mathtt{R}'\mathtt{R}^{\top}\mathbf{t}], \text{ where } \mathtt{H}_{\mathtt{E}}^{-1} = \begin{bmatrix} \mathtt{R}^{\top} & -\mathtt{R}^{\top}\mathbf{t} \\ \mathbf{0}^{\top} & 1 \end{bmatrix}$$

Then

$$\mathtt{E} = [\mathbf{t}' - \mathtt{R}'\mathtt{R}^{\top}\mathbf{t}]_{\times}\mathtt{R}'\mathtt{R}^{\top}$$

The essential matrix $\mathtt{E}$ from canonical cameras $\mathtt{P} = [\mathtt{I} \,|\, \mathbf{0}]$ and $\mathtt{P}' = [\mathtt{R} \,|\, \mathbf{t}]$

$$\mathtt{E} = [\mathbf{t}]_{\times}\mathtt{R}$$

Essential matrix

Note if $\mathtt{F}$ is determined from point correspondences $\mathbf{x}_i \leftrightarrow \mathbf{x}'_i$ in pixel coordinates, then

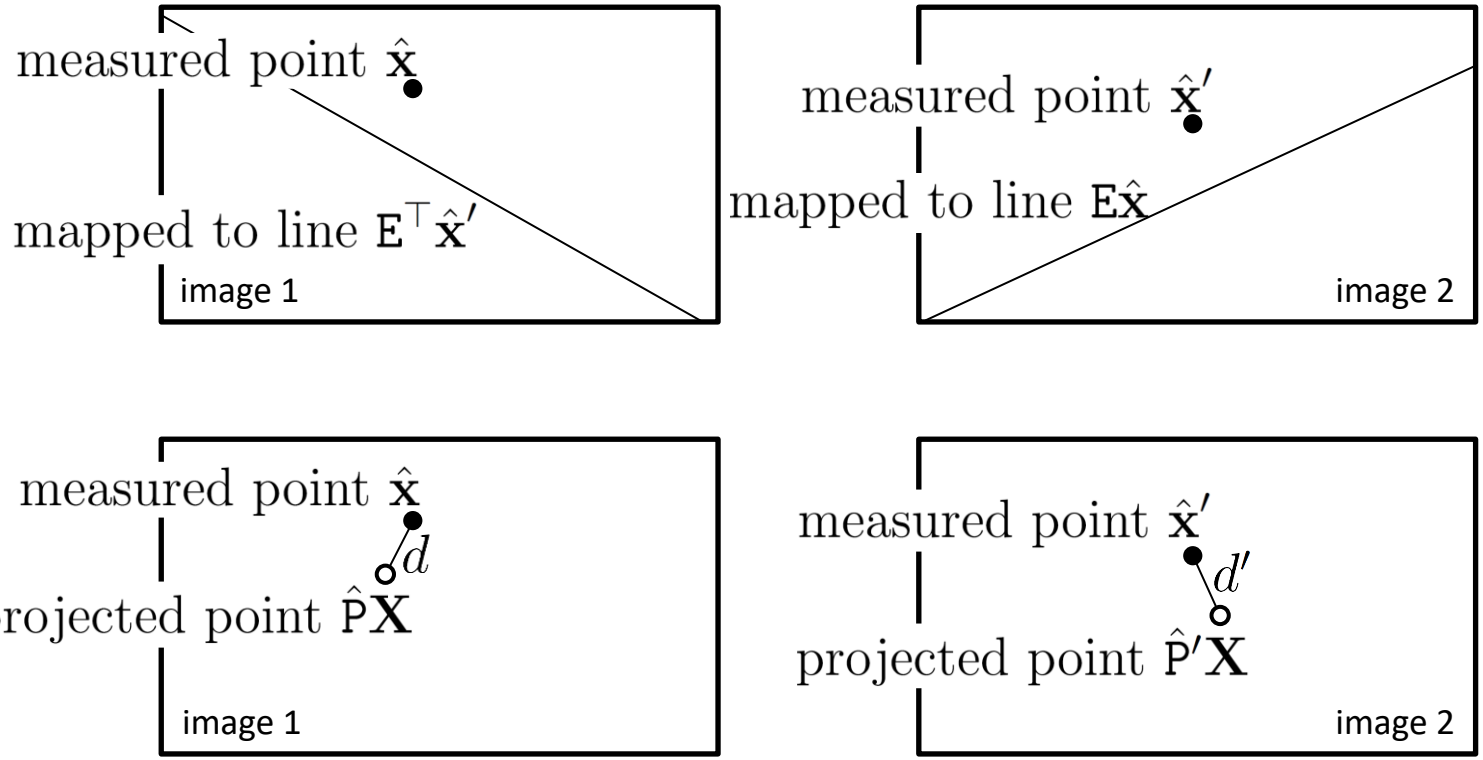
$$\mathtt{F} = \mathtt{K}'^{-\top}\mathtt{E}\mathtt{K}^{-1}$$

$$\mathtt{E} = \mathtt{K}'^{\top}\mathtt{F}\mathtt{K}$$

In general, the resulting $\mathtt{E}$ will not be a valid essential matrix (i.e., singular values may not be $(\sigma, \sigma, 0)^\top$). If the camera calibration matrices are known, then determine $\mathtt{E}$ directly from $\hat{\mathbf{x}}_i \leftrightarrow \hat{\mathbf{x}}'_i$ in normalized coordinates.

Essential matrix

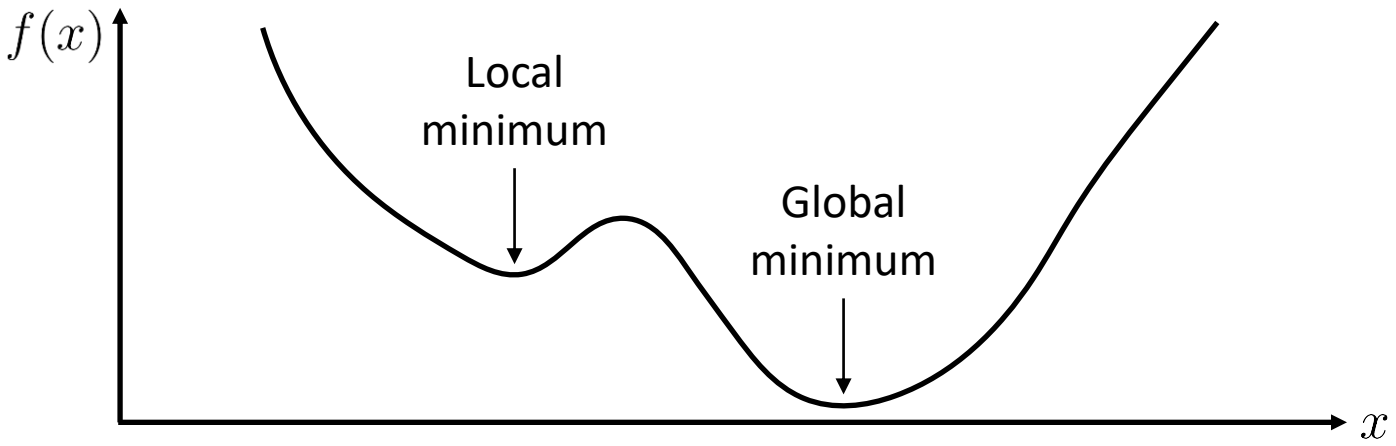
- Error in image point measurements



Estimation of essential matrix

$$\hat{\mathbf{x}}_i = \hat{\mathrm{P}}\mathbf{X}_i \,\forall\, i \text{ and } \hat{\mathbf{x}}'_i = \hat{\mathrm{P}}'\mathbf{X}_i \,\forall\, i, \text{ set } \hat{\mathrm{P}} = [\mathrm{I} \,|\, \mathbf{0}] \text{ and solve for } \hat{\mathrm{P}}'$$

- Given point correspondences $\hat{\mathbf{x}}_i \leftrightarrow \hat{\mathbf{x}}'_i$
- Minimize geometric error $\sum_i \left(d(\hat{\mathbf{x}}_i, \hat{\mathrm{P}}\mathbf{X}_i)^2 + d(\hat{\mathbf{x}}'_i, \hat{\mathrm{P}}'\mathbf{X}_i)^2 \right)$
- A nonlinear mapping requires a nonlinear optimization problem solver
  - Use linear estimation for initial estimate (to mitigate converging to local optimum)
  - Iterative process to determine global optimum

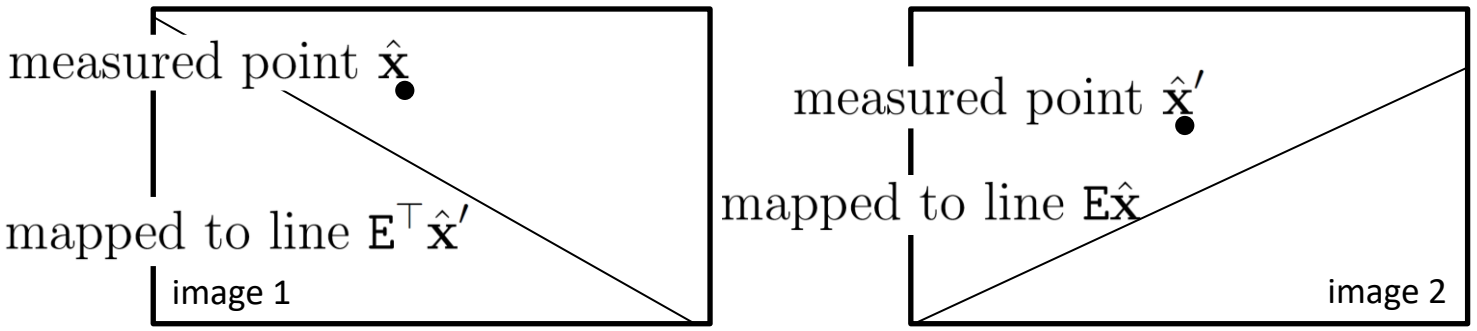


Estimation of essential matrix, linear

Linear estimation of essential matrix using the direct linear transformation (DLT) algorithm

- Minimize algebraic error

$$\hat{\mathbf{x}}'^{\top}\mathbf{E}\hat{\mathbf{x}} = 0 \qquad \text{Epipolar constraint}$$



Linear estimation of essential matrix using the direct linear transformation (DLT) algorithm

$$\hat{\mathbf{x}}'^{\top}\mathtt{E}\hat{\mathbf{x}} = 0$$

$$\hat{\mathbf{x}}'^{\top}\begin{bmatrix}\mathbf{e}^{1\top}\\ \mathbf{e}^{2\top}\\ \mathbf{e}^{3\top}\end{bmatrix}\hat{\mathbf{x}} = 0, \text{ where } \mathtt{E} = \begin{bmatrix}\mathbf{e}^{1\top}\\ \mathbf{e}^{2\top}\\ \mathbf{e}^{3\top}\end{bmatrix}$$

$$\begin{bmatrix}\hat{x}' & \hat{y}' & \hat{w}'\end{bmatrix}\begin{bmatrix}\mathbf{e}^{1\top}\hat{\mathbf{x}}\\ \mathbf{e}^{2\top}\hat{\mathbf{x}}\\ \mathbf{e}^{3\top}\hat{\mathbf{x}}\end{bmatrix} = 0$$

$$\hat{x}'\mathbf{e}^{1\top}\hat{\mathbf{x}} + \hat{y}'\mathbf{e}^{2\top}\hat{\mathbf{x}} + \hat{w}'\mathbf{e}^{3\top}\hat{\mathbf{x}} = 0$$

$$\begin{bmatrix}\hat{x}'\hat{\mathbf{x}}^{\top} & \hat{y}'\hat{\mathbf{x}}^{\top} & \hat{w}'\hat{\mathbf{x}}^{\top}\end{bmatrix}\begin{bmatrix}\mathbf{e}^{1}\\ \mathbf{e}^{2}\\ \mathbf{e}^{3}\end{bmatrix} = 0$$

$$\left(\hat{\mathbf{x}}'^{\top} \otimes \hat{\mathbf{x}}^{\top}\right)\mathbf{e} = 0, \text{ where } \otimes \text{ denotes the Kronecker product}$$

$$\text{and } \mathbf{e} = \text{vec}(\mathtt{E}^{\top}) = \begin{bmatrix}\mathbf{e}^{1}\\ \mathbf{e}^{2}\\ \mathbf{e}^{3}\end{bmatrix}$$

Linear estimation of essential matrix using the direct linear transformation (DLT) algorithm

Given $n \geq 8$ point correspondences $\hat{\mathbf{x}}_i \leftrightarrow \hat{\mathbf{x}}'_i$

$$\hat{\mathbf{x}}_i'^{\top} \mathtt{E} \hat{\mathbf{x}}_i = 0 \, \forall \, i, \text{ solve for } \mathtt{E}$$

$$\left(\hat{\mathbf{x}}_i'^{\top} \otimes \hat{\mathbf{x}}_i^{\top}\right) \mathbf{e} = 0 \, \forall \, i, \text{ solve for } \mathbf{e} = \mathrm{vec}(\mathtt{E}^{\top})$$

$$\begin{bmatrix} \hat{\mathbf{x}}_1'^{\top} \otimes \hat{\mathbf{x}}_1^{\top} \\ \hat{\mathbf{x}}_2'^{\top} \otimes \hat{\mathbf{x}}_2^{\top} \\ \vdots \\ \hat{\mathbf{x}}_n'^{\top} \otimes \hat{\mathbf{x}}_n^{\top} \end{bmatrix} \mathbf{e} = \mathbf{0}$$

$$\mathtt{A}\mathbf{e} = \mathbf{0}, \text{ where } \mathtt{A} = \begin{bmatrix} \hat{\mathbf{x}}_1'^{\top} \otimes \hat{\mathbf{x}}_1^{\top} \\ \hat{\mathbf{x}}_2'^{\top} \otimes \hat{\mathbf{x}}_2^{\top} \\ \vdots \\ \hat{\mathbf{x}}_n'^{\top} \otimes \hat{\mathbf{x}}_n^{\top} \end{bmatrix}$$

Then, enforce constraint (using SVD)

$$\mathtt{E} = \mathtt{U}\Sigma\mathtt{V}^{\top}, \text{ where } \Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \sigma_3)$$

$$\mathtt{E} = \mathtt{U}\Sigma'\mathtt{V}^{\top}, \text{ where } \Sigma' = \mathrm{diag}(1, 1, 0)$$

Linear estimation of essential matrix using the direct linear transformation (DLT) algorithm

Given $n \geq 9$ point correspondences $\hat{\mathbf{x}}_i \leftrightarrow \hat{\mathbf{x}}'_i$

$$\mathtt{A}\mathbf{e} = \mathbf{0}, \text{ where } \mathtt{A} = \begin{bmatrix} \hat{\mathbf{x}}'^{\top}_1 \otimes \hat{\mathbf{x}}^{\top}_1 \\ \hat{\mathbf{x}}'^{\top}_2 \otimes \hat{\mathbf{x}}^{\top}_2 \\ \vdots \\ \hat{\mathbf{x}}'^{\top}_n \otimes \hat{\mathbf{x}}^{\top}_n \end{bmatrix}$$

In general, due to error in measurements, the (right) null space of the matrix $\mathtt{A}$ will be empty (i.e., $\mathtt{A}$ will be full rank). Instead, solve for the vector $\mathbf{e}$ such that $\|\mathtt{A}\mathbf{e}\|$ is minimized.

$$\mathtt{A} = \mathtt{U}\Sigma\mathtt{V}^{\top}$$

The solution vector $\mathbf{e}$ is the column of $\mathtt{V}$ corresponding to the smallest singular value (i.e., the last column of $\mathtt{V}$ (or last row of $\mathtt{V}^{\top}$)).
Then, enforce constraint (using SVD)

Linear estimation of essential matrix using the direct linear transformation (DLT) algorithm

Given $n = 8$ point correspondences $\hat{\mathbf{x}}_i \leftrightarrow \hat{\mathbf{x}}'_i$

$$\mathtt{A}\mathbf{e} = \mathbf{0}, \text{ where } \mathtt{A} = \begin{bmatrix} \hat{\mathbf{x}}'^{\top}_1 \otimes \hat{\mathbf{x}}^{\top}_1 \\ \hat{\mathbf{x}}'^{\top}_2 \otimes \hat{\mathbf{x}}^{\top}_2 \\ \hat{\mathbf{x}}'^{\top}_3 \otimes \hat{\mathbf{x}}^{\top}_3 \\ \hat{\mathbf{x}}'^{\top}_4 \otimes \hat{\mathbf{x}}^{\top}_4 \\ \hat{\mathbf{x}}'^{\top}_5 \otimes \hat{\mathbf{x}}^{\top}_5 \\ \hat{\mathbf{x}}'^{\top}_6 \otimes \hat{\mathbf{x}}^{\top}_6 \\ \hat{\mathbf{x}}'^{\top}_7 \otimes \hat{\mathbf{x}}^{\top}_7 \\ \hat{\mathbf{x}}'^{\top}_8 \otimes \hat{\mathbf{x}}^{\top}_8 \end{bmatrix}$$

The (right) null space of the matrix $\mathtt{A}$ is the vector $\mathbf{e}$.

$$\mathtt{A} = \mathtt{U}\Sigma\mathtt{V}^{\top}$$

The solution vector $\mathbf{e}$ is the column of $\mathtt{V}$ not corresponding to a singular value (i.e., the last column of $\mathtt{V}$ (or last row of $\mathtt{V}^{\top}$)).
Then, enforce constraint (using SVD)

Linear estimation of essential matrix using the direct linear transformation (DLT) algorithm

Given $n = 7$ point correspondences $\hat{\mathbf{x}}_i \leftrightarrow \hat{\mathbf{x}}'_i$

$$\hat{\mathbf{x}}_i'^{\top} \mathtt{E} \hat{\mathbf{x}}_i = 0 \text{ for } i = 1, \ldots, 7, \text{ solve for } \mathtt{E}$$

$$\left(\hat{\mathbf{x}}_i'^{\top} \otimes \hat{\mathbf{x}}_i^{\top}\right) \mathbf{e} = 0 \text{ for } i = 1, \ldots, 7, \text{ solve for } \mathbf{e} = \mathrm{vec}(\mathtt{E}^{\top})$$

$$\begin{bmatrix} \hat{\mathbf{x}}_1'^{\top} \otimes \hat{\mathbf{x}}_1^{\top} \\ \hat{\mathbf{x}}_2'^{\top} \otimes \hat{\mathbf{x}}_2^{\top} \\ \hat{\mathbf{x}}_3'^{\top} \otimes \hat{\mathbf{x}}_3^{\top} \\ \hat{\mathbf{x}}_4'^{\top} \otimes \hat{\mathbf{x}}_4^{\top} \\ \hat{\mathbf{x}}_5'^{\top} \otimes \hat{\mathbf{x}}_5^{\top} \\ \hat{\mathbf{x}}_6'^{\top} \otimes \hat{\mathbf{x}}_6^{\top} \\ \hat{\mathbf{x}}_7'^{\top} \otimes \hat{\mathbf{x}}_7^{\top} \end{bmatrix} \begin{bmatrix} e_{1,1} & e_{2,1} \\ e_{1,2} & e_{2,2} \\ e_{1,3} & e_{2,3} \\ e_{1,4} & e_{2,4} \\ e_{1,5} & e_{2,5} \\ e_{1,6} & e_{2,6} \\ e_{1,7} & e_{2,7} \\ e_{1,8} & e_{2,8} \\ e_{1,9} & e_{2,9} \end{bmatrix} = \mathtt{0}$$

$$\mathtt{A} \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix} = \mathtt{0}, \text{ where } \mathtt{A} = \begin{bmatrix} \hat{\mathbf{x}}_1'^{\top} \otimes \hat{\mathbf{x}}_1^{\top} \\ \hat{\mathbf{x}}_2'^{\top} \otimes \hat{\mathbf{x}}_2^{\top} \\ \vdots \\ \hat{\mathbf{x}}_7'^{\top} \otimes \hat{\mathbf{x}}_7^{\top} \end{bmatrix},$$

$\mathbf{e}_1 = (e_{1,1}, e_{1,2}, e_{1,3}, e_{1,4}, e_{1,5}, e_{1,6}, e_{1,7}, e_{1,8}, e_{1,9})^{\top}$, and
$\mathbf{e}_2 = (e_{2,1}, e_{2,2}, e_{2,3}, e_{2,4}, e_{2,5}, e_{2,6}, e_{2,7}, e_{2,8}, e_{2,9})^{\top}$

Linear estimation of essential matrix using the direct linear transformation (DLT) algorithm

$$\mathtt{A}\begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2\end{bmatrix} = \mathtt{0}, \text{ where } \mathtt{A} = \begin{bmatrix} \hat{\mathbf{x}}_1'^{\top} \otimes \hat{\mathbf{x}}_1^{\top} \\ \hat{\mathbf{x}}_2'^{\top} \otimes \hat{\mathbf{x}}_2^{\top} \\ \hat{\mathbf{x}}_3'^{\top} \otimes \hat{\mathbf{x}}_3^{\top} \\ \hat{\mathbf{x}}_4'^{\top} \otimes \hat{\mathbf{x}}_4^{\top} \\ \hat{\mathbf{x}}_5'^{\top} \otimes \hat{\mathbf{x}}_5^{\top} \\ \hat{\mathbf{x}}_6'^{\top} \otimes \hat{\mathbf{x}}_6^{\top} \\ \hat{\mathbf{x}}_7'^{\top} \otimes \hat{\mathbf{x}}_7^{\top} \end{bmatrix}$$

The (right) null space of the matrix $\mathtt{A}$ is the matrix $\begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2\end{bmatrix}$.

$$\mathtt{A} = \mathtt{U}\Sigma\mathtt{V}^{\top}$$

The solution matrix $\begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2\end{bmatrix}$ are the columns of $\mathtt{V}$ not corresponding to singular values (i.e., the last two columns of $\mathtt{V}$ (or last two rows of $\mathtt{V}^{\top}$)).

Linear estimation of essential matrix using the direct linear transformation (DLT) algorithm

$$\mathbf{e}_1 = (e_{1,1}, e_{1,2}, e_{1,3}, e_{1,4}, e_{1,5}, e_{1,6}, e_{1,7}, e_{1,8}, e_{1,9})^\top, \text{ and}$$
$$\mathbf{e}_2 = (e_{2,1}, e_{2,2}, e_{2,3}, e_{2,4}, e_{2,5}, e_{2,6}, e_{2,7}, e_{2,8}, e_{2,9})^\top$$

$$\mathtt{E}_1 = \begin{bmatrix} e_{1,1} & e_{1,2} & e_{1,3} \\ e_{1,4} & e_{1,5} & e_{1,6} \\ e_{1,7} & e_{1,8} & e_{1,9} \end{bmatrix} \text{ and } \mathtt{E}_2 = \begin{bmatrix} e_{2,1} & e_{2,2} & e_{2,3} \\ e_{2,4} & e_{2,5} & e_{2,6} \\ e_{2,7} & e_{2,8} & e_{2,9} \end{bmatrix}$$

$$\mathtt{E} = \alpha \mathtt{E}_1 + \mathtt{E}_2$$

$$\det(\mathtt{E}) = 0, \text{ solve for } \alpha$$

3 solutions α_1, α_2, and α_3 (1 or 3 will be real solutions, i.e., the real roots of the cubic polynomial)
Then substitute α_i to solve for $\mathtt{E}_i$

$$\mathtt{E}_i = \alpha_i \mathtt{E}_1 + \mathtt{E}_2 \qquad \text{(1 or 3 real solutions)}$$

Then, enforce constraint (using SVD)

$$\mathtt{E}_i = \mathtt{U}\Sigma\mathtt{V}^\top, \text{ where } \Sigma = \text{diag}(\sigma_1, \sigma_2, \sigma_3)$$
$$\mathtt{E}_i = \mathtt{U}\Sigma'\mathtt{V}^\top, \text{ where } \Sigma' = \text{diag}(1, 1, 0)$$

Linear estimation of essential matrix using the direct linear transformation (DLT) algorithm

Given $n = 6$ point correspondences $\hat{\mathbf{x}}_i \leftrightarrow \hat{\mathbf{x}}'_i$

$$\hat{\mathbf{x}}_i'^{\top} \mathtt{E} \hat{\mathbf{x}}_i = 0 \text{ for } i = 1, \ldots, 6, \text{ solve for } \mathtt{E}$$

$$\left(\hat{\mathbf{x}}_i'^{\top} \otimes \hat{\mathbf{x}}_i^{\top}\right) \mathbf{e} = 0 \text{ for } i = 1, \ldots, 6, \text{ solve for } \mathbf{e} = \mathrm{vec}(\mathtt{E}^{\top})$$

$$\begin{bmatrix} \hat{\mathbf{x}}_1'^{\top} \otimes \hat{\mathbf{x}}_1^{\top} \\ \hat{\mathbf{x}}_2'^{\top} \otimes \hat{\mathbf{x}}_2^{\top} \\ \hat{\mathbf{x}}_3'^{\top} \otimes \hat{\mathbf{x}}_3^{\top} \\ \hat{\mathbf{x}}_4'^{\top} \otimes \hat{\mathbf{x}}_4^{\top} \\ \hat{\mathbf{x}}_5'^{\top} \otimes \hat{\mathbf{x}}_5^{\top} \\ \hat{\mathbf{x}}_6'^{\top} \otimes \hat{\mathbf{x}}_6^{\top} \end{bmatrix} \begin{bmatrix} e_{1,1} & e_{2,1} & e_{3,1} \\ e_{1,2} & e_{2,2} & e_{3,2} \\ e_{1,3} & e_{2,3} & e_{3,3} \\ e_{1,4} & e_{2,4} & e_{3,4} \\ e_{1,5} & e_{2,5} & e_{3,5} \\ e_{1,6} & e_{2,6} & e_{3,6} \\ e_{1,7} & e_{2,7} & e_{3,7} \\ e_{1,8} & e_{2,8} & e_{3,8} \\ e_{1,9} & e_{2,9} & e_{3,9} \end{bmatrix} = \mathtt{0}$$

$$\mathtt{A} \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} = \mathtt{0}, \text{ where } \mathtt{A} = \begin{bmatrix} \hat{\mathbf{x}}_1'^{\top} \otimes \hat{\mathbf{x}}_1^{\top} \\ \hat{\mathbf{x}}_2'^{\top} \otimes \hat{\mathbf{x}}_2^{\top} \\ \vdots \\ \hat{\mathbf{x}}_6'^{\top} \otimes \hat{\mathbf{x}}_6^{\top} \end{bmatrix},$$

$$\mathbf{e}_1 = (e_{1,1}, e_{1,2}, e_{1,3}, e_{1,4}, e_{1,5}, e_{1,6}, e_{1,7}, e_{1,8}, e_{1,9})^{\top},$$
$$\mathbf{e}_2 = (e_{2,1}, e_{2,2}, e_{2,3}, e_{2,4}, e_{2,5}, e_{2,6}, e_{2,7}, e_{2,8}, e_{2,9})^{\top}, \text{ and}$$
$$\mathbf{e}_3 = (e_{3,1}, e_{3,2}, e_{3,3}, e_{3,4}, e_{3,5}, e_{3,6}, e_{3,7}, e_{3,8}, e_{3,9})^{\top}$$

Linear estimation of essential matrix using the direct linear transformation (DLT) algorithm

$$\mathtt{A}\begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\end{bmatrix} = \mathtt{0}, \text{ where } \mathtt{A} = \begin{bmatrix} \hat{\mathbf{x}}_1'^{\top} \otimes \hat{\mathbf{x}}_1^{\top} \\ \hat{\mathbf{x}}_2'^{\top} \otimes \hat{\mathbf{x}}_2^{\top} \\ \hat{\mathbf{x}}_3'^{\top} \otimes \hat{\mathbf{x}}_3^{\top} \\ \hat{\mathbf{x}}_4'^{\top} \otimes \hat{\mathbf{x}}_4^{\top} \\ \hat{\mathbf{x}}_5'^{\top} \otimes \hat{\mathbf{x}}_5^{\top} \\ \hat{\mathbf{x}}_6'^{\top} \otimes \hat{\mathbf{x}}_6^{\top} \end{bmatrix}$$

The (right) null space of the matrix $\mathtt{A}$ is the matrix $\begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\end{bmatrix}$.

$$\mathtt{A} = \mathtt{U}\Sigma\mathtt{V}^{\top}$$

The solution matrix $\begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\end{bmatrix}$ are the columns of $\mathtt{V}$ not corresponding to singular values (i.e., the last three columns of $\mathtt{V}$ (or last three rows of $\mathtt{V}^{\top}$)).

Linear estimation of essential matrix using the direct linear transformation (DLT) algorithm

$$\mathbf{e}_1 = (e_{1,1}, e_{1,2}, e_{1,3}, e_{1,4}, e_{1,5}, e_{1,6}, e_{1,7}, e_{1,8}, e_{1,9})^\top,$$
$$\mathbf{e}_2 = (e_{2,1}, e_{2,2}, e_{2,3}, e_{2,4}, e_{2,5}, e_{2,6}, e_{2,7}, e_{2,8}, e_{2,9})^\top, \text{ and}$$
$$\mathbf{e}_3 = (e_{3,1}, e_{3,2}, e_{3,3}, e_{3,4}, e_{3,5}, e_{3,6}, e_{3,7}, e_{3,8}, e_{3,9})^\top$$

$$\mathtt{E}_1 = \begin{bmatrix} e_{1,1} & e_{1,2} & e_{1,3} \\ e_{1,4} & e_{1,5} & e_{1,6} \\ e_{1,7} & e_{1,8} & e_{1,9} \end{bmatrix}, \mathtt{E}_2 = \begin{bmatrix} e_{2,1} & e_{2,2} & e_{2,3} \\ e_{2,4} & e_{2,5} & e_{2,6} \\ e_{2,7} & e_{2,8} & e_{2,9} \end{bmatrix}, \text{ and } \mathtt{E}_3 = \begin{bmatrix} e_{3,1} & e_{3,2} & e_{3,3} \\ e_{3,4} & e_{3,5} & e_{3,6} \\ e_{3,7} & e_{3,8} & e_{3,9} \end{bmatrix}$$

$$\mathtt{E} = \alpha_1 \mathtt{E}_1 + \alpha_2 \mathtt{E}_2 + \mathtt{E}_3$$

The Kruppa equations (9 equations)

$$\mathtt{E}\mathtt{E}^\top \mathtt{E} - \frac{1}{2}\mathrm{Tr}(\mathtt{E}\mathtt{E}^\top)\mathtt{E} = \mathtt{0}, \text{ solve for } \alpha_1 \text{ and } \alpha_2$$
$$\mathtt{M}\mathbf{v} = \mathtt{0}, \text{ solve for } \mathbf{v}$$

where $\mathtt{M}$ is a 9×10 matrix of coefficients and

$$\mathbf{v} = (\alpha_1^3, \alpha_2^3, \alpha_1^2\alpha_2, \alpha_1\alpha_2^2, \alpha_1^2, \alpha_2^2, \alpha_1\alpha_2, \alpha_1, \alpha_2, 1)^\top \text{ to nonzero scale}$$

Then substitute to solve for $\mathtt{E}$

$$\mathtt{E} = \alpha_1 \mathtt{E}_1 + \alpha_2 \mathtt{E}_2 + \mathtt{E}_3$$

Then, enforce constraint (using SVD)

$$\mathtt{E} = \mathtt{U}\Sigma\mathtt{V}^\top, \text{ where } \Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \sigma_3)$$
$$\mathtt{E} = \mathtt{U}\Sigma'\mathtt{V}^\top, \text{ where } \Sigma' = \mathrm{diag}(1, 1, 0)$$

Linear estimation of essential matrix using the direct linear transformation (DLT) algorithm

Given $n = 5$ point correspondences $\hat{\mathbf{x}}_i \leftrightarrow \hat{\mathbf{x}}'_i$

$$\hat{\mathbf{x}}_i'^{\top} \mathtt{E} \hat{\mathbf{x}}_i = 0 \text{ for } i = 1, \ldots, 5, \text{ solve for } \mathtt{E}$$

$$\left(\hat{\mathbf{x}}_i'^{\top} \otimes \hat{\mathbf{x}}_i^{\top}\right) \mathbf{e} = 0 \text{ for } i = 1, \ldots, 5, \text{ solve for } \mathbf{e} = \mathrm{vec}(\mathtt{E}^{\top})$$

$$\begin{bmatrix} \hat{\mathbf{x}}_1'^{\top} \otimes \hat{\mathbf{x}}_1^{\top} \\ \hat{\mathbf{x}}_2'^{\top} \otimes \hat{\mathbf{x}}_2^{\top} \\ \hat{\mathbf{x}}_3'^{\top} \otimes \hat{\mathbf{x}}_3^{\top} \\ \hat{\mathbf{x}}_4'^{\top} \otimes \hat{\mathbf{x}}_4^{\top} \\ \hat{\mathbf{x}}_5'^{\top} \otimes \hat{\mathbf{x}}_5^{\top} \end{bmatrix} \begin{bmatrix} e_{1,1} & e_{2,1} & e_{3,1} & e_{4,1} \\ e_{1,2} & e_{2,2} & e_{3,2} & e_{4,2} \\ e_{1,3} & e_{2,3} & e_{3,3} & e_{4,3} \\ e_{1,4} & e_{2,4} & e_{3,4} & e_{4,4} \\ e_{1,5} & e_{2,5} & e_{3,5} & e_{4,5} \\ e_{1,6} & e_{2,6} & e_{4,6} & e_{3,6} \\ e_{1,7} & e_{2,7} & e_{3,7} & e_{4,7} \\ e_{1,8} & e_{2,8} & e_{3,8} & e_{4,8} \\ e_{1,9} & e_{2,9} & e_{3,9} & e_{4,9} \end{bmatrix} = \mathtt{0}$$

$$\mathtt{A} \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \end{bmatrix} = \mathtt{0}, \text{ where } \mathtt{A} = \begin{bmatrix} \hat{\mathbf{x}}_1'^{\top} \otimes \hat{\mathbf{x}}_1^{\top} \\ \hat{\mathbf{x}}_2'^{\top} \otimes \hat{\mathbf{x}}_2^{\top} \\ \vdots \\ \hat{\mathbf{x}}_5'^{\top} \otimes \hat{\mathbf{x}}_5^{\top} \end{bmatrix},$$

$$\mathbf{e}_1 = (e_{1,1}, e_{1,2}, e_{1,3}, e_{1,4}, e_{1,5}, e_{1,6}, e_{1,7}, e_{1,8}, e_{1,9})^{\top},$$
$$\mathbf{e}_2 = (e_{2,1}, e_{2,2}, e_{2,3}, e_{2,4}, e_{2,5}, e_{2,6}, e_{2,7}, e_{2,8}, e_{2,9})^{\top},$$
$$\mathbf{e}_3 = (e_{3,1}, e_{3,2}, e_{3,3}, e_{3,4}, e_{3,5}, e_{3,6}, e_{3,7}, e_{3,8}, e_{3,9})^{\top}, \text{ and}$$
$$\mathbf{e}_4 = (e_{4,1}, e_{4,2}, e_{4,3}, e_{4,4}, e_{4,5}, e_{4,6}, e_{4,7}, e_{4,8}, e_{4,9})^{\top}$$

Linear estimation of essential matrix using the direct linear transformation (DLT) algorithm

$$\mathtt{A}\begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4\end{bmatrix} = \mathtt{0}, \text{ where } \mathtt{A} = \begin{bmatrix}\hat{\mathbf{x}}_1'^{\top} \otimes \hat{\mathbf{x}}_1^{\top} \\ \hat{\mathbf{x}}_2'^{\top} \otimes \hat{\mathbf{x}}_2^{\top} \\ \hat{\mathbf{x}}_3'^{\top} \otimes \hat{\mathbf{x}}_3^{\top} \\ \hat{\mathbf{x}}_4'^{\top} \otimes \hat{\mathbf{x}}_4^{\top} \\ \hat{\mathbf{x}}_5'^{\top} \otimes \hat{\mathbf{x}}_5^{\top}\end{bmatrix}$$

The (right) null space of the matrix $\mathtt{A}$ is the matrix $\begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4\end{bmatrix}$.

$$\mathtt{A} = \mathtt{U}\Sigma\mathtt{V}^{\top}$$

The solution matrix $\begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4\end{bmatrix}$ are the columns of $\mathtt{V}$ not corresponding to singular values (i.e., the last four columns of $\mathtt{V}$ (or last four rows of $\mathtt{V}^{\top}$)).

Linear estimation of essential matrix using the direct linear transformation (DLT) algorithm

$$\mathbf{e}_1 = (e_{1,1}, e_{1,2}, e_{1,3}, e_{1,4}, e_{1,5}, e_{1,6}, e_{1,7}, e_{1,8}, e_{1,9})^\top,$$
$$\mathbf{e}_2 = (e_{2,1}, e_{2,2}, e_{2,3}, e_{2,4}, e_{2,5}, e_{2,6}, e_{2,7}, e_{2,8}, e_{2,9})^\top,$$
$$\mathbf{e}_3 = (e_{3,1}, e_{3,2}, e_{3,3}, e_{3,4}, e_{3,5}, e_{3,6}, e_{3,7}, e_{3,8}, e_{3,9})^\top, \text{ and}$$
$$\mathbf{e}_4 = (e_{4,1}, e_{4,2}, e_{4,3}, e_{4,4}, e_{4,5}, e_{4,6}, e_{4,7}, e_{4,8}, e_{4,9})^\top$$

$$\mathtt{E}_1 = \begin{bmatrix} e_{1,1} & e_{1,2} & e_{1,3} \\ e_{1,4} & e_{1,5} & e_{1,6} \\ e_{1,7} & e_{1,8} & e_{1,9} \end{bmatrix}, \mathtt{E}_2 = \begin{bmatrix} e_{2,1} & e_{2,2} & e_{2,3} \\ e_{2,4} & e_{2,5} & e_{2,6} \\ e_{2,7} & e_{2,8} & e_{2,9} \end{bmatrix}, \mathtt{E}_3 = \begin{bmatrix} e_{3,1} & e_{3,2} & e_{3,3} \\ e_{3,4} & e_{3,5} & e_{3,6} \\ e_{3,7} & e_{3,8} & e_{3,9} \end{bmatrix}, \text{ and}$$

$$\mathtt{E}_4 = \begin{bmatrix} e_{4,1} & e_{4,2} & e_{4,3} \\ e_{4,4} & e_{4,5} & e_{4,6} \\ e_{4,7} & e_{4,8} & e_{4,9} \end{bmatrix}$$

$$\mathtt{E} = \alpha_1 \mathtt{E}_1 + \alpha_2 \mathtt{E}_2 + \alpha_3 \mathtt{E}_3 + \mathtt{E}_4$$

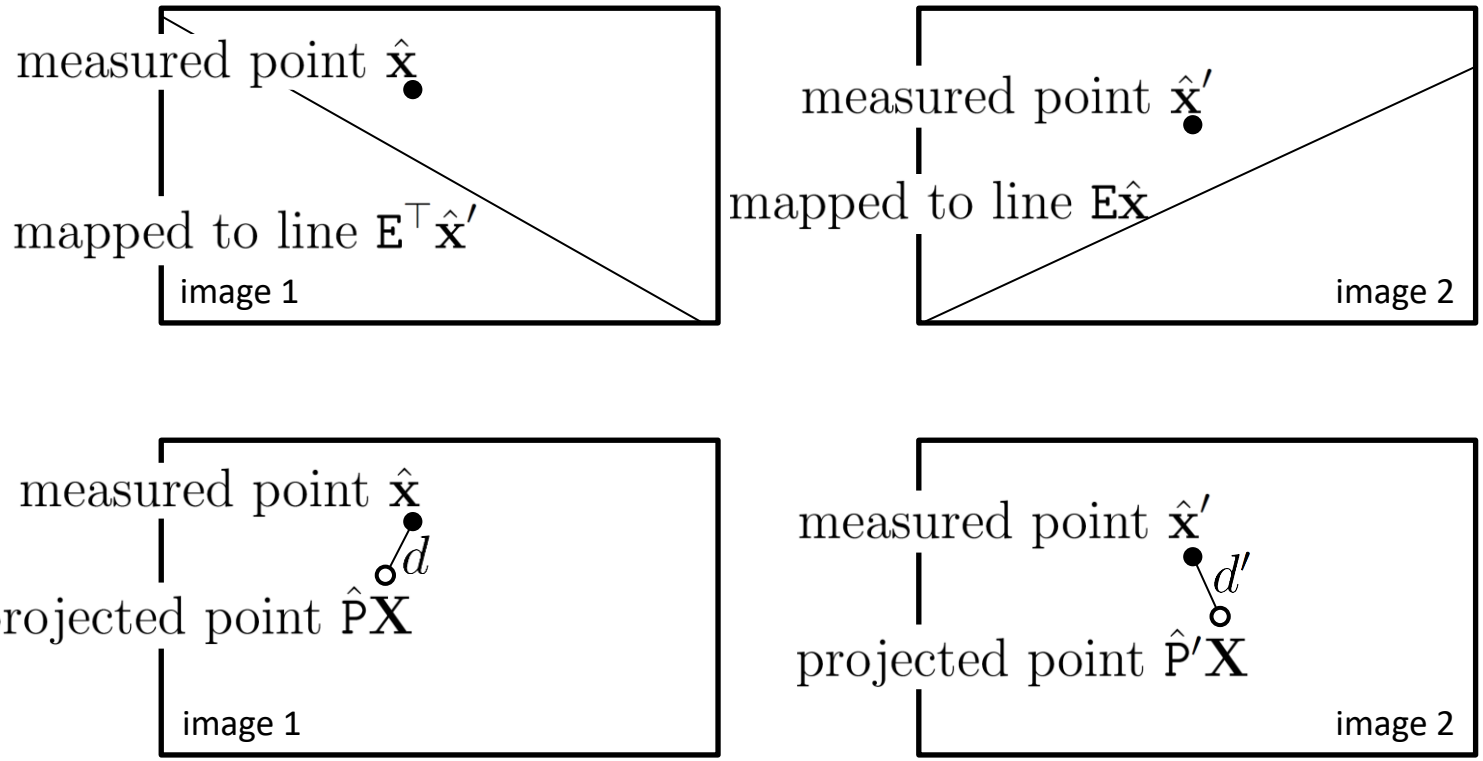
$$\det(\mathtt{E}) = 0 \text{ and } \mathtt{E}\mathtt{E}^\top\mathtt{E} - \frac{1}{2}\operatorname{Tr}(\mathtt{E}\mathtt{E}^\top)\mathtt{E} = \mathtt{0} \text{ (10 equations), solve for } \alpha_1, \alpha_2 \text{ and } \alpha_3$$

There are up to 10 real solutions for the essential matrix. Requires calculating the solution to a polynomial eigenvalue problem (see Kukelova et al. for details).

Sampson correction and Sampson error

Essential matrix

- Error in image point measurements



Sampson cost

Sampson error uses a cost function that lies between algebraic and geometric cost functions (in terms of complexity) and gives a close approximation to geometric error.
Measurement vector $\mathbf{X} = (\hat{\hat{x}}, \hat{\hat{y}}, \hat{\hat{x}}', \hat{\hat{y}}')^\top$
$\mathcal{C}_{\mathtt{E}}(\mathbf{X})$ is the cost associated with $\mathbf{X}$

$$\mathcal{C}_{\mathtt{E}}(\mathbf{X}) = 0$$

$$\mathcal{C}_{\mathtt{E}}(\mathbf{X} + \boldsymbol{\delta}_{\mathbf{X}}) = \mathcal{C}_{\mathtt{E}}(\mathbf{X}) + \frac{\mathrm{d}\mathcal{C}_{\mathtt{E}}(\mathbf{X})}{\partial \mathbf{X}}\boldsymbol{\delta}_{\mathbf{X}} \qquad \text{first order Taylor expansion}$$

$$\text{if } \hat{\mathbf{X}} = \mathbf{X} + \boldsymbol{\delta}_{\mathbf{X}}$$

$$\hat{\mathbf{X}} - \mathbf{X} = \boldsymbol{\delta}_{\mathbf{X}} \text{ and desire } \hat{\mathbf{X}} \text{ such that } \mathcal{C}_{\mathtt{E}}(\hat{\mathbf{X}}) = 0, \text{ then}$$

$$\mathcal{C}_{\mathtt{E}}(\mathbf{X}) + \frac{\mathrm{d}\mathcal{C}_{\mathtt{E}}(\mathbf{X})}{\partial \mathbf{X}}\boldsymbol{\delta}_{\mathbf{X}} = \mathbf{0}$$

$$\epsilon + \mathtt{J}\boldsymbol{\delta}_{\mathbf{X}} = 0, \text{ where } \epsilon = \mathcal{C}_{\mathtt{E}}(\mathbf{X}) \text{ and } \mathtt{J} = \frac{\mathrm{d}\mathcal{C}_{\mathtt{E}}(\mathbf{X})}{\partial \mathbf{X}}$$

$$\mathtt{J}\boldsymbol{\delta}_{\mathbf{X}} = -\epsilon, \text{ solve for } \boldsymbol{\delta}_{\mathbf{X}} \text{ that minimizes } \|\boldsymbol{\delta}_{\mathbf{X}}\|$$

Sampson corrected and Sampson error

$$\mathtt{J}\boldsymbol{\delta}_{\mathbf{X}} = -\epsilon, \text{ solve for } \boldsymbol{\delta}_{\mathbf{X}} \text{ that minimizes } \|\boldsymbol{\delta}_{\mathbf{X}}\|$$

Use a Lagrange multiplier λ and find extrema of $\boldsymbol{\delta}_{\mathbf{X}}^{\top}\boldsymbol{\delta}_{\mathbf{X}} - 2\lambda(\mathtt{J}\boldsymbol{\delta}_{\mathbf{X}} + \epsilon)$

$$\mathtt{J}\mathtt{J}^{\top}\lambda = -\epsilon, \text{ solve for } \lambda$$

then substitute λ to solve for $\boldsymbol{\delta}_{\mathbf{X}}$

$$\boldsymbol{\delta}_{\mathbf{X}} = \mathtt{J}^{\top}\lambda$$

Sampson corrected	$\hat{\mathbf{X}} = \mathbf{X} + \boldsymbol{\delta}_{\mathbf{X}}$
Squared sampson error	$\|\boldsymbol{\delta}_{\mathbf{X}}\|^2 = \boldsymbol{\delta}_{\mathbf{X}}^{\top}\boldsymbol{\delta}_{\mathbf{X}}$

Sampson corrected and Sampson error

Determine $\mathtt{J}$ and ϵ

$$\hat{\mathbf{x}}_i'^{\top}\mathtt{E}\hat{\mathbf{x}}_i = 0\,\forall\, i$$

$$\hat{\mathbf{x}}_i'^{\top}\begin{bmatrix}\mathbf{e}^{1\top}\\ \mathbf{e}^{2\top}\\ \mathbf{e}^{3\top}\end{bmatrix}\hat{\mathbf{x}}_i = 0\,\forall\, i, \text{ where } \mathtt{E} = \begin{bmatrix}\mathbf{e}^{1\top}\\ \mathbf{e}^{2\top}\\ \mathbf{e}^{3\top}\end{bmatrix}$$

$$\begin{bmatrix}\hat{x}_i'\hat{\mathbf{x}}_i^{\top} & \hat{y}_i'\hat{\mathbf{x}}_i^{\top} & \hat{w}_i'\hat{\mathbf{x}}_i^{\top}\end{bmatrix}\begin{bmatrix}\mathbf{e}^1\\ \mathbf{e}^2\\ \mathbf{e}^3\end{bmatrix} = 0\,\forall\, i$$

$$\mathtt{A}_i\mathbf{e} = 0\,\forall\, i, \text{ where } \mathtt{A}_i = \begin{bmatrix}\hat{x}_i'\hat{\mathbf{x}}_i^{\top} & \hat{y}_i'\hat{\mathbf{x}}_i^{\top} & \hat{w}_i'\hat{\mathbf{x}}_i^{\top}\end{bmatrix} \text{ and } \mathbf{e} = \mathrm{vec}(\mathtt{E}^{\top}) = \begin{bmatrix}\mathbf{e}^1\\ \mathbf{e}^2\\ \mathbf{e}^3\end{bmatrix}$$

Sampson corrected and Sampson error

$$\mathtt{A}_i\mathbf{e} = \begin{bmatrix}\hat{\tilde{x}}_i'\hat{\mathbf{x}}_i^\top & \hat{\tilde{y}}_i'\hat{\mathbf{x}}_i^\top & \hat{\mathbf{x}}_i^\top\end{bmatrix}\begin{bmatrix}\mathbf{e}^1\\ \mathbf{e}^2\\ \mathbf{e}^3\end{bmatrix}\ \forall\, i$$

$$\mathtt{A}_i\mathbf{e} = \hat{\tilde{x}}_i'\hat{\mathbf{e}}^{1\top}\hat{\mathbf{x}}_i + \hat{\tilde{y}}_i'\mathbf{e}^{2\top}\hat{\mathbf{x}}_i + \mathbf{e}^{3\top}\hat{\mathbf{x}}_i\,\forall\, i$$

$$\mathbf{X}_i = (\hat{\tilde{x}}_i, \hat{\tilde{y}}_i, \hat{\tilde{x}}_i', \hat{\tilde{y}}_i')^\top$$

$$\epsilon_i = \mathcal{C}_{\mathtt{E}}(\mathbf{X}_i) = \mathtt{A}_i\mathbf{e}$$

$$\epsilon_i = \hat{\tilde{x}}_i\hat{\tilde{x}}_i'e_{11} + \hat{\tilde{x}}_i\hat{\tilde{y}}_i'e_{21} + \hat{\tilde{x}}_ie_{31} + \hat{\tilde{y}}_i\hat{\tilde{x}}_i'e_{12} + \hat{\tilde{y}}_i\hat{\tilde{y}}_i'e_{22} + \hat{\tilde{y}}_ie_{32} + \hat{\tilde{x}}_i'e_{13} + \hat{\tilde{y}}_i'e_{23} + e_{33}$$

$$\mathtt{J}_i = \frac{\mathrm{d}\mathcal{C}_{\mathtt{E}}(\mathbf{X}_i)}{\partial\mathbf{X}_i} = \frac{\mathrm{d}(\mathtt{A}_i\mathbf{e})}{\partial(\hat{\tilde{x}}_i, \hat{\tilde{y}}_i, \hat{\tilde{x}}_i', \hat{\tilde{y}}_i')}$$

$$\mathtt{J}_i = \begin{bmatrix}\hat{\tilde{x}}_i'e_{11} + \hat{\tilde{y}}_i'e_{21} + e_{31} & \hat{\tilde{x}}_i'e_{12} + \hat{\tilde{y}}_i'e_{22} + e_{32} & \hat{\tilde{x}}_ie_{11} + \hat{\tilde{y}}_ie_{12} + e_{13} & \hat{\tilde{x}}_ie_{21} + \hat{\tilde{y}}_ie_{22} + e_{23}\end{bmatrix}$$

$\mathtt{J}_i\mathtt{J}_i^\top\lambda_i = -\epsilon_i$, solve for $\lambda_i = -\frac{\epsilon_i}{\mathtt{J}_i\mathtt{J}_i^\top}$, then $\boldsymbol{\delta}_{\mathbf{X}_i} = \mathtt{J}_i^\top\lambda_i$

Sampson corrected points $\quad \hat{\mathbf{X}}_i = (\hat{\hat{\tilde{x}}}_i, \hat{\hat{\tilde{y}}}_i, \hat{\hat{\tilde{x}}}_i', \hat{\hat{\tilde{y}}}_i')^\top = \mathbf{X}_i + \boldsymbol{\delta}_{\mathbf{X}_i}$

Squared Sampson error $\quad \|\boldsymbol{\delta}_{\mathbf{X}_i}\|^2 = \boldsymbol{\delta}_{\mathbf{X}_i}^\top\boldsymbol{\delta}_{\mathbf{X}_i} = \epsilon_i(\mathtt{J}_i\mathtt{J}_i^\top)^{-1}\epsilon_i = \dfrac{\epsilon_i^2}{\mathtt{J}_i\mathtt{J}_i^\top}$

Outlier rejection

Outlier rejection

- Even the presence of a single outlier may result in an inaccurate estimate
- Linear estimate
  - Minimizes sum of squared error and, in general, outliers have large error relative to inliers
  - Resulting estimate may not be near global minimum
- Nonlinear estimate
  - Cost function minimizes sum of squared error, so may not converge to accurate solution
    - Alternatively, use a robust cost function

Random sample consensus (RANSAC) and M-estimator sample consensus (MSAC)

Objective is to determine the consensus set with the minimum cost

tol is the tolerance for establishing datum/model compatibility
τ_{cost} is the upper bound on the cost of an acceptable consensus set
maxTrials is the maximum number of attempts to find a consensus set

$\text{consensus}_{\text{cost}} = \infty$
for (trials $= 0$; trials $<$ maxTrials && $\text{consensus}_{\text{cost}} > \tau_{\text{cost}}$; ++trials)
    Select a random sample of unique data points
    Calculate the model using the random sample
    Calculate the error for each data point using model
    Calculate the cost
    if cost $< \text{consensus}_{\text{cost}}$
        $\text{consensus}_{\text{cost}} = \text{cost}$
        $\text{consensus}_{\text{model}} = \text{model}$
Calculate the error for each data point using $\text{consensus}_{\text{model}}$
Calculate the set of inliers (i.e., data points with error $\leq$ tol)

If multiple solutions, try each one, but it is still a single trial

Outlier rejection

- RANSAC cost

count is the total number of data points
tol is the tolerance for establishing datum/model compatibility

```
cost = 0
for (n = 0; n < count; ++n)
    cost += error[n] ≤ tol ? 0 : 1
```

- MSAC cost

count is the total number of data points
tol is the tolerance for establishing datum/model compatibility

```
cost = 0
for (n = 0; n < count; ++n)
    cost += error[n] ≤ tol ? error[n] : tol
```

RANSAC and MSAC

- Maximum number of trials
  - Adaptive maximum number of trials
- The tolerance for establishing datum/model compatibility

RANSAC and MSAC

- Maximum number of trials

The smaller the sample size, the smaller maximum number of trials. Use minimum solution!

Sample size	Proportion of outliers ϵ						
s	5%	10%	20%	25%	30%	40%	50%
2	2	3	5	6	7	11	17
3	3	4	7	9	11	19	35
4	3	5	9	13	17	34	72
5	4	6	12	17	26	57	146
6	4	7	16	24	37	97	293
7	4	8	20	33	54	163	588
8	5	9	26	44	78	272	1177

Table 4.3. *The number N of samples required to ensure, with a probability $p = 0.99$, that at least one sample has no outliers for a given size of sample, s, and proportion of outliers, ϵ.*

Adaptive maximum number of trials

s is the sample size

p is the assumed probability that at least one of the random samples does not contain any outliers

tol is the tolerance for establishing datum/model compatibility

τ_{cost} is the upper bound on the cost of an acceptable consensus set

$\text{maxTrials} = \infty$

$\text{consensus}_{\text{cost}} = \infty$

for (trials = 0; trials < maxTrials && $\text{consensus}_{\text{cost}} > \tau_{\text{cost}}$; ++trials)

- Select a random sample of unique data points
- Calculate the model using the random sample
- Calculate the error for each data point using model
- Calculate the cost
- if cost < $\text{consensus}_{\text{cost}}$
  - $\text{consensus}_{\text{cost}} = \text{cost}$
  - $\text{consensus}_{\text{model}} = \text{model}$
  - Calculate the number of inliers
  - $w = \dfrac{\text{number of inliers}}{\text{total number of data points}}$
  - $\text{maxTrials} = \dfrac{\log(1-p)}{\log(1-w^s)}$

If multiple solutions, try each one, but it is still a single trial

Calculate the error for each data point using $\text{consensus}_{\text{model}}$

Calculate the set of inliers (i.e., data points with error $\leq$ tol)

RANSAC and MSAC

- The tolerance for establishing datum/model compatibility

Probability α that a data point is an inlier (usually α is chosen to be 0.95)
Variance σ^2 of the measurement error (if unkonwn, assumed to be 1)
Codimension m
Tolerance is square distance $t^2 = F_m^{-1}(\alpha)\sigma^2$, where $F_m^{-1}(\alpha)$ is the inverse chi-squared cumulative distribution function with m degress of freedom at the probability α

- Codimension

If w is a subspace of a finite-dimensional vector space v, then $\text{codim}(w) = \dim(v) - \dim(w)$
The essential matrix is a variety of dimension 3 in $(\hat{\tilde{x}}, \hat{\tilde{y}}, \hat{\tilde{x}}', \hat{\tilde{y}}') \in \mathbb{R}^4$, so has codimension 1

Use the square Sampson error

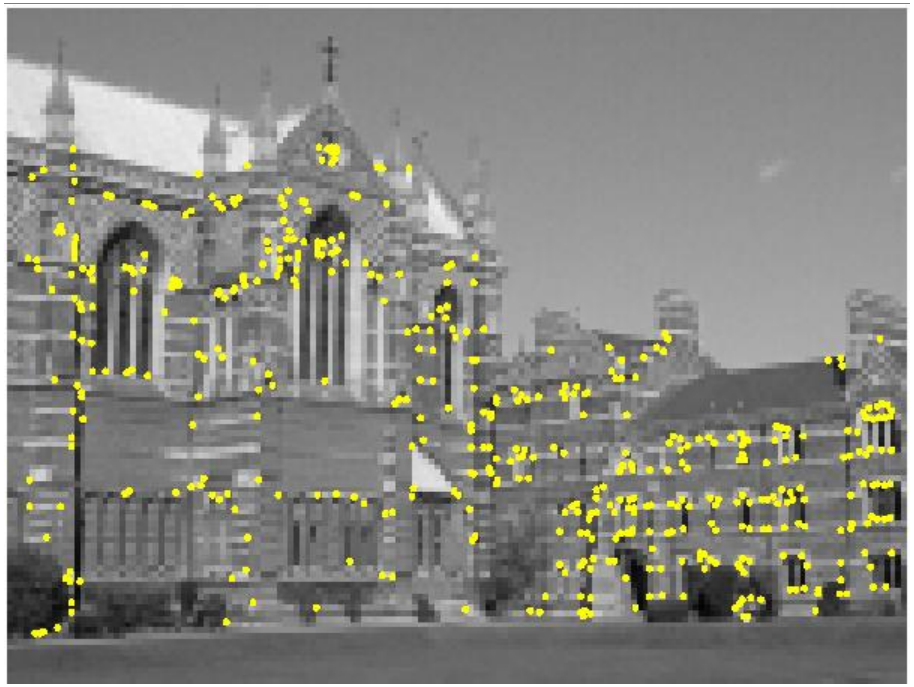
Feature detection and matching

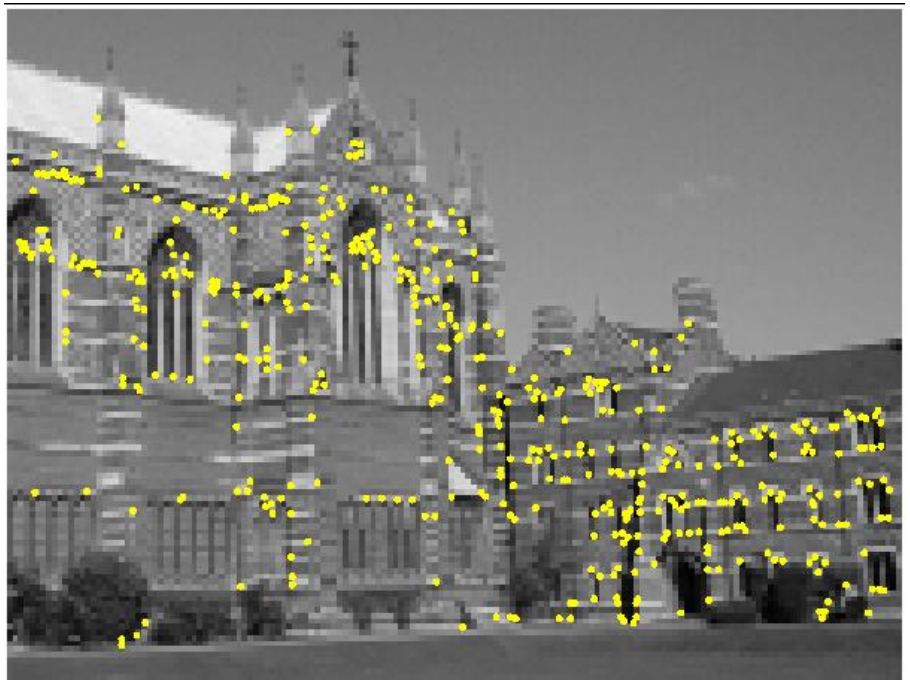




Input Images

Feature detection and matching

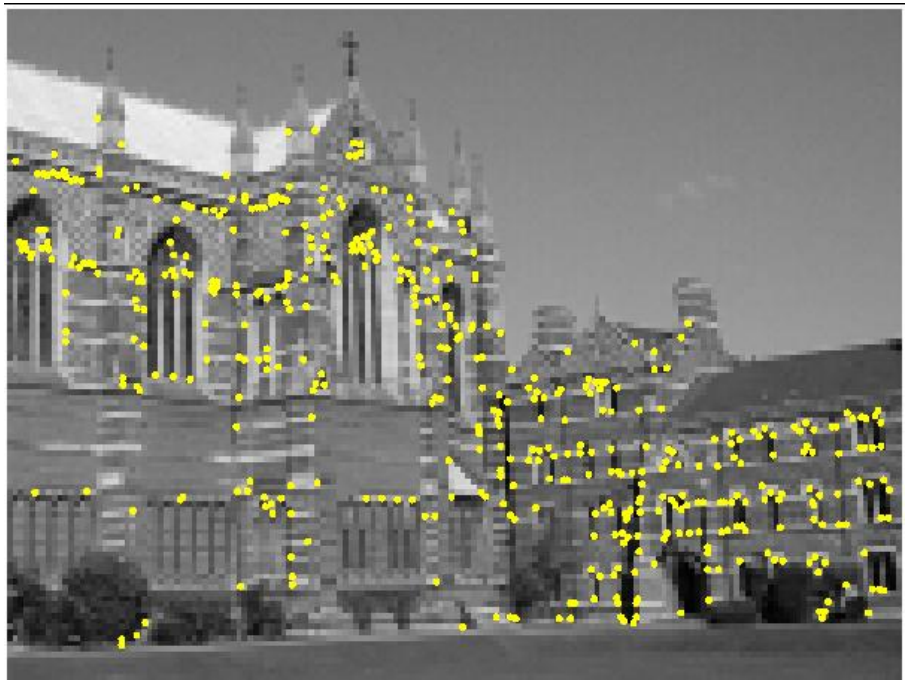




Detected Corners

Feature detection and matching

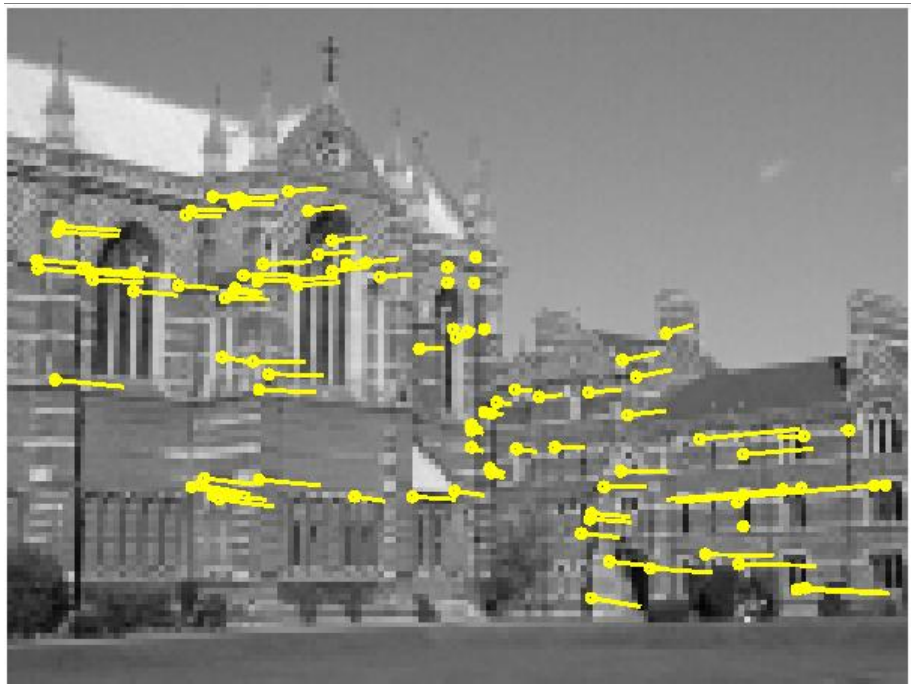


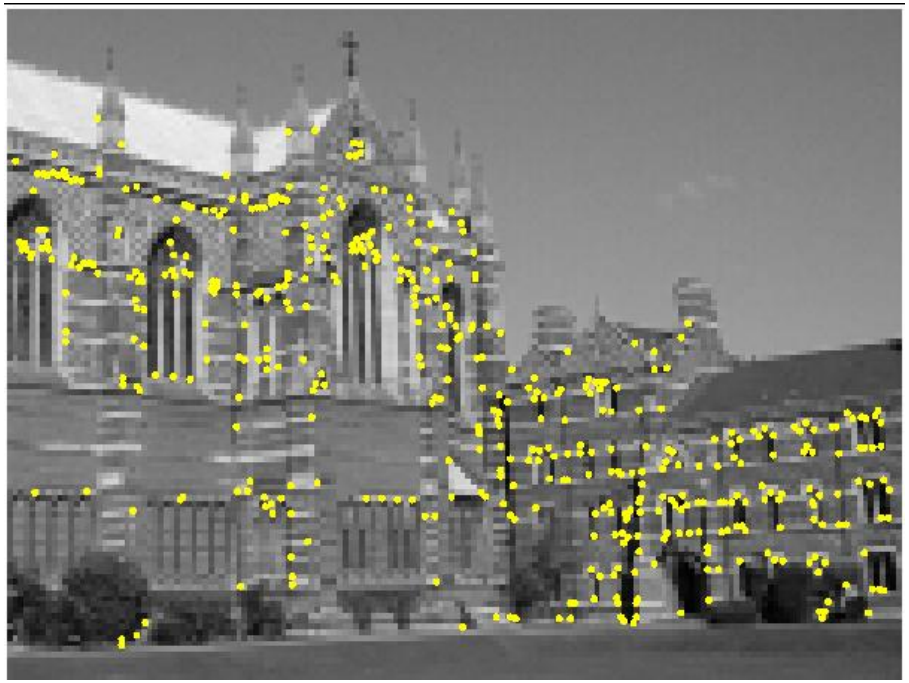


Simple Matching

Expect some false matches

Feature detection and matching





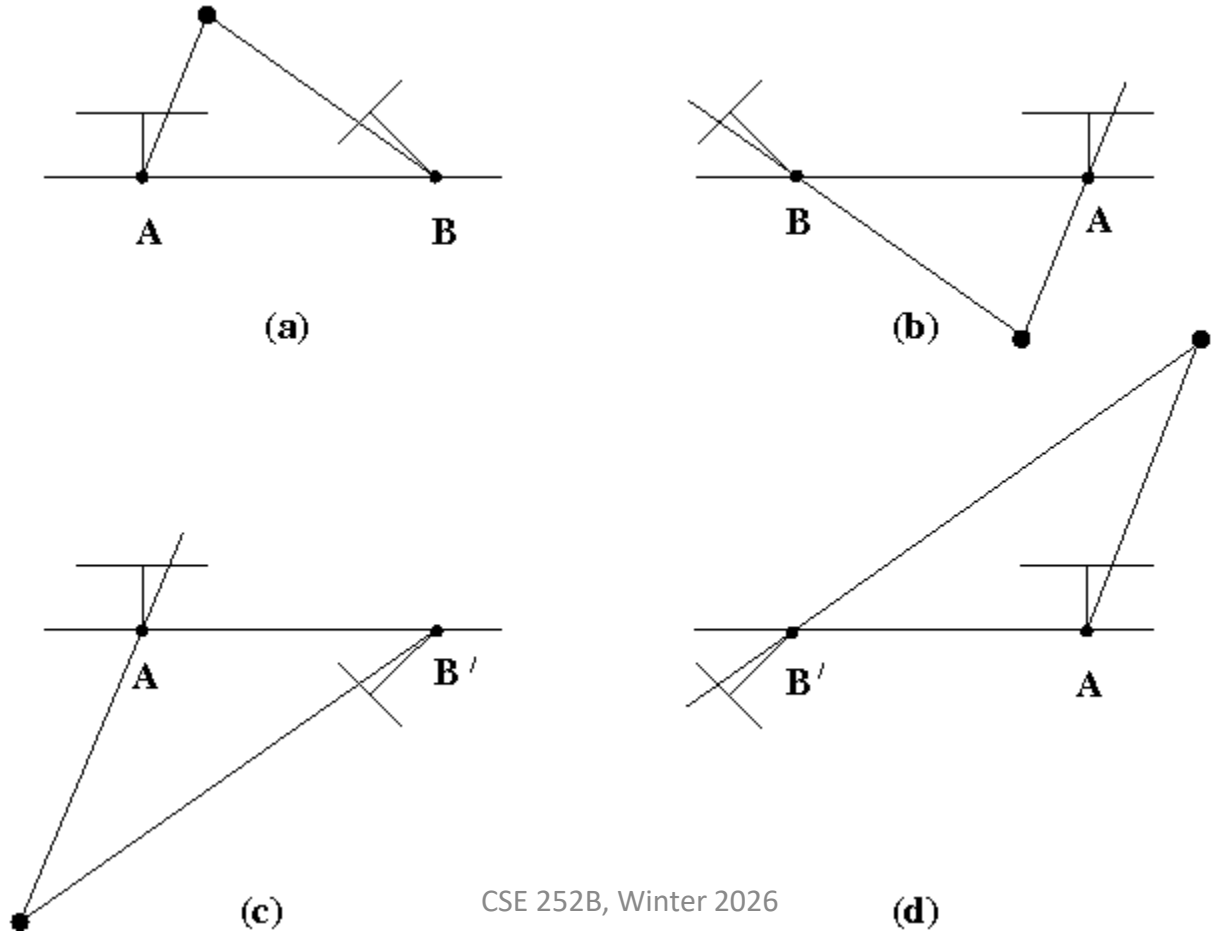
Simple Matching
Including Outlier Rejection

No geometric outliers
But still some false matches

Decompose essential matrix

$$\hat{\mathtt{P}} = [\mathtt{I} \,|\, \mathbf{0}] \text{ and } \hat{\mathtt{P}}' = [\mathtt{R} \,|\, \mathbf{t}]$$

Four solutions, but only one where the reconstructed scene point is in front of both cameras



Decompose essential matrix

$$\hat{\mathtt{P}} = [\mathtt{I} \,|\, \mathbf{0}] \text{ and } \hat{\mathtt{P}}' = [\mathtt{R} \,|\, \mathbf{t}]$$

Four solutions, but only one where the reconstructed scene point is in front of both cameras

$$\mathtt{E} = \mathtt{U}\mathtt{D}\mathtt{V}^\top \quad \text{singular value decomposition, where } \mathtt{D} = \text{diag}(1, 1, 0) \text{ to scale}$$

Two possible choices of second camera rotation matrix $\mathtt{R}$

$$\mathtt{R}_1 = \mathtt{U}\mathtt{Z}\mathtt{V}^\top \text{ or } \mathtt{R}_2 = \mathtt{U}\mathtt{Z}^\top\mathtt{V}^\top, \text{ where } \mathtt{Z} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ensure $\mathtt{R}_1$ and $\mathtt{R}_2$ are special orthogonal. If $\det(\mathtt{R}_1) = -1$, then negate $\mathtt{R}_1$. If $\det(\mathtt{R}_2) = -1$, then negate $\mathtt{R}_2$.

Two possible choices of second camera translation vector $\mathbf{t}$ (to positive scale)

$$\mathbf{t}_1 = \mathbf{u}_3 \text{ or } \mathbf{t}_2 = -\mathbf{u}_3, \text{ where } \mathbf{u}_3 \text{ is the last column of } \mathtt{U}$$

Four possible choices of the second normalized camera projection matrix $\hat{\mathtt{P}}'$

$$\hat{\mathtt{P}}'_1 = [\mathtt{R}_1 \,|\, \mathbf{t}_1], \hat{\mathtt{P}}'_2 = [\mathtt{R}_1 \,|\, \mathbf{t}_2], \hat{\mathtt{P}}'_3 = [\mathtt{R}_2 \,|\, \mathbf{t}_1], \text{ or } \hat{\mathtt{P}}'_4 = [\mathtt{R}_2 \,|\, \mathbf{t}_2]$$

For each possible choice, perform triangulation using one 2D point correspondence $\hat{\mathbf{x}} \leftrightarrow \hat{\mathbf{x}}'$ and choose the $\hat{\mathtt{P}}'$ where the reconstructed scene point is in front of both cameras

Depth of point

The depth of a point $\mathbf{X} = (X, Y, Z, T)^\top$ with respect to a camera $\mathtt{P}$ is given by

$$\text{depth}(\mathbf{X}; \mathtt{P}) = \frac{\text{sign}(\det(\mathtt{M}))}{\|\mathbf{m}^3\|} \frac{w}{T}, \text{ where } w = \mathbf{p}^{3\top}\mathbf{X}$$

and

$$\mathtt{P} = [\mathtt{M} \,|\, \mathbf{v}], \text{ where } \mathtt{M} \text{ is nonsingular}$$

$$\begin{bmatrix} \mathbf{p}^{1\top} \\ \mathbf{p}^{2\top} \\ \mathbf{p}^{3\top} \end{bmatrix} = \begin{bmatrix} \mathbf{m}^{1\top} & v_1 \\ \mathbf{m}^{2\top} & v_2 \\ \mathbf{m}^{3\top} & v_3 \end{bmatrix}$$

Chirality of point

$$\text{sign}(wT \det(\mathtt{M})) = \begin{cases} 1 & \text{if point } \mathbf{X} \text{ is in front of camera } \mathtt{P} \\ -1 & \text{if point } \mathbf{X} \text{ is behind camera } \mathtt{P} \end{cases}$$

Next lecture

- Estimation of essential matrix, nonlinear