Estimation of Fundamental Matrix, Nonlinear

Computer Vision II

CSE 252B

Lecture 13

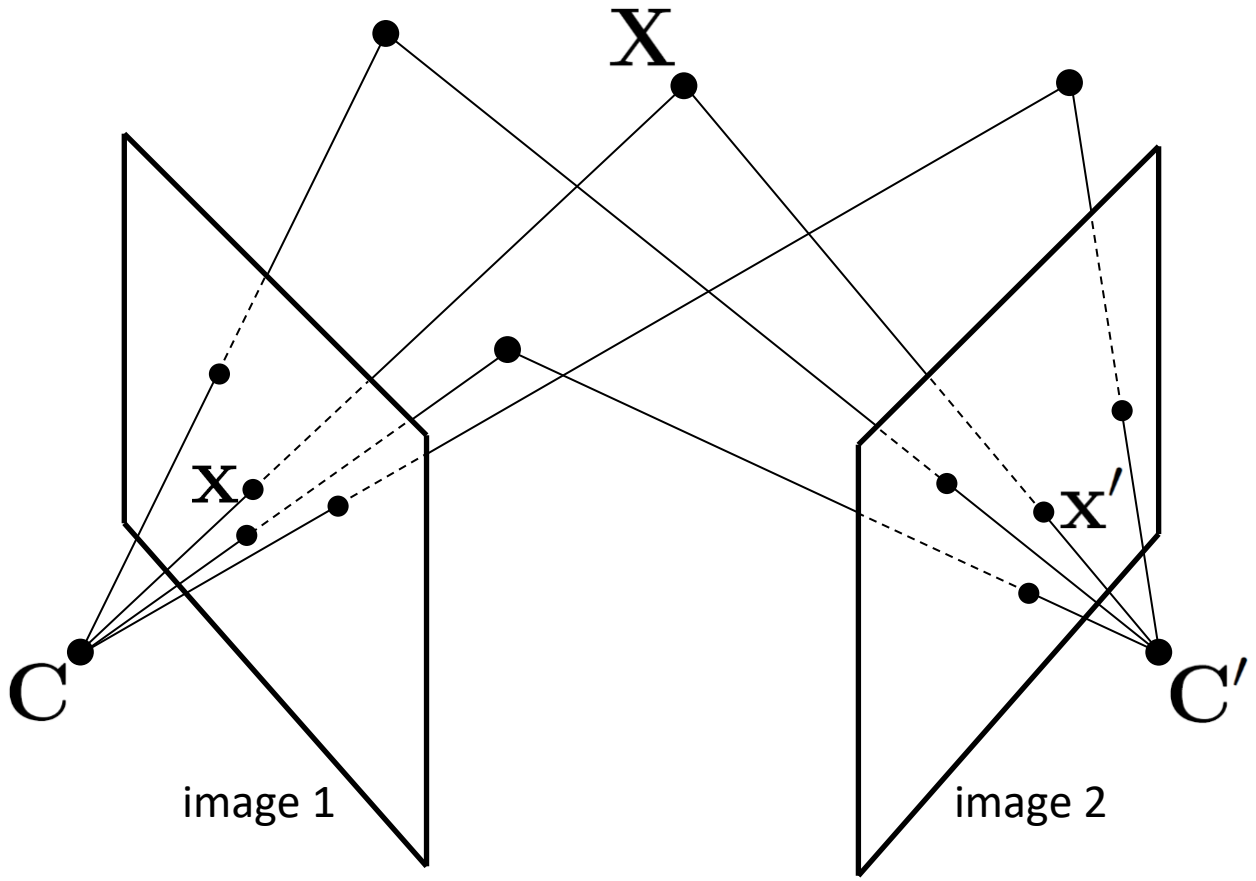
Announcements

- Assignment 4 is due Mar 4, 11:59 PM
- Reading
  - Sections 10.1, 9.5, 11.4, 12.5, 10.5

Two view geometry

Two view geometry

- Imaging a 3D scene



Fundamental matrix

$$\boldsymbol{\ell}' = \mathtt{F}\mathbf{x} \text{ and } \mathbf{x}'^{\top}\boldsymbol{\ell}' = 0$$

$$\mathbf{x}'^{\top}\mathtt{F}\mathbf{x} = 0 \qquad \text{Epipolar constraint}$$

$\boldsymbol{\ell}' = \mathtt{F}\mathbf{x}$	The epipolar line corresponding to the point $\mathbf{x}$
$\boldsymbol{\ell} = \mathtt{F}^{\top}\mathbf{x}'$	The epipolar line corresponding to the point $\mathbf{x}'$
$\mathtt{F}\mathbf{e} = \mathbf{0}$	The epipole $\mathbf{e}$ is the null space of $\mathtt{F}$
$\mathbf{e}'^{\top}\mathtt{F} = \mathbf{0}^{\top}$	The epipole $\mathbf{e}'$ is the left null space of $\mathtt{F}$
$\mathtt{F}^{\top}\mathbf{e}' = \mathbf{0}$	The epipole $\mathbf{e}'$ is the null space of $\mathtt{F}^{\top}$

The fundamental matrix $\mathtt{F}$ from general cameras $\mathtt{P}$ and $\mathtt{P}'$

$$\mathtt{F} = [\mathbf{e}']_{\times}\mathtt{P}'\mathtt{P}^{+}, \text{ where } \mathbf{e}' = \mathtt{P}'\mathbf{C} \text{ and } \mathtt{P}\mathbf{C} = \mathbf{0} \text{ (i.e., } \mathbf{C} \text{ is the null space of } \mathtt{P})$$

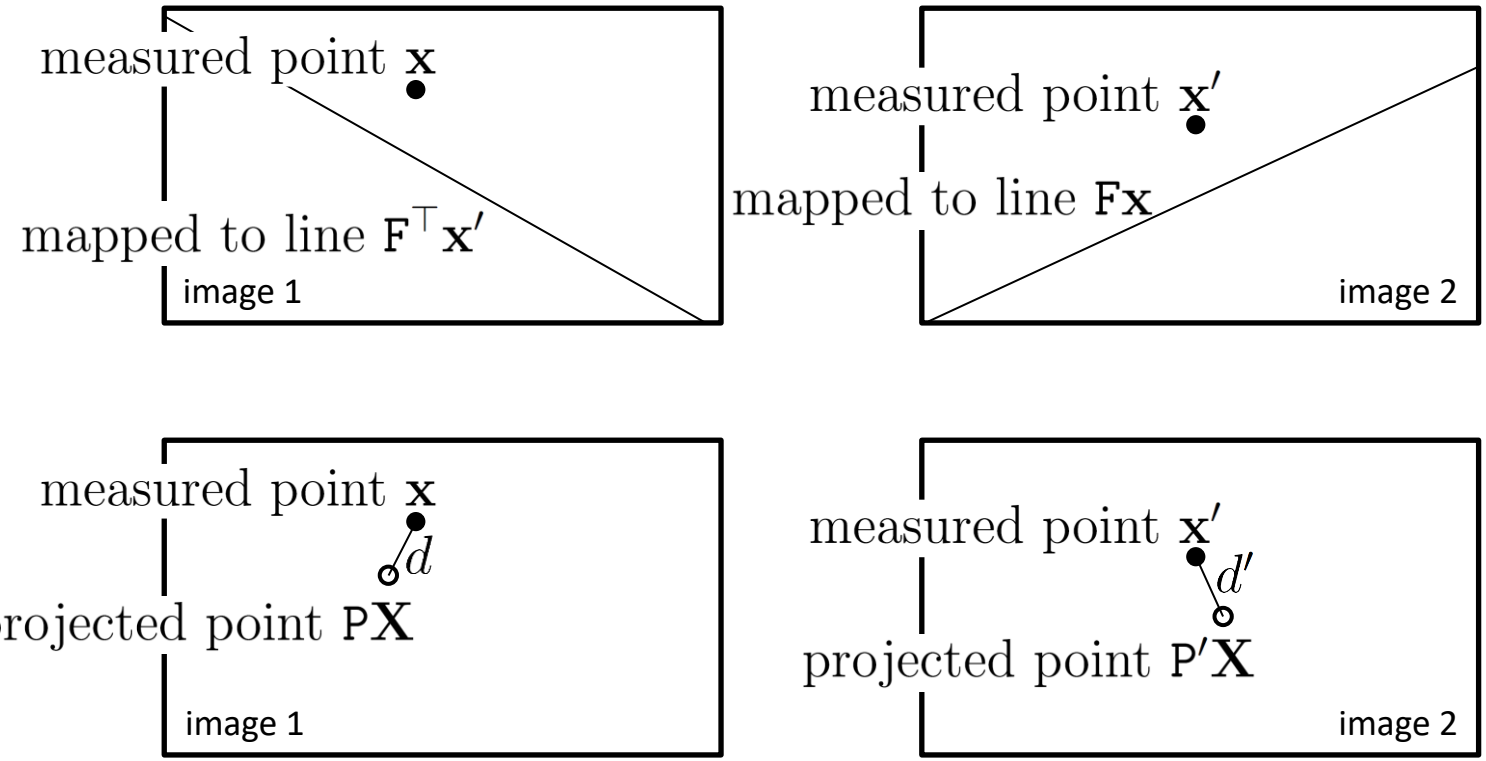
The fundamental matrix $\mathtt{F}$ from canonical cameras $\mathtt{P} = [\mathtt{I} \,|\, \mathbf{0}]$ and $\mathtt{P}' = [\mathtt{M} \,|\, \mathbf{v}]$

$$\mathtt{F} = [\mathbf{v}]_{\times}\mathtt{M}$$

$$\mathtt{F} = [\mathbf{e}']_{\times}\mathtt{M} = \mathtt{M}^{-\top}[\mathbf{e}]_{\times}, \text{ where } \mathbf{e}' = \mathbf{v} \text{ and } \mathbf{e} = \mathtt{M}^{-1}\mathbf{v}$$

Fundamental matrix

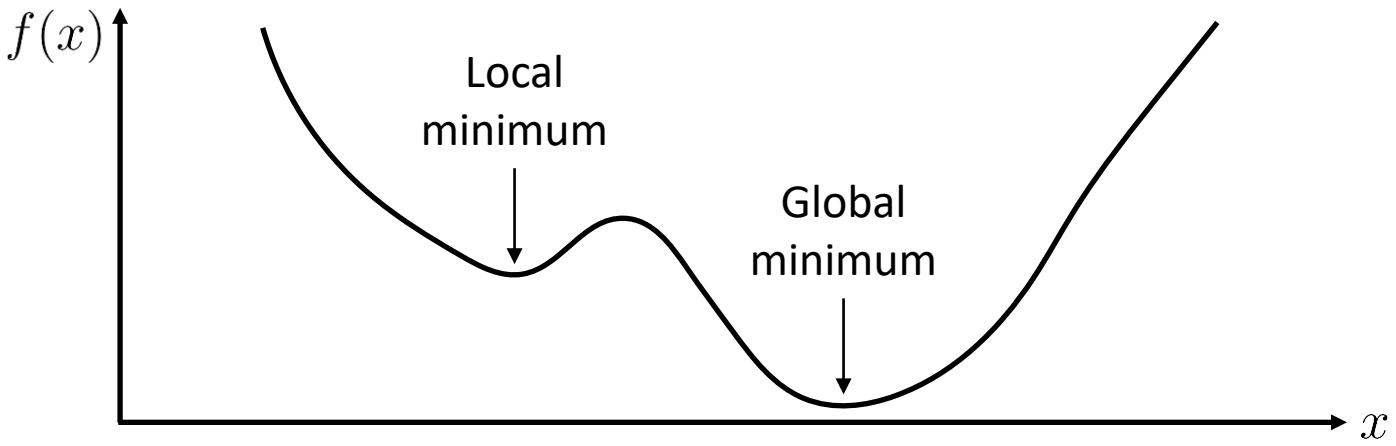
- Error in image point measurements



Estimation of fundamental matrix

$$\mathbf{x}_i = \mathtt{P}\mathbf{X}_i \,\forall\, i \text{ and } \mathbf{x}'_i = \mathtt{P}'\mathbf{X}_i \,\forall\, i, \text{ set } \mathtt{P} = [\mathtt{I} \,|\, \mathbf{0}] \text{ and solve for } \mathtt{P}'$$

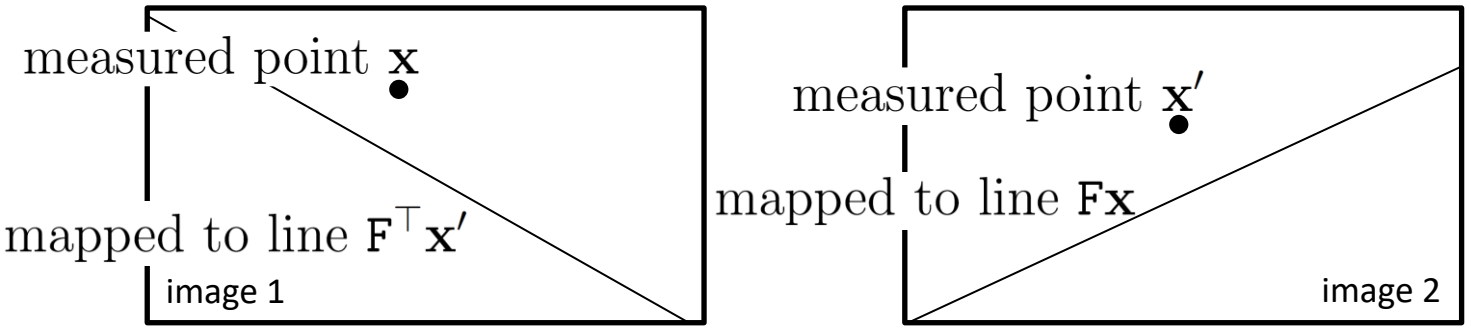
- Given point correspondences $\mathbf{x}_i \leftrightarrow \mathbf{x}'_i$
- Minimize geometric error $\sum_i \left(d(\mathbf{x}_i, \mathtt{P}\mathbf{X}_i)^2 + d(\mathbf{x}'_i, \mathtt{P}'\mathbf{X}_i)^2 \right)$
- A nonlinear mapping requires a nonlinear optimization problem solver
  - Use linear estimation for initial estimate (to mitigate converging to local optimum)
  - Iterative process to determine global optimum



Linear estimation of fundamental matrix using the direct linear transformation (DLT) algorithm

- Minimize algebraic error

$$\mathbf{x}'^{\top}\mathbf{F}\mathbf{x} = 0 \qquad \text{Epipolar constraint}$$



Linear estimation of fundamental matrix using the direct linear transformation (DLT) algorithm

Given $n \geq 8$ point correspondences $\mathbf{x}_i \leftrightarrow \mathbf{x}'_i$

See previous lecture estimation from 7 point correspondences

$$\mathbf{x}_i'^{\top} \mathtt{F} \mathbf{x}_i = 0 \, \forall \, i, \text{ solve for } \mathtt{F}$$

$$\left(\mathbf{x}_i'^{\top} \otimes \mathbf{x}_i^{\top}\right) \mathbf{f} = 0 \, \forall \, i, \text{ solve for } \mathbf{f} = \mathrm{vec}(\mathtt{F}^{\top})$$

$$\begin{bmatrix} \mathbf{x}_1'^{\top} \otimes \mathbf{x}_1^{\top} \\ \mathbf{x}_2'^{\top} \otimes \mathbf{x}_2^{\top} \\ \vdots \\ \mathbf{x}_n'^{\top} \otimes \mathbf{x}_n^{\top} \end{bmatrix} \mathbf{f} = \mathbf{0}$$

$$\mathtt{A}\mathbf{f} = \mathbf{0}, \text{ where } \mathtt{A} = \begin{bmatrix} \mathbf{x}_1'^{\top} \otimes \mathbf{x}_1^{\top} \\ \mathbf{x}_2'^{\top} \otimes \mathbf{x}_2^{\top} \\ \vdots \\ \mathbf{x}_n'^{\top} \otimes \mathbf{x}_n^{\top} \end{bmatrix}$$

Then, enforce rank 2 constraint (simple and convenient method using SVD)

See textbook for alternative method

$$\mathtt{F} = \mathtt{U}\Sigma\mathtt{V}^{\top}, \text{ where } \Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \sigma_3)$$

$$\mathtt{F} = \mathtt{U}\Sigma'\mathtt{V}^{\top}, \text{ where } \Sigma' = \mathrm{diag}(\sigma_1, \sigma_2, 0)$$

But, data normalization must be used

Linear estimation of fundamental matrix using the direct linear transformation (DLT) algorithm

- Algorithm

1. Data normalize the points

$$\mathbf{x}_{\mathrm{DN},i} = \mathtt{T}\mathbf{x}_i \,\forall\, i$$
$$\mathbf{x}'_{\mathrm{DN},i} = \mathtt{T}'\mathbf{x}'_i \,\forall\, i$$

2. Estimate the data normalized fundamental matrix $\mathtt{F}_{\mathrm{DN}}$ from the data normalized points

3. Data denormalize the data normalized fundamental matrix

$$\mathbf{x}'^{\top}_{\mathrm{DN},i}\mathtt{F}_{\mathrm{DN}}\mathbf{x}_{\mathrm{DN},i} = 0 \,\forall\, i$$
$$(\mathtt{T}'\mathbf{x}'_i)^{\top}\mathtt{F}_{\mathrm{DN}}\mathtt{T}\mathbf{x}_i = 0 \,\forall\, i$$
$$\mathbf{x}'^{\top}_i\mathtt{T}'^{\top}\mathtt{F}_{\mathrm{DN}}\mathtt{T}\mathbf{x}_i = 0 \,\forall\, i$$
$$\mathbf{x}'^{\top}_i\mathtt{F}\mathbf{x}_i = 0 \,\forall\, i, \text{ where } \mathtt{F} = \mathtt{T}'^{\top}\mathtt{F}_{\mathrm{DN}}\mathtt{T}$$

Retrieve camera projection matrices from fundamental matrix

Camera projection matrices must be compatible with the fundamental matrix.
General formula

$$\mathtt{P} = [\mathtt{I} \,|\, \mathbf{0}] \text{ and } \mathtt{P}' = \left[[\mathbf{e}']_\times \mathtt{F} + \mathbf{e}'\mathbf{v}^\top \,|\, \lambda \mathbf{e}'\right] \text{ up to 3D projective transformation}$$

where $\mathtt{F}^\top \mathbf{e}' = \mathbf{0}$ (i.e., $\mathbf{e}'$ is the null space of $\mathtt{F}^\top$), $\mathbf{v}$ is any 3-vector, and λ is any nonzero scalar.
If $\mathbf{v} = 0$ and $\lambda = 1$, then

Rank 2

$$\mathtt{P} = [\mathtt{I} \,|\, \mathbf{0}] \text{ and } \mathtt{P}' = \left[\overbrace{[\mathbf{e}']_\times \mathtt{F}}^{\text{Rank 2}} \,|\, \mathbf{e}'\right] \text{ up to 3D projective transformation}$$

Up to 3D projective transformation

$$\mathbf{x} = \mathtt{P}\mathbf{X} \qquad \mathbf{x}' = \mathtt{P}'\mathbf{X}$$
$$\mathbf{x} = \mathtt{P}\mathtt{H}^{-1}\mathtt{H}\mathbf{X} \qquad \mathbf{x}' = \mathtt{P}'\mathtt{H}^{-1}\mathtt{H}\mathbf{X}, \text{ where } \mathtt{H} \text{ is any 3D projective transformation}$$

Both pairs of camera projection matrices, $\mathtt{P}$ and $\mathtt{P}'$, and $\mathtt{P}\mathtt{H}^{-1}$ and $\mathtt{P}'\mathtt{H}^{-1}$ have the same fundamental matrix $\mathtt{F}$

Retrieve camera projection matrices from fundamental matrix

$$\mathtt{P} = [\mathtt{I} \,|\, \mathbf{0}] \text{ and } \mathtt{P}' = [\mathtt{M} \,|\, \mathbf{v}] \text{ up to 3D projective transformation}$$

$\mathtt{M}$ of full rank (i.e., rank 3) and as far away from rank 2 as possible is obtained as follows

$\mathtt{F} = \mathtt{U}\mathtt{D}\mathtt{V}^\top$ singular value decomposition, where $\mathtt{U} = [\mathbf{u}_1 \,|\, \mathbf{u}_2 \,|\, \mathbf{u}_3]$ and $\mathtt{D} = \text{diag}(s, t, 0)$

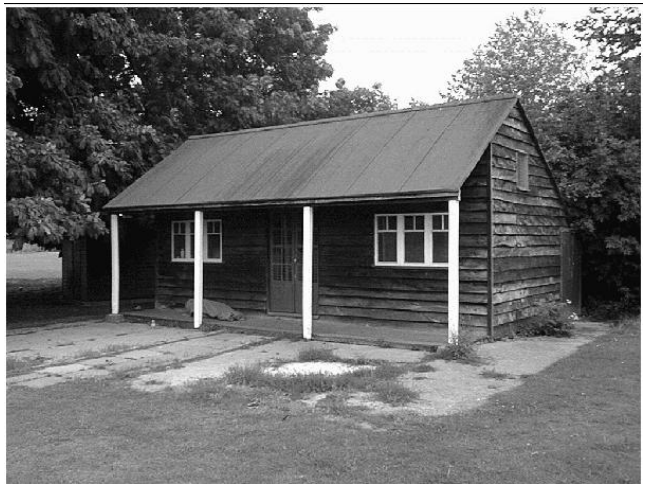
$\mathtt{F} = \mathtt{U}\mathtt{W}\mathtt{Z}\mathtt{D}'\mathtt{V}^\top$, where $\mathtt{D} = \mathtt{W}\mathtt{Z}\mathtt{D}'$ and $\mathtt{W}\mathtt{Z} = \text{diag}(1, 1, 0)$

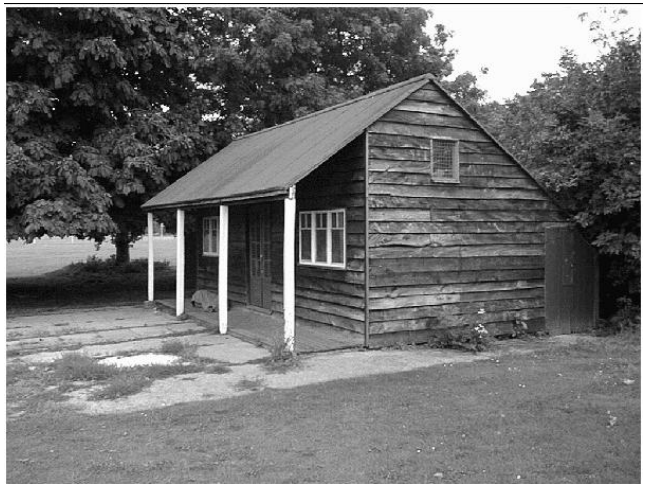
$$\mathtt{W} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathtt{Z} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } \mathtt{D}' = \text{diag}(s, t, (s+t)/2)$$

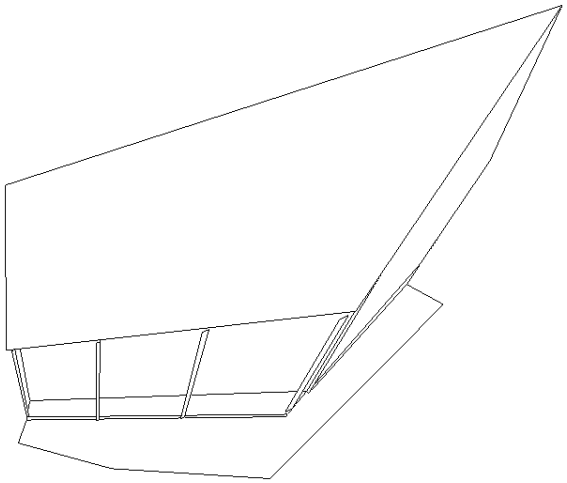
$\mathtt{F} = (\mathtt{U}\mathtt{W}\mathtt{U}^\top)(\mathtt{U}\mathtt{Z}\mathtt{D}'\mathtt{V}^\top)$

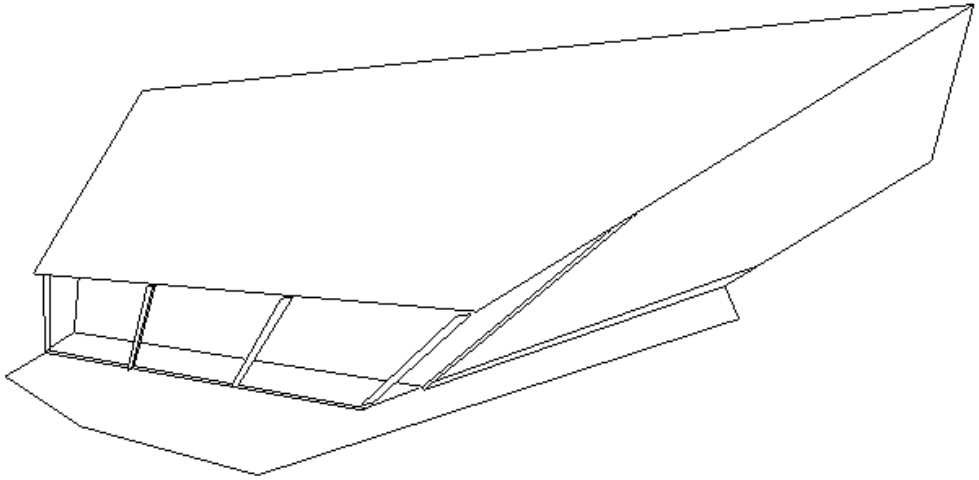
$\mathtt{F} = \mathtt{S}\mathtt{M}$ and $\mathtt{P}' = [\mathtt{M} \,|\, \mathbf{v}]$, where $\mathtt{S} = \mathtt{U}\mathtt{W}\mathtt{U}^\top = -[\mathbf{v}]_\times = -[\mathbf{u}_3]_\times$ and $\mathtt{M} = \mathtt{U}\mathtt{Z}\mathtt{D}'\mathtt{V}^\top$ and $\mathbf{v} = \mathbf{u}_3 = \mathbf{e}'$

Up to 3D projective transformation



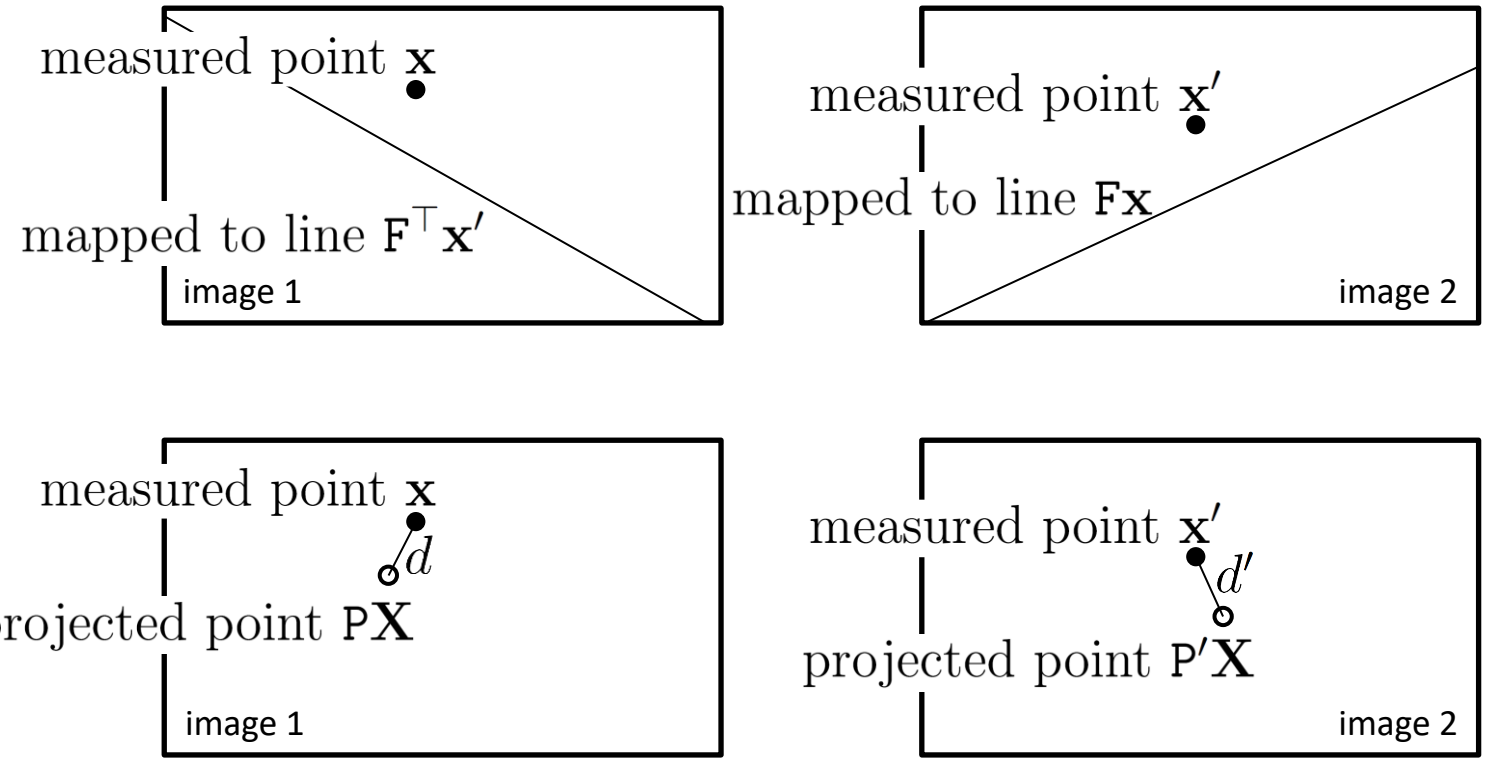






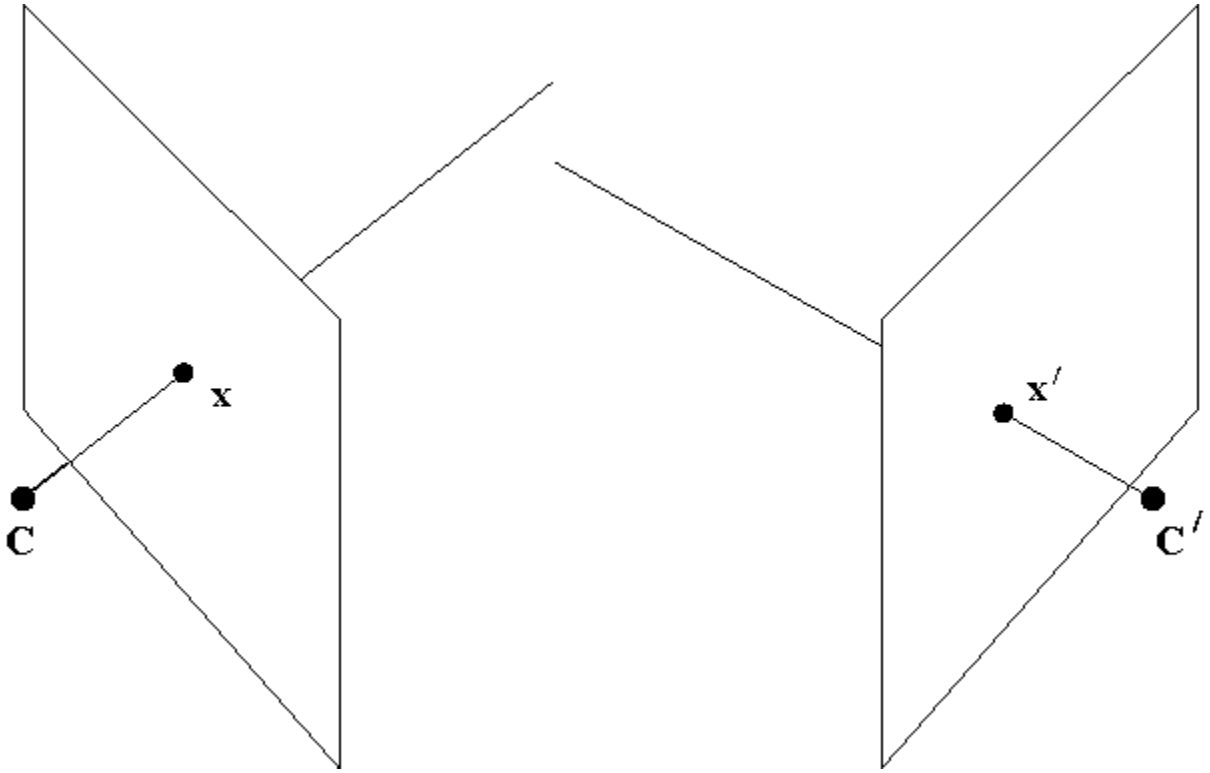
Fundamental matrix

- Error in image point measurements



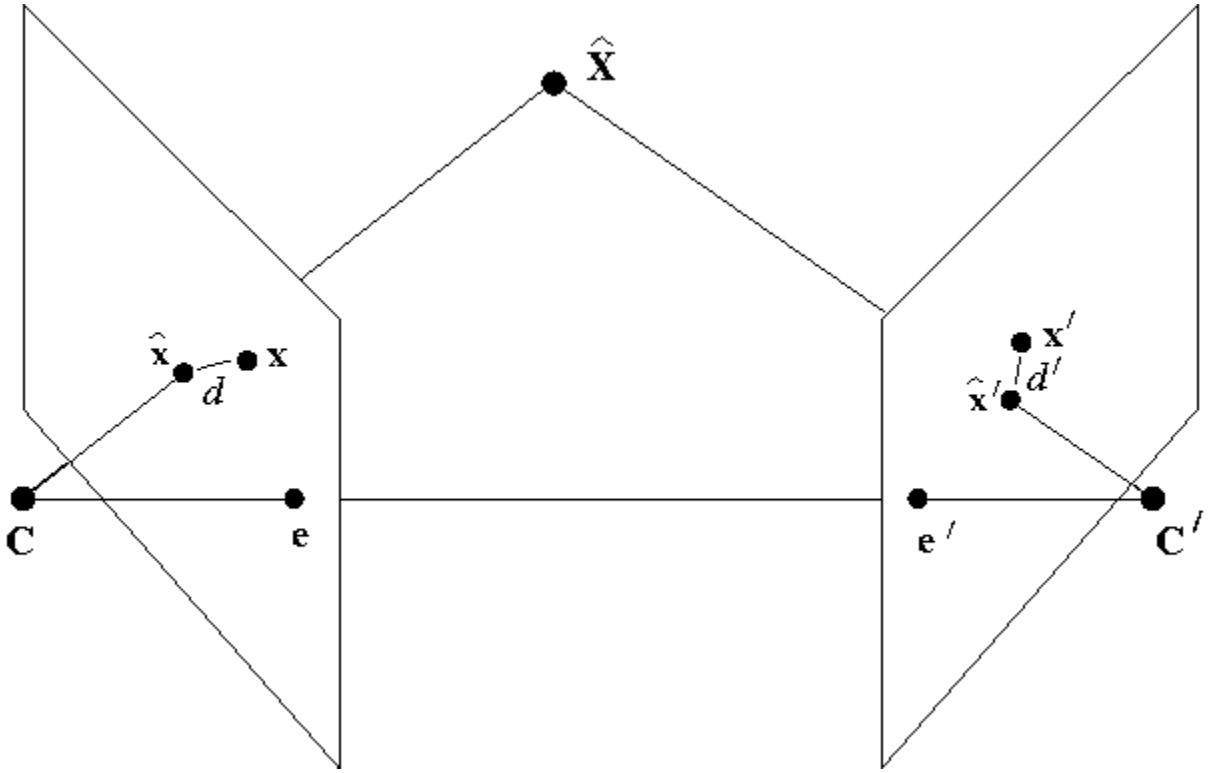
Triangulation

- In general, corresponding 2D points backprojected to 3D lines will not intersect



Triangulation

- Correct image points such that 3D lines intersect



“Optimal” correction

Given the fundamental matrix $\mathtt{F}$ and corresponding points $\mathbf{x} \leftrightarrow \mathbf{x}'$, determine the “optimal” corrected corresponding points $\hat{\mathbf{x}} \leftrightarrow \hat{\mathbf{x}}'$ that exactly satisfy $\hat{\mathbf{x}}_i'^{\top} \mathtt{F} \hat{\mathbf{x}}_i = 0$ as follows.
Determine the fundamental matrix $\mathtt{F}_{\mathrm{s}}$ in a special form.
First, $\mathtt{F}_{\mathrm{s}} = \mathtt{T}'^{-\top} \mathtt{F} \mathtt{T}^{-1}$, where

$$\mathtt{T} = \begin{bmatrix} w & 0 & -x \\ 0 & w & -y \\ 0 & 0 & w \end{bmatrix} \text{ and } \mathtt{T}' = \begin{bmatrix} w' & 0 & -x' \\ 0 & w' & -y' \\ 0 & 0 & w' \end{bmatrix}$$

are the transformations that translate the points to the origin. Then, calculate its associated epipoles $\mathbf{e}$ (i.e., the null space of $\mathtt{F}_{\mathrm{s}}$) and $\mathbf{e}'$ (i.e., the null space of $\mathtt{F}_{\mathrm{s}}^{\top}$).
Then, scale $\mathbf{e} = (e_1, e_2, e_3)^{\top}$ and $\mathbf{e}' = (e_1', e_2', e_3')^{\top}$ such that $e_1^2 + e_2^2 = 1$ and $e_1'^2 + e_2'^2 = 1$, i.e.,

$$\mathbf{e} = \frac{1}{\sqrt{e_1^2 + e_2^2}} \mathbf{e} \text{ and } \mathbf{e}' = \frac{1}{\sqrt{e_1'^2 + e_2'^2}} \mathbf{e}'$$

“Optimal” correction

Last, $\mathtt{F}_\mathrm{s} = \mathtt{R}'\mathtt{F}_\mathrm{s}\mathtt{R}^\top = \mathtt{R}'\mathtt{T}'^{-\top}\mathtt{F}\mathtt{T}^{-1}\mathtt{R}^\top$, where

$$\mathtt{R} = \begin{bmatrix} e_1 & e_2 & 0 \\ -e_2 & e_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \mathtt{R}' = \begin{bmatrix} e_1' & e_2' & 0 \\ -e_2' & e_1' & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are the transformations that rotate the scaled epipoles to the x-axis such that $\mathtt{R}\mathbf{e} = (1, 0, e_3)^\top$ and $\mathtt{R}'\mathbf{e}' = (1, 0, e_3')^\top$. Note $\mathtt{F}_\mathrm{s}(1, 0, e_3)^\top = \mathbf{0}$ and $(1, 0, e_3')\mathtt{F}_\mathrm{s} = \mathbf{0}^\top$, so $\mathtt{F}_\mathrm{s}$ has the form

$$\mathtt{F}_\mathrm{s} = \begin{bmatrix} ff'd & -f'c & -f'd \\ -fb & a & b \\ -fd & c & d \end{bmatrix}$$

where $f = e_3$ and $f' = e_3'$.

“Optimal” correction

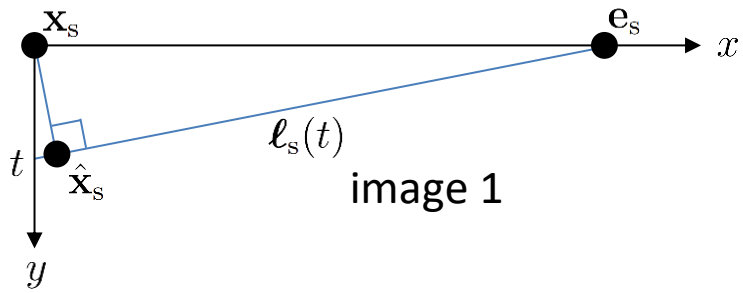
Consider a transformed epipolar line in the first image passing through (the transformed epipole $(1, 0, f)^\top$ and) a point $(0, t, 1)^\top$ on the y-axis, which also maps to a transformed epipolar line in the second image, i.e.,

$$\boldsymbol{\ell}_{\mathrm{s}}(t) = \begin{bmatrix} 0 \\ t \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ f \end{bmatrix} = \begin{bmatrix} tf \\ 1 \\ -t \end{bmatrix} \text{ and } \boldsymbol{\ell}'_{\mathrm{s}}(t) = \mathtt{F}_{\mathrm{s}} \begin{bmatrix} 0 \\ t \\ 1 \end{bmatrix} = \begin{bmatrix} -f'(ct+d) \\ at+b \\ ct+d \end{bmatrix}$$

The sum of squared distance from the origin (i.e., the transformed points) to the transformed epipolar lines is given by

image 1 image 2

$$s(t) = \frac{t^2}{1+f^2t^2} + \frac{(ct+d)^2}{(at+b)^2 + f'^2(ct+d)^2}$$



“Optimal” correction

$$s(t) = \frac{t^2}{1 + f^2 t^2} + \frac{(ct + d)^2}{(at + b)^2 + f'^2 (ct + d)^2}$$

Find the minimum of $s(t)$ by first setting its derivative to 0, which simplifies to

$$g(t) = t((at + b)^2 + f'^2 (ct + d)^2)^2 - (ad - bc)(1 + f^2 t^2)^2 (at + b)(ct + d) = 0,$$

solve for t. The 6 roots of $g(t)$ are the maxima and minima of $s(t)$. Note the minimum may also be at $t = \infty$.

$$\lim_{t \to \infty} s(t) = \frac{1}{f^2} + \frac{c^2}{a^2 + f'^2 c^2}$$

Evaluate $\lim_{t \to \infty} s(t)$ and $s(t)$ at the real roots (or the real part of the 6 roots) of $g(t)$, and select $t_{\min}$ that yields the smallest value.
Determine the transformed points $\hat{\mathbf{x}}_{\mathrm{s}}$ and $\hat{\mathbf{x}}'_{\mathrm{s}}$ as the closest points on the transformed lines $\boldsymbol{\ell}_{\mathrm{s}}(t_{\min})$ and $\boldsymbol{\ell}'_{\mathrm{s}}(t_{\min})$, noting $\boldsymbol{\ell}_{\mathrm{s}}(t \to \infty) = (f, 0, -1)^\top$ and $\boldsymbol{\ell}'_{\mathrm{s}}(t \to \infty) = (-f'c, a, c)^\top$, to the origin and inverse transform the points back to their original coordinates.

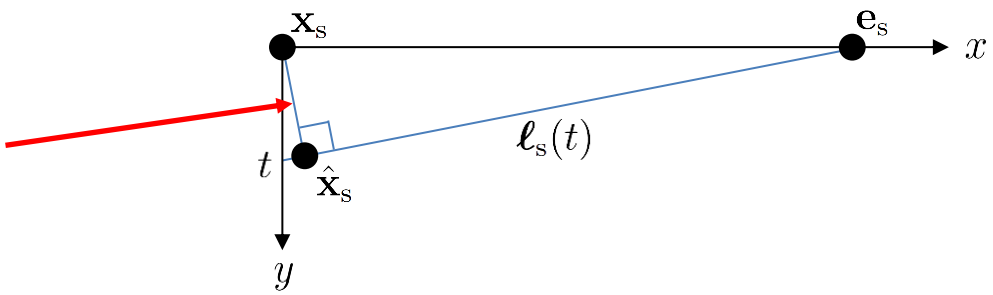
$$\hat{\mathbf{x}} = \mathtt{T}^{-1} \mathtt{R}^\top \hat{\mathbf{x}}_{\mathrm{s}} \text{ and } \hat{\mathbf{x}}' = \mathtt{T}'^{-1} \mathtt{R}'^\top \hat{\mathbf{x}}'_{\mathrm{s}}$$

Line orthogonal to hyperplane and through point

Hyperplane $\boldsymbol{\pi} = (\mathbf{n}^\top, d)^\top$, where $\mathbf{n}$ is normal vector
Point $\mathbf{X}'$ on line orthogonal to $\boldsymbol{\pi}$ and through $\mathbf{X}$ is $\mathbf{X}$ translated by $\mathbf{n}$

$$\mathbf{X}' = \begin{bmatrix} \mathtt{I} & \mathbf{n} \\ \mathbf{0}^\top & 1 \end{bmatrix} \mathbf{X}$$

Line orthogonal to $\boldsymbol{\pi}$ and through $\mathbf{X}$ is join of $\mathbf{X}$ and $\mathbf{X}'$



Point on hyperplane closest to point

Point $\mathbf{X}_\pi$ on hyperplane $\boldsymbol{\pi}$ closest to point $\mathbf{X}$ is intersection of line orthogonal to $\boldsymbol{\pi}$ and through $\mathbf{X}$. Represent line as a pencil of points and solve for intersection point $\mathbf{X}_\pi$.

$$\mathbf{X}(\lambda) = \lambda\mathbf{X} + (1-\lambda)\mathbf{X}' \text{ and } \quad \boldsymbol{\pi}^\top\mathbf{X}(\lambda) = 0, \text{ solve for } \lambda$$

$$\boldsymbol{\pi}^\top(\lambda\mathbf{X} + (1-\lambda)\mathbf{X}') = 0$$

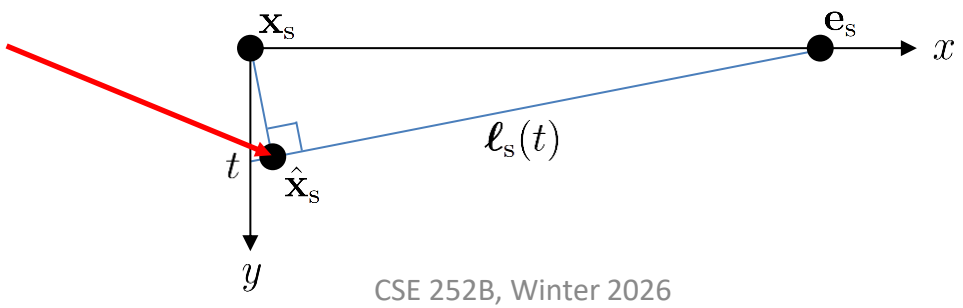
$$\lambda\boldsymbol{\pi}^\top\mathbf{X} + \boldsymbol{\pi}^\top\mathbf{X}' - \lambda\boldsymbol{\pi}^\top\mathbf{X}' = 0$$

$$\lambda(\boldsymbol{\pi}^\top\mathbf{X} - \boldsymbol{\pi}^\top\mathbf{X}') = -\boldsymbol{\pi}^\top\mathbf{X}'$$

$$\lambda = -\frac{\boldsymbol{\pi}^\top\mathbf{X}'}{\boldsymbol{\pi}^\top\mathbf{X} - \boldsymbol{\pi}^\top\mathbf{X}'}$$

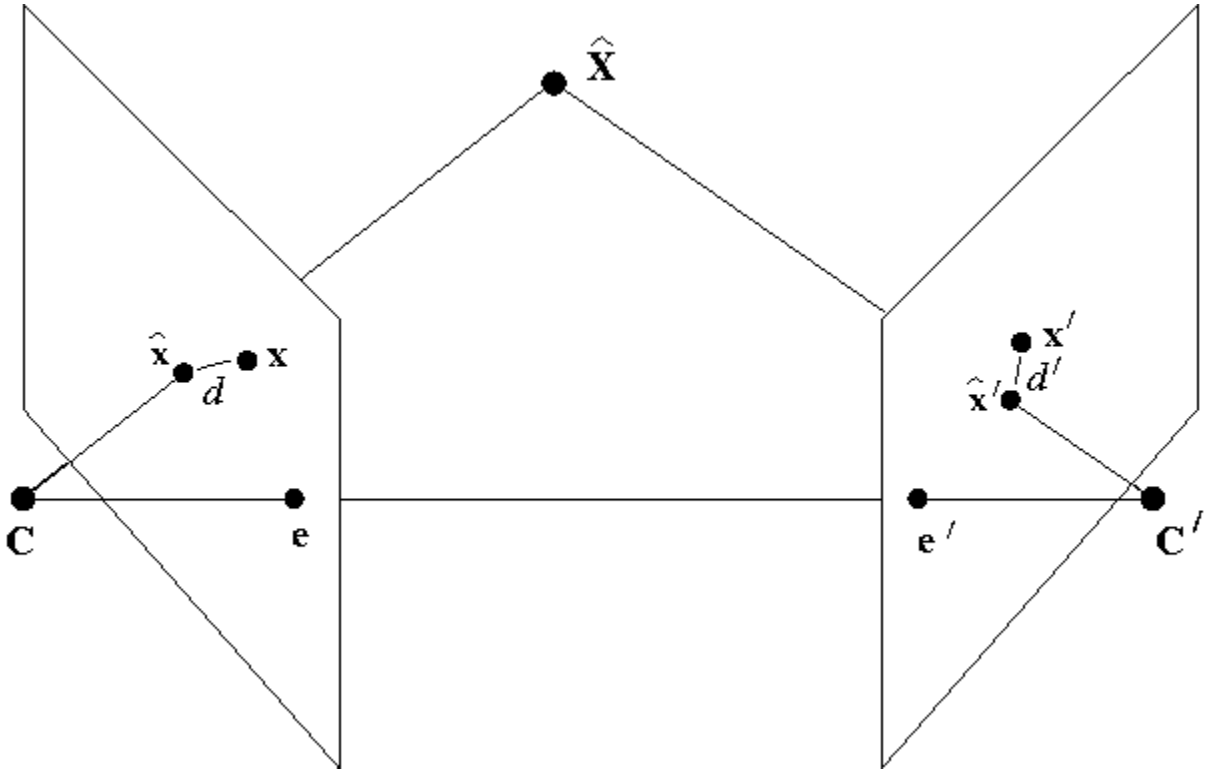
Substitute λ into $\mathbf{X}(\lambda) = \lambda\mathbf{X} + (1-\lambda)\mathbf{X}'$ to determine point on hyperplane $\boldsymbol{\pi}$ closest to point $\mathbf{X}$

$$\mathbf{X}_\pi = \lambda\mathbf{X} + (1-\lambda)\mathbf{X}'$$

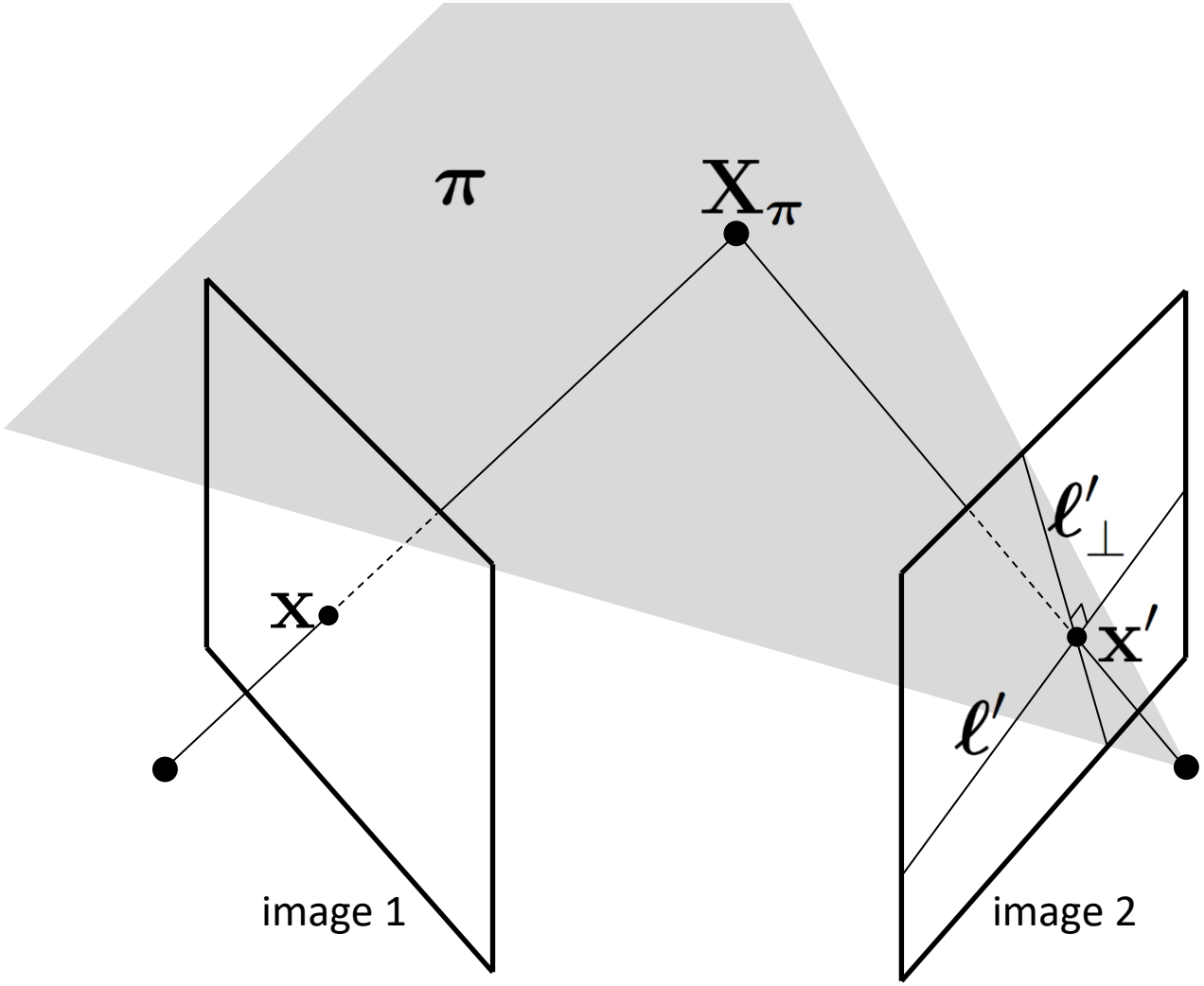


Triangulation

- Corrected image points exactly satisfy epipolar constraint, so 3D lines intersect



Fundamental matrix



Triangulation

Perform two view triangulation using the corresponding points $\mathbf{x} \leftrightarrow \mathbf{x}'$ that exactly satisfy the epipolar constraint.
Map point $\mathbf{x} = (x, y, w)^\top$ to line $\boldsymbol{\ell}' = \mathtt{F}\mathbf{x} = (a', b', c')^\top$ under the fundamental matrix $\mathtt{F}$
Determine line $\boldsymbol{\ell}'_\perp = (-b'w', a'w', b'x' - a'y')^\top$ orthogonal to $\boldsymbol{\ell}'$ and through $\mathbf{x}' = (x', y', w')^\top$
Backproject line $\boldsymbol{\ell}'_\perp$ to 3D plane $\boldsymbol{\pi} = \mathtt{P}'^\top \boldsymbol{\ell}'_\perp = (a, b, c, d)^\top = (\mathbf{n}^\top, d)^\top$
Backproject point $\mathbf{x}$ to 3D line defined by two 3D points

$$\mathtt{P}\mathbf{C} = \mathbf{0}, \text{ where camera center } \mathbf{C} \text{ is the null space of } \mathtt{P}$$

$$\mathbf{X} = \mathtt{P}^+\mathbf{x}, \text{ where } \mathtt{P}^+ = \mathtt{P}^\top(\mathtt{P}\mathtt{P}^\top)^{-1}$$

Intersection of 3D line and 3D plane

$$\mathbf{X}_{\boldsymbol{\pi}} = \begin{bmatrix} X_2(bY_1 + cZ_1 + dT_1) - X_1(bY_2 + cZ_2 + dT_2) \\ Y_2(aX_1 + cZ_1 + dT_1) - Y_1(aX_2 + cZ_2 + dT_2) \\ Z_2(aX_1 + bY_1 + dT_1) - Z_1(aX_2 + bY_2 + dT_2) \\ T_2(aX_1 + bY_1 + cZ_1) - T_1(aX_2 + bY_2 + cZ_2) \end{bmatrix}, \text{ where } \mathbf{X}_1 = \mathbf{C} \text{ and } \mathbf{X}_2 = \mathbf{X}$$

Triangulation

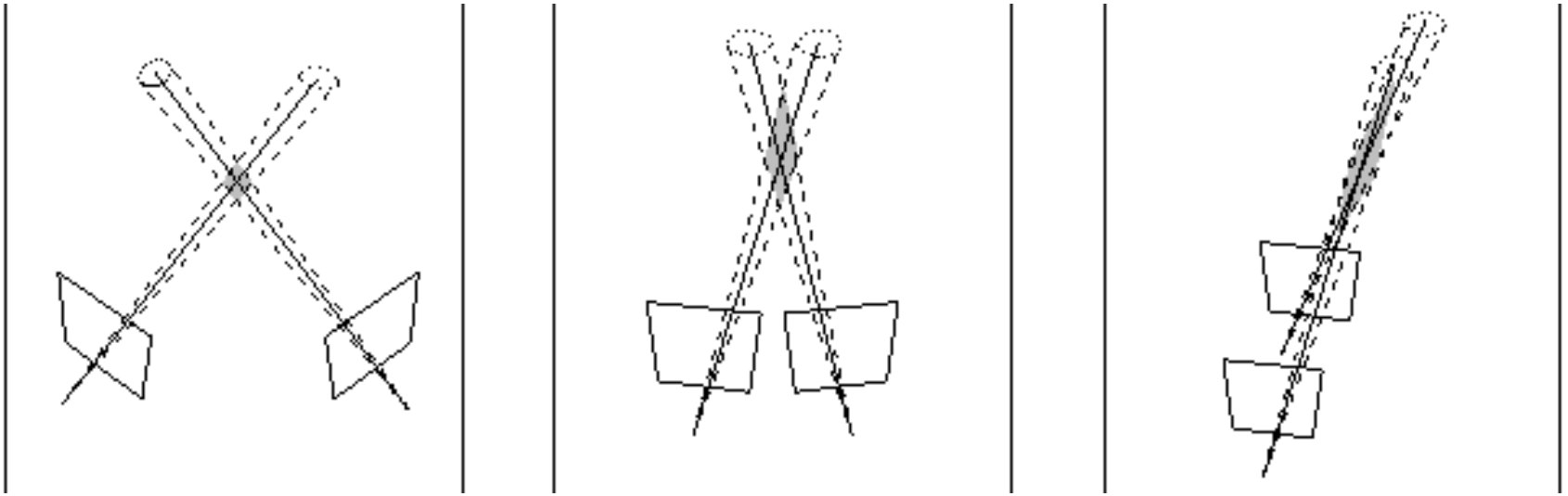
- Canonical cameras

In the case of canonical cameras, $\mathtt{P} = [\mathtt{I} \,|\, \mathbf{0}]$ and $\mathtt{P}^{+} = \mathtt{P}^{\top}$, so $\mathbf{C} = (0, 0, 0, 1)^{\top}$, $\mathbf{X} = (x, y, w, 0)^{\top}$, and

$$\mathbf{X}_{\boldsymbol{\pi}} = \begin{bmatrix} dx \\ dy \\ dw \\ -(ax + by + cw) \end{bmatrix} = \begin{bmatrix} d\mathbf{x} \\ -\mathbf{n}^{\top}\mathbf{x} \end{bmatrix}$$

Triangulation uncertainty

- The more orthogonal the 3D lines, the less uncertainty



Nonlinear estimation using the Levenberg-Marquardt algorithm

Given

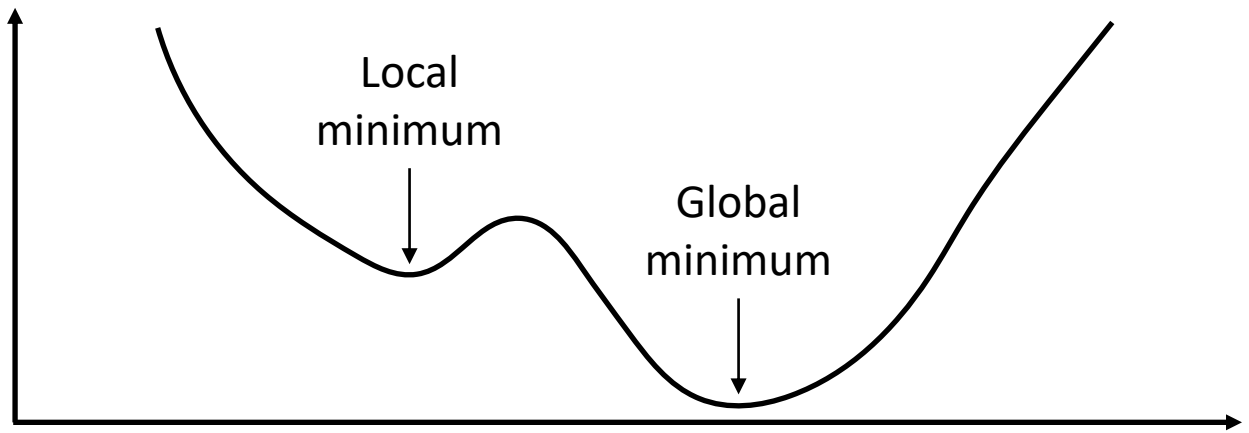
Measurement vector $\mathbf{X}$ with associated covariance matrix $\Sigma_{\mathbf{X}}$

Parameter vector $\hat{\mathbf{P}}$ (initial estimate)

Nonlinear function f, where $\hat{\mathbf{X}} = f(\hat{\mathbf{P}})$

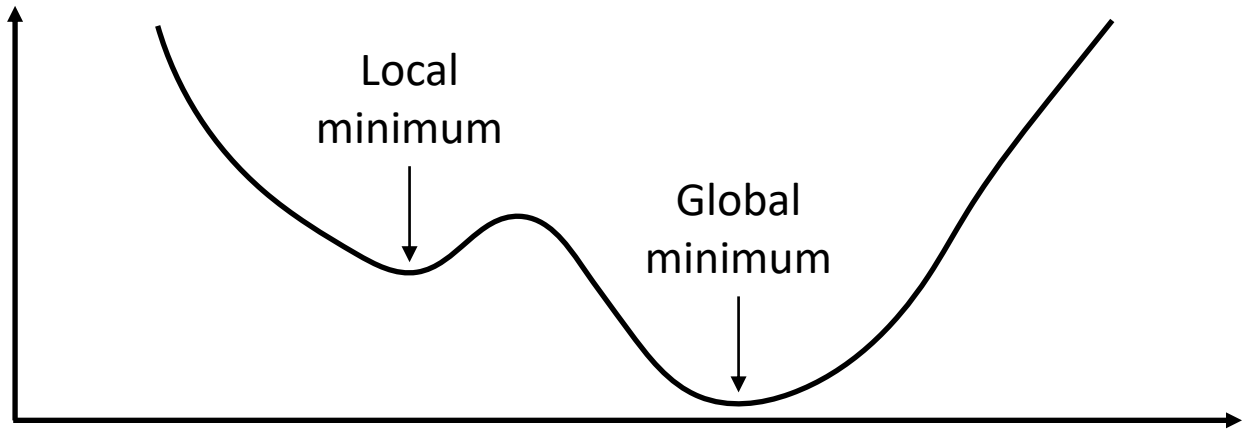
Objective

Find $\hat{\mathbf{P}}$ that minimizes $\boldsymbol{\epsilon}^\top \Sigma_{\mathbf{X}}^{-1} \boldsymbol{\epsilon}$, where $\boldsymbol{\epsilon} = \mathbf{X} - \hat{\mathbf{X}}$



Parameter vector

- May get stuck in local minimum of high dimensional space
- **Use minimal parameterizations towards mitigating this**



Parameter vector

- The fundamental matrix has 7 degrees of freedom, so can be parameterized with 7 parameters
  - Advanced topic (come to office hours)
- Alternatively, fix the first camera projection matrix and adjust the second one

Parameterization of a homogeneous vector

Let the homogeneous vector $\bar{\mathbf{v}} = (a, \mathbf{b}^\top)^\top \in \mathbb{R}^n$, where $\|\bar{\mathbf{v}}\| = 1$ (i.e., $\bar{\mathbf{v}}$ is a unit vector), be parameterized as

$$\mathbf{v} = \frac{2}{\operatorname{sinc}(\cos^{-1}(a))}\mathbf{b} \in \mathbb{R}^{n-1}$$

then, if $\|\mathbf{v}\| > \pi$, normalized by

$$\mathbf{v} = \left(1 - \frac{2\pi}{\|\mathbf{v}\|}\left\lceil \frac{\|\mathbf{v}\| - \pi}{2\pi} \right\rceil\right)\mathbf{v}$$

Always use the parameterized homogeneous vector, not the input homogeneous vector or matrix

Parameterization of a homogeneous vector

The parameterized homogeneous vector $\mathbf{v}$ is deparameterized as the homogeneous vector

$$\bar{\mathbf{v}} = \left(\cos\left(\frac{\|\mathbf{v}\|}{2}\right), \frac{\operatorname{sinc}\left(\frac{\|\mathbf{v}\|}{2}\right)}{2}\mathbf{v}^\top \right)^\top \in \mathbb{R}^n$$

Unit vector with nonnegative first element

$$\bar{\mathbf{v}} = \left(a, \mathbf{b}^\top\right)^\top, \text{ where } a = \cos\left(\frac{\|\mathbf{v}\|}{2}\right) \text{ and } \mathbf{b} = \frac{\operatorname{sinc}\left(\frac{\|\mathbf{v}\|}{2}\right)}{2}\mathbf{v}$$

where $\|\bar{\mathbf{v}}\| = 1$ and a is nonnegative. For the deparameterization,

$$\frac{\partial \bar{\mathbf{v}}}{\partial \mathbf{v}} = \frac{\partial (a, \mathbf{b}^\top)}{\partial \mathbf{v}} = \begin{bmatrix} \frac{\mathrm{d}a}{\partial \mathbf{v}} \\ \frac{\partial \mathbf{b}}{\partial \mathbf{v}} \end{bmatrix} \in \mathbb{R}^{n \times (n-1)}$$

where

$$\frac{\mathrm{d}a}{\partial \mathbf{v}} = \begin{cases} \mathbf{0}^\top & \text{if } \|\mathbf{v}\| = 0 \\ -\frac{1}{2}\mathbf{b}^\top & \text{otherwise} \end{cases}$$

and

$$\frac{\partial \mathbf{b}}{\partial \mathbf{v}} = \begin{cases} \frac{1}{2}\mathtt{I} & \text{if } \|\mathbf{v}\| = 0 \\ \frac{\operatorname{sinc}\left(\frac{\|\mathbf{v}\|}{2}\right)}{2}\mathtt{I} + \frac{1}{4\|\mathbf{v}\|}\frac{\mathrm{d}\operatorname{sinc}\left(\frac{\|\mathbf{v}\|}{2}\right)}{\mathrm{d}\frac{\|\mathbf{v}\|}{2}}\mathbf{v}\mathbf{v}^\top & \text{otherwise} \end{cases}$$

Parameterization of a homogeneous vector

- Sinc function

The sinc function

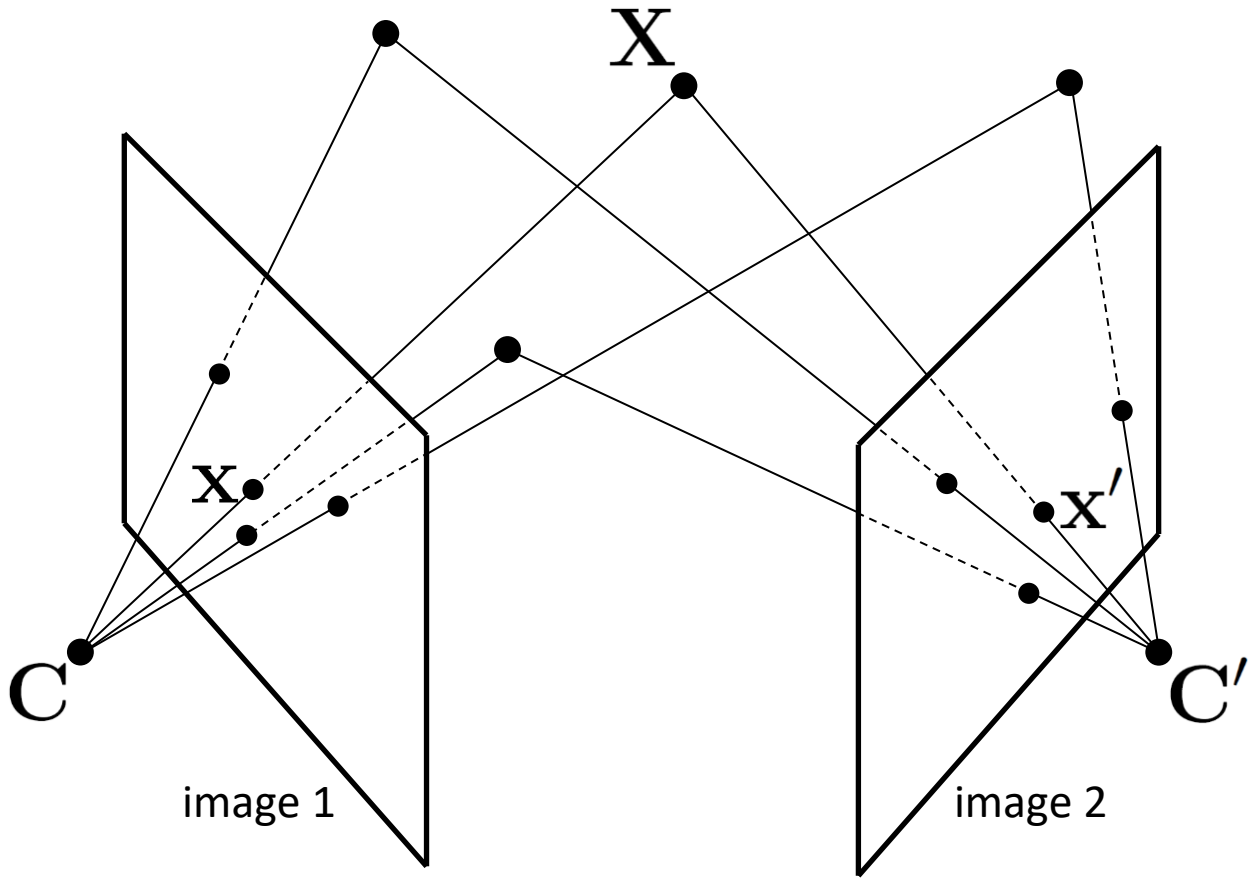
$$\operatorname{sinc}(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{\sin(x)}{x} & \text{otherwise} \end{cases}$$

The derivative is given by

$$\frac{\mathrm{d}\operatorname{sinc}(x)}{\mathrm{d}x} = \begin{cases} 0 & \text{if } x = 0 \\ \frac{\cos(x)}{x} - \frac{\sin(x)}{x^2} & \text{otherwise} \end{cases}$$

Two view geometry

- Imaging a 3D scene



Nonlinear estimation of fundamental matrix using the Levenberg-Marquardt algorithm

Parameterization of homogeneous vector

Parameter vector $(11 + 3n)$ x 1

$$\left(\hat{\mathbf{p}}'^{\top}, \hat{\mathbf{X}}_1^{\top}, \hat{\mathbf{X}}_2^{\top}, \ldots, \hat{\mathbf{X}}_n^{\top}\right)^{\top}$$

Always use the parameterized homogeneous vector, not the input homogeneous vector or matrix

Mapping

$$\hat{\mathbf{x}}_i = \mathrm{P}\hat{\mathbf{X}}_i \,\forall\, i$$
$$\hat{\mathbf{x}}'_i = \hat{\mathrm{P}}'\hat{\mathbf{X}}_i \,\forall\, i$$

Measurement vector $4n$ x 1

$$\left(\tilde{\mathbf{x}}_1^{\top}, \tilde{\mathbf{x}}_2^{\top}, \ldots, \tilde{\mathbf{x}}_n^{\top}, \tilde{\mathbf{x}}_1'^{\top}, \tilde{\mathbf{x}}_2'^{\top}, \ldots, \tilde{\mathbf{x}}_n'^{\top}\right)^{\top}$$

with associated covariance matrices $\Sigma_{\tilde{\mathbf{x}}_1}, \Sigma_{\tilde{\mathbf{x}}_2}, \ldots, \Sigma_{\tilde{\mathbf{x}}_n}, \Sigma_{\tilde{\mathbf{x}}'_1}, \Sigma_{\tilde{\mathbf{x}}'_2}, \ldots, \Sigma_{\tilde{\mathbf{x}}'_n}$

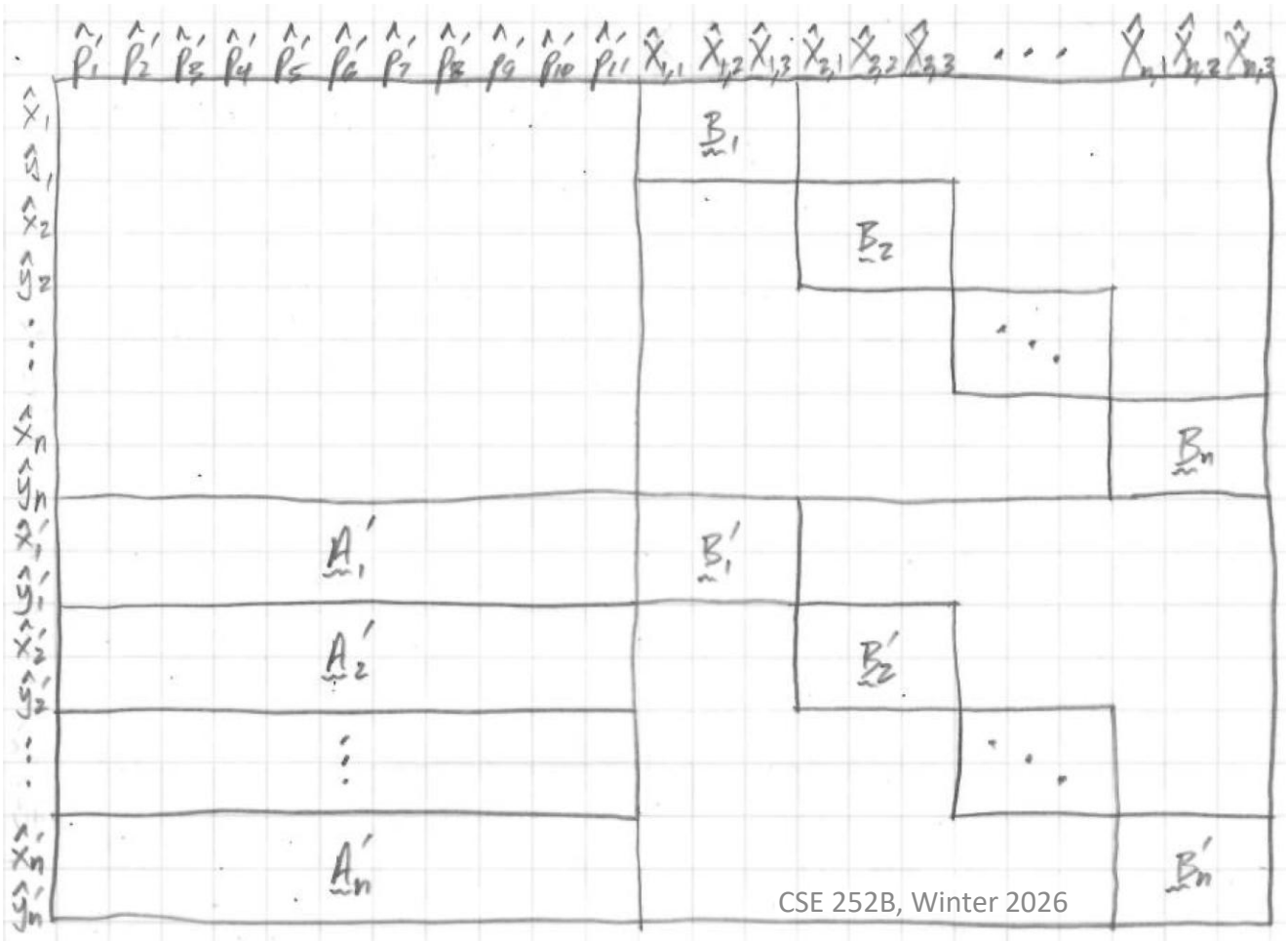
Cost

$$\sum_i \left(\boldsymbol{\epsilon}_i^{\top}\Sigma_{\tilde{\mathbf{x}}_i}^{-1}\boldsymbol{\epsilon}_i + \boldsymbol{\epsilon}_i'^{\top}\Sigma_{\tilde{\mathbf{x}}'_i}^{-1}\boldsymbol{\epsilon}'_i\right), \text{ where } \boldsymbol{\epsilon}_i = \tilde{\mathbf{x}}_i - \hat{\tilde{\mathbf{x}}}_i \text{ and } \boldsymbol{\epsilon}'_i = \tilde{\mathbf{x}}'_i - \hat{\tilde{\mathbf{x}}}'_i$$

But, use data normalization

Nonlinear estimation of fundamental matrix using the Levenberg-Marquardt algorithm

- Jacobian



$$\mathrm{A}'_i = \frac{\partial \hat{\tilde{\mathbf{x}}}'_i}{\partial \hat{\mathbf{p}}'} = \frac{\partial \hat{\tilde{\mathbf{x}}}'_i}{\partial \bar{\mathbf{p}}'} \frac{\partial \bar{\mathbf{p}}'}{\partial \hat{\mathbf{p}}'}$$

$$\mathrm{B}_i = \frac{\partial \hat{\tilde{\mathbf{x}}}_i}{\partial \hat{\mathbf{X}}_i} = \frac{\partial \hat{\tilde{\mathbf{x}}}_i}{\partial \bar{\mathbf{X}}_i} \frac{\partial \bar{\mathbf{X}}_i}{\partial \hat{\mathbf{X}}_i}$$

$$\mathrm{B}'_i = \frac{\partial \hat{\tilde{\mathbf{x}}}'_i}{\partial \hat{\mathbf{X}}_i} = \frac{\partial \hat{\tilde{\mathbf{x}}}'_i}{\partial \bar{\mathbf{X}}_i} \frac{\partial \bar{\mathbf{X}}_i}{\partial \hat{\mathbf{X}}_i}$$

2 x 12 12 x 11

2 x 4 4 x 3

Projection of a point under the camera projection matrix

The homogeneous 3D point $\mathbf{X}$ is projected to the homogeneous 2D point $\mathbf{x}$ under the (homogeneous) camera projection matrix $\mathtt{P}$ by

$$\mathbf{x} = \mathtt{P}\mathbf{X}$$

Dehomogenizing the 2D point results in the mapping $\mathbf{X} \mapsto \tilde{\mathbf{x}}$. For this mapping

$$\frac{\partial \tilde{\mathbf{x}}}{\partial \mathbf{p}} = \frac{1}{w}\begin{bmatrix} \mathbf{X}^\top & \mathbf{0}^\top & -\tilde{x}\mathbf{X}^\top \\ \mathbf{0}^\top & \mathbf{X}^\top & -\tilde{y}\mathbf{X}^\top \end{bmatrix} \qquad 2 \text{ x } 12$$

and

$$\frac{\partial \tilde{\mathbf{x}}}{\partial \mathbf{X}} = \frac{1}{w}\begin{bmatrix} \mathbf{p}^{1\top} - \tilde{x}\mathbf{p}^{3\top} \\ \mathbf{p}^{2\top} - \tilde{y}\mathbf{p}^{3\top} \end{bmatrix} \qquad 2 \text{ x } 4$$

where $w = \mathbf{p}^{3\top}\mathbf{X}$ and $\mathbf{p}^{i\top}$ is the ith row of $\mathtt{P}$.

Nonlinear estimation of fundamental matrix using the Levenberg-Marquardt algorithm

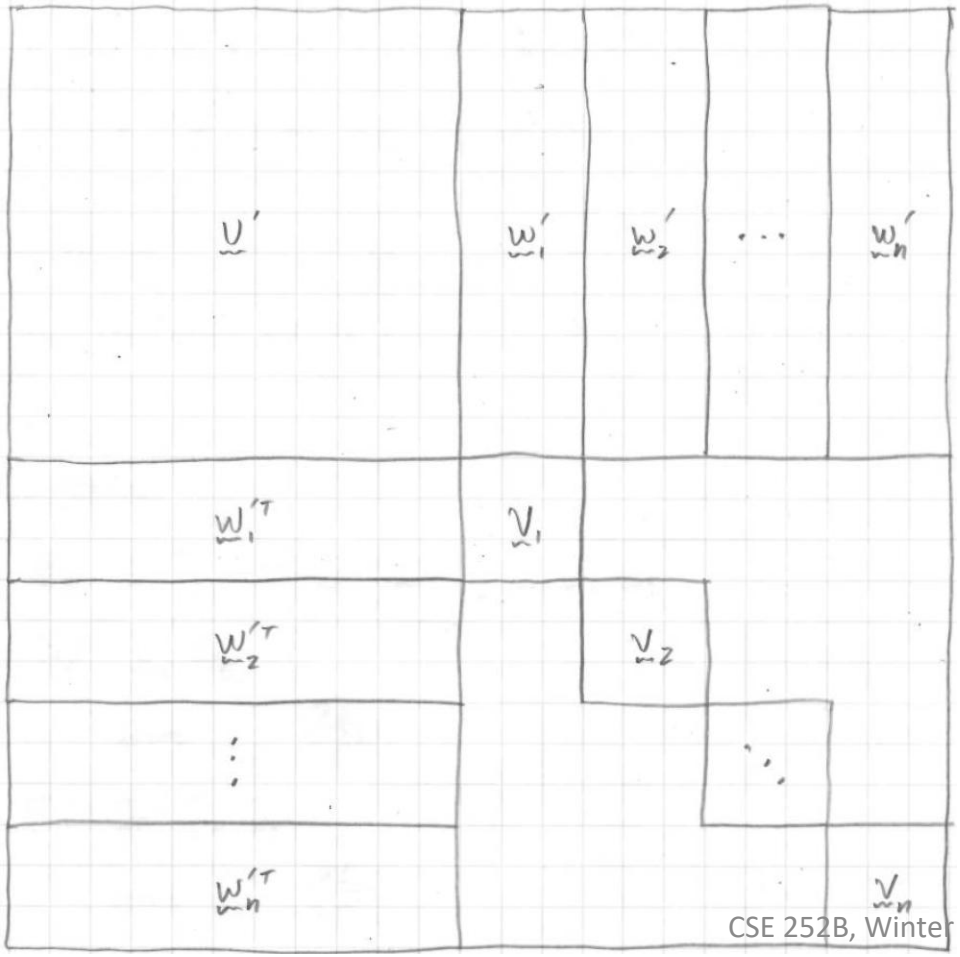
- Normal equations

$$\left(\mathtt{J}^\top \Sigma_{\mathbf{X}}^{-1} \mathtt{J}\right) \boldsymbol{\delta} = \mathtt{J}^\top \Sigma_{\mathbf{X}}^{-1} \boldsymbol{\epsilon}$$

$$\begin{bmatrix} \mathtt{U}' & \mathtt{W}' \\ \mathtt{W}'^\top & \mathtt{V} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta}_{\mathbf{a}'} \\ \boldsymbol{\delta}_{\mathbf{b}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\epsilon}_{\mathbf{a}'} \\ \boldsymbol{\epsilon}_{\mathbf{b}} \end{bmatrix}$$

Nonlinear estimation of fundamental matrix using the Levenberg-Marquardt algorithm

- Normal equations matrix $\begin{bmatrix} \mathrm{U}' & \mathrm{W}' \\ \mathrm{W}'^{\top} & \mathrm{V} \end{bmatrix}$



$$\mathrm{U}' = \sum_i \mathrm{A}_i'^{\top} \Sigma_{\tilde{\mathbf{x}}_i'}^{-1} \mathrm{A}_i' \qquad 11 \text{ x } 11$$

$$\mathrm{V}_i = \mathrm{B}_i^{\top} \Sigma_{\tilde{\mathbf{x}}_i}^{-1} \mathrm{B}_i + \mathrm{B}_i'^{\top} \Sigma_{\tilde{\mathbf{x}}_i'}^{-1} \mathrm{B}_i' \qquad 3 \text{ x } 3$$

$$\mathrm{W}_i' = \mathrm{A}_i'^{\top} \Sigma_{\tilde{\mathbf{x}}_i'}^{-1} \mathrm{B}_i' \qquad 11 \text{ x } 3$$

Nonlinear estimation of fundamental matrix using the Levenberg-Marquardt algorithm

- Normal equations vector $\begin{bmatrix} \boldsymbol{\epsilon}_{\mathbf{a}'} \\ \boldsymbol{\epsilon}_{\mathbf{b}} \end{bmatrix}$

$$\left(\boldsymbol{\epsilon}_{\mathbf{a}'}^\top, \boldsymbol{\epsilon}_{\mathbf{b}_1}^\top, \boldsymbol{\epsilon}_{\mathbf{b}_2}^\top, \ldots, \boldsymbol{\epsilon}_{\mathbf{b}_n}^\top\right)^\top$$

where

$$\boldsymbol{\epsilon}_{\mathbf{a}'} = \sum_i \mathtt{A}_i'^\top \Sigma_{\tilde{\mathbf{x}}_i'}^{-1} \boldsymbol{\epsilon}_i' \qquad 11 \text{ x } 1$$

$$\boldsymbol{\epsilon}_{\mathbf{b}_i} = \mathtt{B}_i^\top \Sigma_{\tilde{\mathbf{x}}_i}^{-1} \boldsymbol{\epsilon}_i + \mathtt{B}_i'^\top \Sigma_{\tilde{\mathbf{x}}_i'}^{-1} \boldsymbol{\epsilon}_i' \qquad 3 \text{ x } 1$$

where $\boldsymbol{\epsilon}_i = \tilde{\mathbf{x}}_i - \hat{\tilde{\mathbf{x}}}_i$ and $\boldsymbol{\epsilon}_i' = \tilde{\mathbf{x}}_i' - \hat{\tilde{\mathbf{x}}}_i'$

Nonlinear estimation of fundamental matrix using the Levenberg-Marquardt algorithm

Augmented normal equations

$$\begin{bmatrix} \mathtt{U}'^{*} & \mathtt{W}' \\ \mathtt{W}'^{\top} & \mathtt{V}^{*} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta}_{\mathbf{a}'} \\ \boldsymbol{\delta}_{\mathbf{b}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\epsilon}_{\mathbf{a}'} \\ \boldsymbol{\epsilon}_{\mathbf{b}} \end{bmatrix}$$

where $\mathtt{U}'^{*} = \mathtt{U}' + \lambda \mathtt{I}$ and $\mathtt{V}_i^{*} = \mathtt{V}_i + \lambda \mathtt{I}$

Solve for $\left(\boldsymbol{\delta}_{\mathbf{a}'}^{\top}, \boldsymbol{\delta}_{\mathbf{b}}^{\top}\right)^{\top}$ using blockwise methods

$$\begin{bmatrix} \mathtt{I} & -\mathtt{W}'\mathtt{V}^{*-1} \\ \mathtt{0} & \mathtt{I} \end{bmatrix} \begin{bmatrix} \mathtt{U}'^{*} & \mathtt{W}' \\ \mathtt{W}'^{\top} & \mathtt{V}^{*} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta}_{\mathbf{a}'} \\ \boldsymbol{\delta}_{\mathbf{b}} \end{bmatrix} = \begin{bmatrix} \mathtt{I} & -\mathtt{W}'\mathtt{V}^{*-1} \\ \mathtt{0} & \mathtt{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\epsilon}_{\mathbf{a}'} \\ \boldsymbol{\epsilon}_{\mathbf{b}} \end{bmatrix}$$

$$\begin{bmatrix} \mathtt{U}'^{*} - \mathtt{W}'\mathtt{V}^{*-1}\mathtt{W}'^{\top} & \mathtt{0} \\ \mathtt{W}'^{\top} & \mathtt{V}^{*} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta}_{\mathbf{a}'} \\ \boldsymbol{\delta}_{\mathbf{b}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\epsilon}_{\mathbf{a}'} - \mathtt{W}'\mathtt{V}^{*-1}\boldsymbol{\epsilon}_{\mathbf{b}} \\ \boldsymbol{\epsilon}_{\mathbf{b}} \end{bmatrix}$$

$$\begin{bmatrix} (\mathtt{U}'^{*} - \mathtt{W}'\mathtt{V}^{*-1}\mathtt{W}'^{\top})\boldsymbol{\delta}_{\mathbf{a}'} \\ \mathtt{W}'^{\top}\boldsymbol{\delta}_{\mathbf{a}'} + \mathtt{V}^{*}\boldsymbol{\delta}_{\mathbf{b}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\epsilon}_{\mathbf{a}'} - \mathtt{W}'\mathtt{V}^{*-1}\boldsymbol{\epsilon}_{\mathbf{b}} \\ \boldsymbol{\epsilon}_{\mathbf{b}} \end{bmatrix}$$

Nonlinear estimation of fundamental matrix using the Levenberg-Marquardt algorithm

$$\begin{bmatrix} (\mathtt{U}'^{*} - \mathtt{W}'\mathtt{V}^{*-1}\mathtt{W}'^{\top})\boldsymbol{\delta}_{\mathbf{a}'} \\ \mathtt{W}'^{\top}\boldsymbol{\delta}_{\mathbf{a}'} + \mathtt{V}^{*}\boldsymbol{\delta}_{\mathbf{b}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\epsilon}_{\mathbf{a}'} - \mathtt{W}'\mathtt{V}^{*-1}\boldsymbol{\epsilon}_{\mathbf{b}} \\ \boldsymbol{\epsilon}_{\mathbf{b}} \end{bmatrix}$$

1. Solve for $\boldsymbol{\delta}_{\mathbf{a}'}$

$$(\mathtt{U}'^{*} - \mathtt{W}'\mathtt{V}^{*-1}\mathtt{W}'^{\top})\boldsymbol{\delta}_{\mathbf{a}'} = \boldsymbol{\epsilon}_{\mathbf{a}'} - \mathtt{W}'\mathtt{V}^{*-1}\boldsymbol{\epsilon}_{\mathbf{b}}$$

$$\mathtt{S}'\boldsymbol{\delta}_{\mathbf{a}'} = \mathbf{e}', \text{ solve for } \boldsymbol{\delta}_{\mathbf{a}'} \qquad 11 \text{ x } 1$$

where

$$\mathtt{S}' = \mathtt{U}'^{*} - \sum_i \mathtt{W}'_i\mathtt{V}_i^{*-1}\mathtt{W}_i'^{\top} \qquad 11 \text{ x } 11$$

$$\mathbf{e}' = \boldsymbol{\epsilon}_{\mathbf{a}'} - \sum_i \mathtt{W}'_i\mathtt{V}_i^{*-1}\boldsymbol{\epsilon}_{\mathbf{b}_i} \qquad 11 \text{ x } 1$$

Nonlinear estimation of fundamental matrix using the Levenberg-Marquardt algorithm

$$\begin{bmatrix} (\mathtt{U}'^{*} - \mathtt{W}'\mathtt{V}^{*-1}\mathtt{W}'^{\top})\boldsymbol{\delta}_{\mathbf{a}'} \\ \mathtt{W}'^{\top}\boldsymbol{\delta}_{\mathbf{a}'} + \mathtt{V}^{*}\boldsymbol{\delta}_{\mathbf{b}} \end{bmatrix} - \begin{bmatrix} \boldsymbol{\epsilon}_{\mathbf{a}'} - \mathtt{W}'\mathtt{V}^{*-1}\boldsymbol{\epsilon}_{\mathbf{b}} \\ \boldsymbol{\epsilon}_{\mathbf{b}} \end{bmatrix}$$

2. Solve for $\boldsymbol{\delta}_{\mathbf{b}}$

$$\begin{aligned} \mathtt{W}'^{\top}\boldsymbol{\delta}_{\mathbf{a}'} + \mathtt{V}^{*}\boldsymbol{\delta}_{\mathbf{b}} &= \boldsymbol{\epsilon}_{\mathbf{b}} \\ \mathtt{V}^{*}\boldsymbol{\delta}_{\mathbf{b}} &= \boldsymbol{\epsilon}_{\mathbf{b}} - \mathtt{W}'^{\top}\boldsymbol{\delta}_{\mathbf{a}'} \\ \boldsymbol{\delta}_{\mathbf{b}} &= \mathtt{V}^{*-1}\left(\boldsymbol{\epsilon}_{\mathbf{b}} - \mathtt{W}'^{\top}\boldsymbol{\delta}_{\mathbf{a}'}\right) \end{aligned}$$

where

$$\boldsymbol{\delta}_{\mathbf{b}_i} = \mathtt{V}_i^{*-1}(\boldsymbol{\epsilon}_{\mathbf{b}_i} - \mathtt{W}_i'^{\top}\boldsymbol{\delta}_{\mathbf{a}'}) \quad 3 \text{ x } 1$$

Candidate parameter vector

$$\begin{aligned} \hat{\mathbf{p}}'_0 &= \hat{\mathbf{p}}' + \boldsymbol{\delta}_{\mathbf{a}'} \\ \hat{\mathbf{X}}_{i_0} &= \hat{\mathbf{X}}_i + \boldsymbol{\delta}_{\mathbf{b}_i} \end{aligned}$$

Adjustment

Normalize parameterization of homogeneous vector after adjustment

Data normalization with covariance propagation

Determine the similarity transformation $\mathtt{T}$ such that the mean (i.e., centroid) of the transformed points is at the origin and their standard deviation from the origin is $\sqrt{2}$.

$$\mathtt{T} = \begin{bmatrix} s & 0 & -s\mu_{\tilde{x}} \\ 0 & s & -s\mu_{\tilde{y}} \\ 0 & 0 & 1 \end{bmatrix}, \text{ where } s = \sqrt{\frac{2}{\sigma_{\tilde{x}}^2 + \sigma_{\tilde{y}}^2}}$$

where $\mu_{\tilde{x}}$ and $\sigma_{\tilde{x}}^2$, and $\mu_{\tilde{y}}$ and $\sigma_{\tilde{y}}^2$ are the mean and variance of the $\tilde{x}$ and $\tilde{y}$ coordinates, respectively.

$$\mathbf{x}_{\text{DN}} = \mathtt{T}\mathbf{x}$$

$$\begin{bmatrix} \tilde{\mathbf{x}}_{\text{DN}} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}} \\ 1 \end{bmatrix}, \text{ where } \mathtt{A} = s\mathtt{I} \text{ and } \mathbf{t} = \begin{bmatrix} -s\mu_{\tilde{x}} \\ -s\mu_{\tilde{y}} \end{bmatrix}$$

$$\begin{bmatrix} \tilde{\mathbf{x}}_{\text{DN}} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathtt{A}\tilde{\mathbf{x}} + \mathbf{t} \\ 1 \end{bmatrix}$$

$$\tilde{\mathbf{x}}_{\text{DN}} = \mathtt{A}\tilde{\mathbf{x}} + \mathbf{t}$$

If unknown, then assume to be identity

Covariance propagation

$$\Sigma_{\tilde{\mathbf{x}}_{\text{DN}}} = \mathtt{J}\Sigma_{\tilde{\mathbf{x}}}\mathtt{J}^\top, \text{ where } \mathtt{J} = \frac{\partial \tilde{\mathbf{x}}_{\text{DN}}}{\partial \tilde{\mathbf{x}}} = \mathtt{A} = s\mathtt{I}$$

Nonlinear estimation of fundamental matrix using the Levenberg-Marquardt algorithm

- Algorithm

1. Initialize the scene points using two-view optimal triangulation
2. Data normalize the image points

$$\tilde{\mathbf{x}}_{\mathrm{DN},i} = \mathtt{A}\tilde{\mathbf{x}}_i + \mathbf{t} \text{ and associated covariance matrices } \Sigma_{\tilde{\mathbf{x}}_{\mathrm{DN},i}} = s^2\Sigma_{\tilde{\mathbf{x}}_i}$$
$$\tilde{\mathbf{x}}'_{\mathrm{DN},i} = \mathtt{A}'\tilde{\mathbf{x}}'_i + \mathbf{t}' \text{ and associated covariance matrices } \Sigma_{\tilde{\mathbf{x}}'_{\mathrm{DN},i}} = s'^2\Sigma_{\tilde{\mathbf{x}}'_i}$$

scene points

$$\mathbf{X}_{\mathrm{DN},i} = \mathtt{U}\mathbf{X}_i \,\forall\, i$$

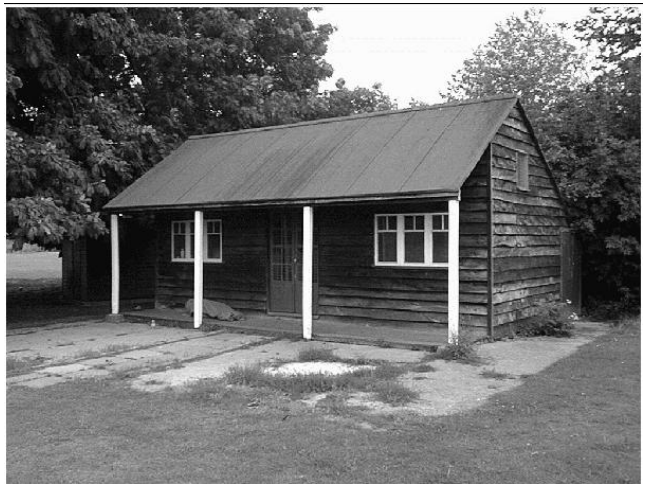
and camera projection matrices

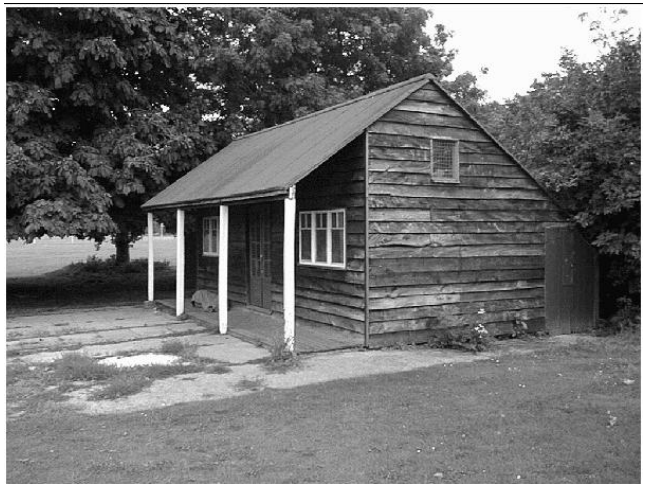
$$\mathtt{P}_{\mathrm{DN}} = \mathtt{T}[\mathtt{I} \mid \mathbf{0}]\mathtt{U}^{-1}$$
$$\mathtt{P}'_{\mathrm{DN}} = \mathtt{T}'\mathtt{P}'\mathtt{U}^{-1}$$

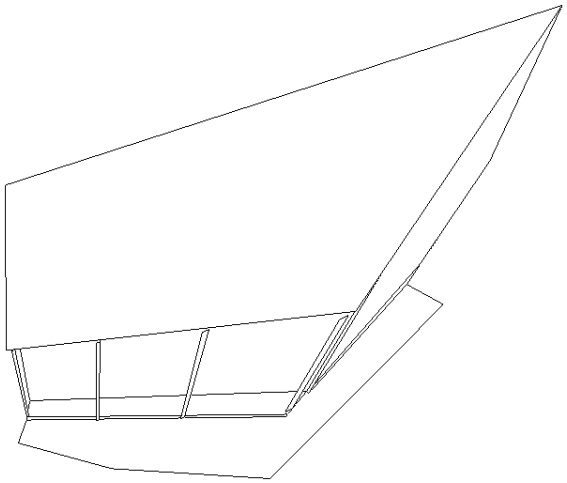
3. Estimate the data normalized camera projection matrix $\mathtt{P}'_{\mathrm{DN}}$ and data normalized scene points from the data normalized image points
4. Data denormalize the final estimate of the data normalized camera projection matrix

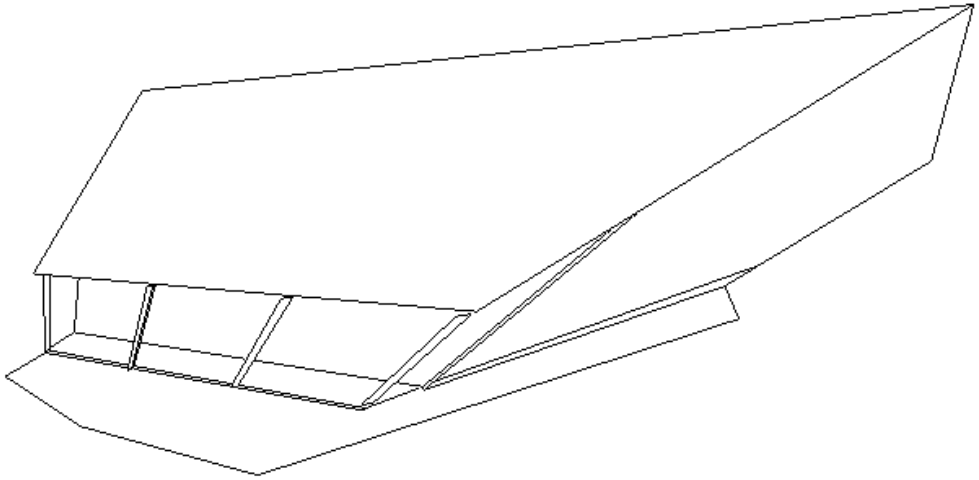
$$\mathtt{P}' = \mathtt{T}'^{-1}\mathtt{P}'_{\mathrm{DN}}\mathtt{U}$$

Up to 3D projective transformation









3D geometric transformations

- Projective

4×4 homogeneous (i.e., defined up to nonzero scale) matrix (15 degrees of freedom)

$$\begin{bmatrix} X' \\ Y' \\ Z' \\ T' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & t_1 \\ a_{21} & a_{22} & a_{23} & t_2 \\ a_{31} & a_{32} & a_{33} & t_3 \\ v_1 & v_2 & v_3 & u \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ T \end{bmatrix}$$

$$\mathbf{X}' = \begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^\top & u \end{bmatrix} \mathbf{X}, \text{ where } \mathtt{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \mathbf{t} = (t_1, t_2, t_3)^\top, \text{ and } \mathbf{v} = (v_1, v_2, v_3)^\top$$

$$\mathbf{X}' = \mathtt{H}_\mathrm{P}\mathbf{X}, \text{ where } \mathtt{H}_\mathrm{P} = \begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^\top & u \end{bmatrix}$$

3D geometric transformations

Linear transformation of points in homogeneous coordinates

$$\mathbf{X}' = \mathtt{H}\mathbf{X}$$

$$\begin{bmatrix} X' \\ Y' \\ Z' \\ T' \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ T \end{bmatrix}$$

$$\begin{bmatrix} X' \\ Y' \\ Z' \\ T' \end{bmatrix} = \begin{bmatrix} Xh_{11} + Yh_{12} + Zh_{13} + Th_{14} \\ Xh_{21} + Yh_{22} + Zh_{23} + Th_{24} \\ Xh_{31} + Yh_{32} + Zh_{33} + Th_{34} \\ Xh_{41} + Yh_{42} + Zh_{43} + Th_{44} \end{bmatrix}$$

Nonlinear transformation of points in inhomogeneous coordinates

$$\tilde{X} = \frac{X'}{T'} = \frac{Xh_{11} + Yh_{12} + Zh_{13} + Th_{14}}{Xh_{41} + Yh_{42} + Zh_{43} + Th_{44}},$$

$$\tilde{Y} = \frac{Y'}{T'} = \frac{Xh_{21} + Yh_{22} + Zh_{23} + Th_{24}}{Xh_{41} + Yh_{42} + Zh_{43} + Th_{44}}, \text{ and}$$

$$\tilde{Z} = \frac{Z'}{T'} = \frac{Xh_{31} + Yh_{32} + Zh_{33} + Th_{34}}{Xh_{41} + Yh_{42} + Zh_{43} + Th_{44}}$$

Do not forget this is a nonlinear transformation

3D projective transformation

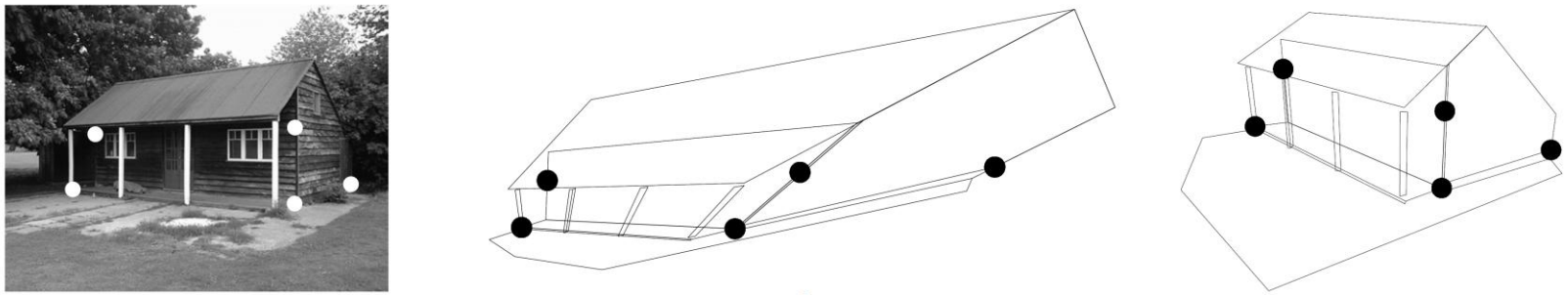


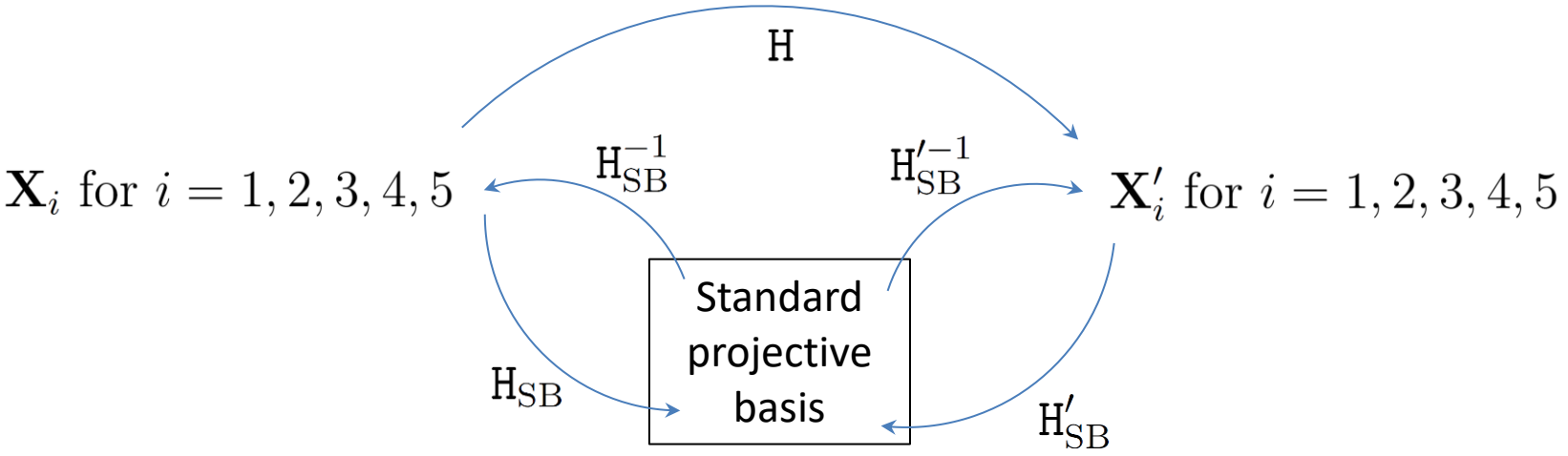
Fig. 10.6. **Direct reconstruction.** *The projective reconstruction of figure 10.3 may be upgraded to metric by specifying the position of five (or more) world points: (a) the five points used; (b) the corresponding points on the projective reconstruction of figure 10.3; (c) the reconstruction after the five points are mapped to their world positions.*

3D projective transformation, minimum solution

- Transformation from 5 points in projective space to 5 points in metric space

The minimum number of point correspondences $\mathbf{X}_i \leftrightarrow \mathbf{X}'_i$ needed to exactly solve for the 3D projective transformation (15 parameters) is 5 (i.e., 15 measurements).

$$\mathbf{X}'_i = \mathtt{H}'^{-1}_{\mathrm{SB}} \mathtt{H}_{\mathrm{SB}} \mathbf{X}_i \text{ for } i = 1, 2, 3, 4, 5$$
$$\mathbf{X}'_i = \mathtt{H} \mathbf{X}_i \text{ for } i = 1, 2, 3, 4, 5 \text{ where } \mathtt{H} = \mathtt{H}'^{-1}_{\mathrm{SB}} \mathtt{H}_{\mathrm{SB}}$$



3D projective transformation, 5 points to standard basis

Points at infinity along each axis

Origin

"Unit point"

Standard basis of $\mathbb{P}^3$

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \text{ and } \mathbf{e}_5 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

5 points can be mapped to standard basis

$$\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 & \mathbf{e}_5 \end{bmatrix} = \mathtt{H}_{\mathrm{SB}} \begin{bmatrix} s_1\mathbf{X}_1 & s_2\mathbf{X}_2 & s_3\mathbf{X}_3 & s_4\mathbf{X}_4 & s_5\mathbf{X}_5 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 & \mathbf{e}_5 \end{bmatrix} = \mathtt{H}_{\mathrm{SB}} \begin{bmatrix} \lambda_1\mathbf{X}_1 & \lambda_2\mathbf{X}_2 & \lambda_3\mathbf{X}_3 & \lambda_4\mathbf{X}_4 & \mathbf{X}_5 \end{bmatrix}$$

$$\text{where } \lambda_1 = \frac{s_1}{s_5}, \lambda_2 = \frac{s_2}{s_5}, \lambda_3 = \frac{s_3}{s_5} \text{ and } \lambda_4 = \frac{s_4}{s_5}$$

3D projective transformation, 5 points to standard basis

$$\begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4\end{bmatrix} = \mathtt{H}_{\mathrm{SB}} \begin{bmatrix}\lambda_1\mathbf{X}_1 & \lambda_2\mathbf{X}_2 & \lambda_3\mathbf{X}_3 & \lambda_4\mathbf{X}_4\end{bmatrix}$$
$$\mathtt{I} = \mathtt{H}_{\mathrm{SB}} \begin{bmatrix}\lambda_1\mathbf{X}_1 & \lambda_2\mathbf{X}_2 & \lambda_3\mathbf{X}_3 & \lambda_4\mathbf{X}_4\end{bmatrix}$$
$$\mathtt{H}_{\mathrm{SB}}^{-1} = \begin{bmatrix}\lambda_1\mathbf{X}_1 & \lambda_2\mathbf{X}_2 & \lambda_3\mathbf{X}_3 & \lambda_4\mathbf{X}_4\end{bmatrix}$$

$$\mathbf{e}_5 = \mathtt{H}_{\mathrm{SB}}\mathbf{X}_5$$
$$\mathtt{H}_{\mathrm{SB}}^{-1}\mathbf{e}_5 = \mathbf{X}_5$$
$$\begin{bmatrix}\lambda_1\mathbf{X}_1 & \lambda_2\mathbf{X}_2 & \lambda_3\mathbf{X}_3 & \lambda_4\mathbf{X}_4\end{bmatrix}\mathbf{e}_5 = \mathbf{X}_5$$
$$\begin{bmatrix}\mathbf{X}_1 & \mathbf{X}_2 & \mathbf{X}_3 & \mathbf{X}_4\end{bmatrix}\boldsymbol{\lambda} = \mathbf{X}_5, \text{ solve for } \boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)^\top$$

then

$$\mathtt{H}_{\mathrm{SB}}^{-1} = \begin{bmatrix}\lambda_1\mathbf{X}_1 & \lambda_2\mathbf{X}_2 & \lambda_3\mathbf{X}_3 & \lambda_4\mathbf{X}_4\end{bmatrix}$$

Linear estimation of 3D projective transformation using the direct linear transformation (DLT) algorithm

Given $n \geq 5$ point correspondences $\mathbf{X}_i \leftrightarrow \mathbf{X}'_i$

$$\mathbf{X}'_i = \mathtt{H}\mathbf{X}_i \,\forall\, i, \text{ solve for } \mathtt{H}$$

$$[\mathbf{X}'_i]^{\perp}\mathtt{H}\mathbf{X}_i = \mathbf{0} \,\forall\, i, \text{ solve for } \mathtt{H}$$

$$\left([\mathbf{X}'_i]^{\perp} \otimes \mathbf{X}_i^{\top}\right)\mathbf{h} = \mathbf{0} \,\forall\, i, \text{ solve for } \mathbf{h} = \mathrm{vec}(\mathtt{H}^{\top})$$

$$\begin{bmatrix} [\mathbf{X}'_1]^{\perp} \otimes \mathbf{X}_1^{\top} \\ [\mathbf{X}'_2]^{\perp} \otimes \mathbf{X}_2^{\top} \\ \vdots \\ [\mathbf{X}'_n]^{\perp} \otimes \mathbf{X}_n^{\top} \end{bmatrix} \mathbf{h} = 0$$

$$\mathtt{A}\mathbf{h} = \mathbf{0}, \text{ where } \mathtt{A} = \begin{bmatrix} [\mathbf{X}'_1]^{\perp} \otimes \mathbf{X}_1^{\top} \\ [\mathbf{X}'_2]^{\perp} \otimes \mathbf{X}_2^{\top} \\ \vdots \\ [\mathbf{X}'_n]^{\perp} \otimes \mathbf{X}_n^{\top} \end{bmatrix}$$

But, data normalization must be used

Next lecture

- Estimation of essential matrix
- Reading
  - Section 9.6
  - Kukelova et al. “Polynomial Eigenvalue Solutions to the 5-pt and 6-pt Relative Pose Problems”