Estimation of Fundamental Matrix, Linear

Computer Vision II

CSE 252B

Lecture 12

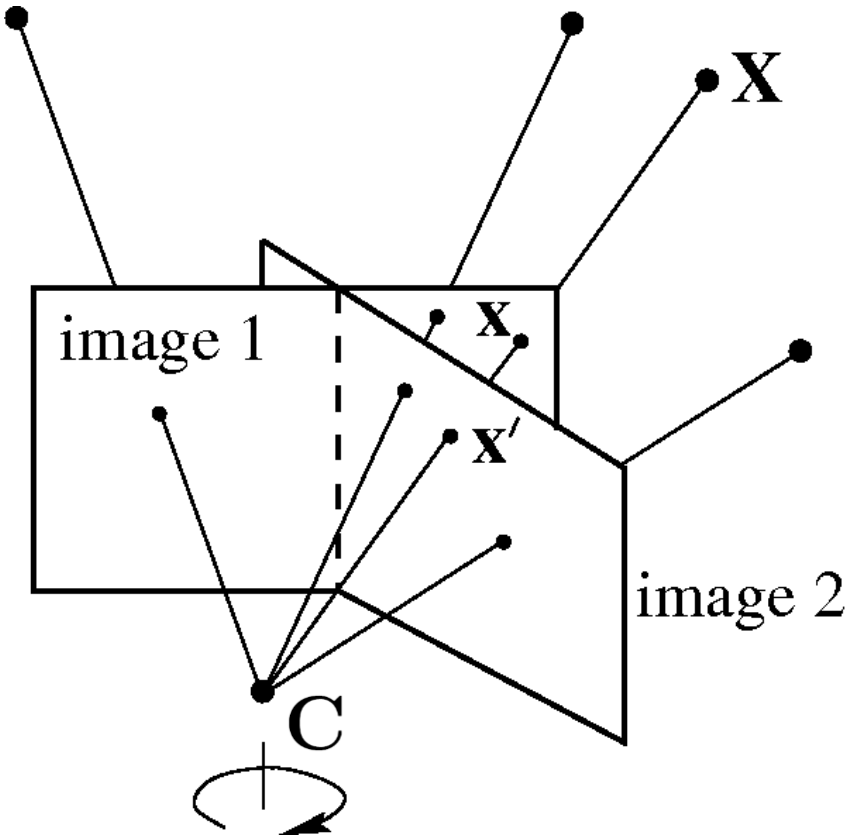
Announcements

- Assignment 3 is due today, 11:59 PM
- Assignment 4 will be released today
  - Due Mar 4, 11:59 PM
- Reading
  - Sections 9.1, 9.2, 11.1, 12.4, 11.2, and 11.3

Two view geometry

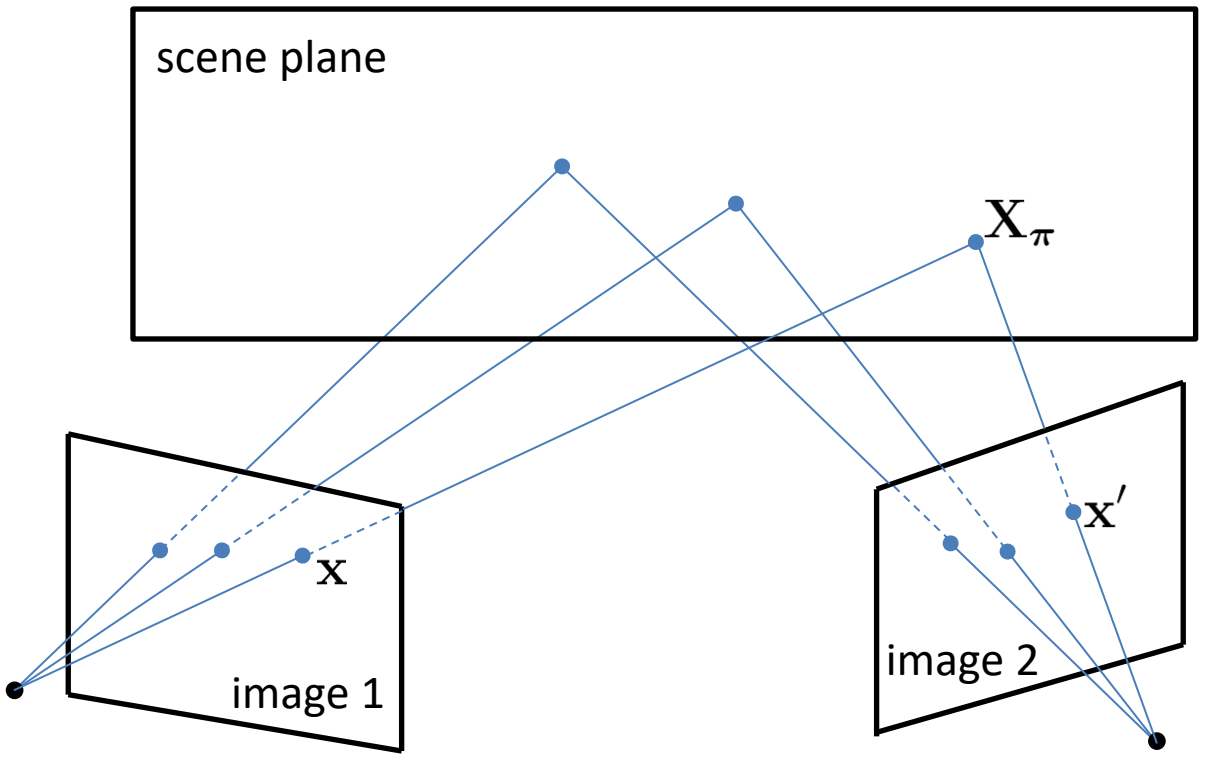
Two view geometry

- Rotation about the same camera center



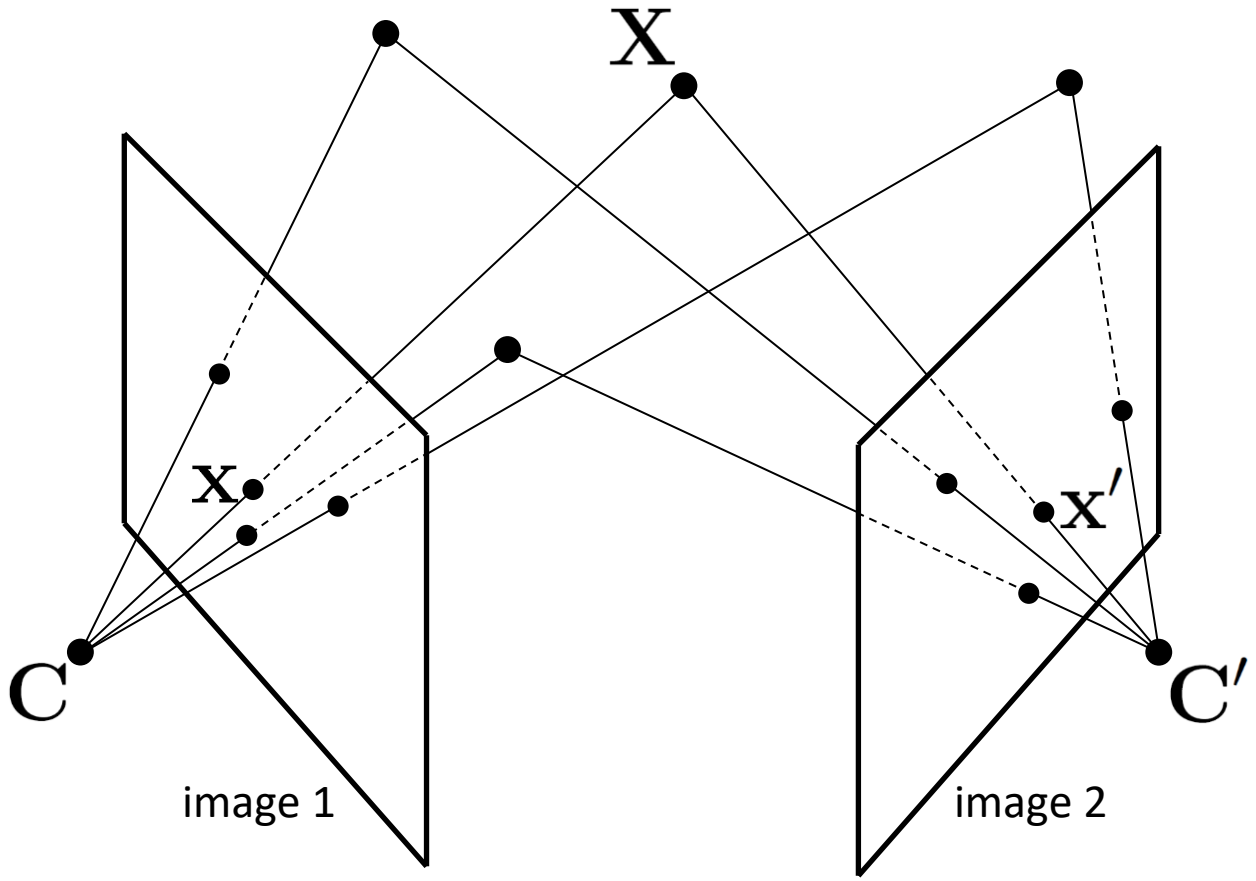
Two view geometry

- Imaging a plane



Two view geometry

- Imaging a 3D scene



Imaging geometry models

- Single view
  - Uncalibrated: camera projection matrix
  - Calibrated: normalized camera projection matrix
- Two views
  - Rotation about the same camera center
    - Uncalibrated: 2D projective transformation matrix
    - Calibrated: 3D rotation matrix
  - Imaging a plane
    - Uncalibrated: 2D projective transformation matrix
    - Calibrated: 2D projective transformation matrix
  - Imaging a 3D scene
    - Uncalibrated: fundamental matrix

Fundamental matrix

Given the canonical cameras $\mathtt{P} = [\mathtt{I} \,|\, \mathbf{0}]$ and $\mathtt{P}' = [\mathtt{M} \,|\, \mathbf{v}]$, backproject the point $\mathbf{x} = (x, y, w)^\top$ in the first image to a line in 3D
First point on the 3D line

$$\mathbf{C} = \begin{bmatrix} \tilde{\mathbf{C}} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} \qquad \text{Center of first camera}$$

Second point on the 3D line

$$\mathbf{x} = \mathtt{P}\mathbf{X}$$

$$\mathbf{X} = \mathtt{P}^+\mathbf{x} = \begin{bmatrix} \mathtt{I} \\ \mathbf{0}^\top \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{x} \\ 0 \end{bmatrix}, \text{ where } \mathtt{P}^+ = \mathtt{P}^\top(\mathtt{P}\mathtt{P}^\top)^{-1} = \begin{bmatrix} \mathtt{I} \\ \mathbf{0}^\top \end{bmatrix} \text{ is the pseudoinverse of } \mathtt{P}$$

3D line $\mathtt{L}$ joining the two 3D points

$$\mathtt{L} = \mathbf{X}\mathbf{C}^\top - \mathbf{C}\mathbf{X}^\top = \begin{bmatrix} \mathtt{0} & \mathbf{x} \\ -\mathbf{x}^\top & 0 \end{bmatrix}$$

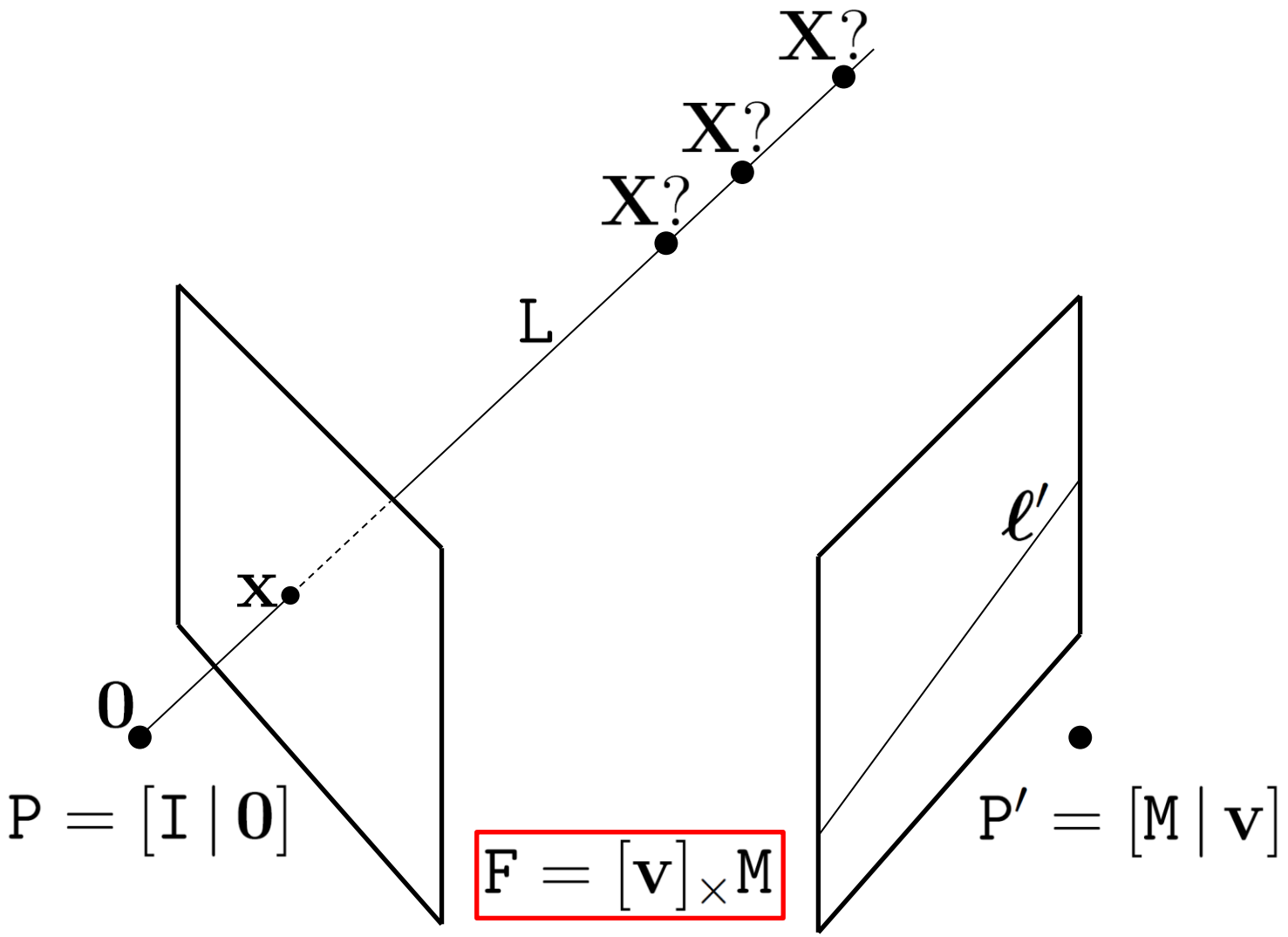
The projection of the 3D line $\mathtt{L}$ to a 2D line $\boldsymbol{\ell}'$ in the second image is given by

$$[\boldsymbol{\ell}']_\times = \mathtt{P}'\mathtt{L}\mathtt{P}'^\top = \begin{bmatrix} \mathtt{M} & \mathbf{v} \end{bmatrix} \begin{bmatrix} \mathtt{0} & \mathbf{x} \\ -\mathbf{x}^\top & 0 \end{bmatrix} \begin{bmatrix} \mathtt{M}^\top \\ \mathbf{v}^\top \end{bmatrix}$$

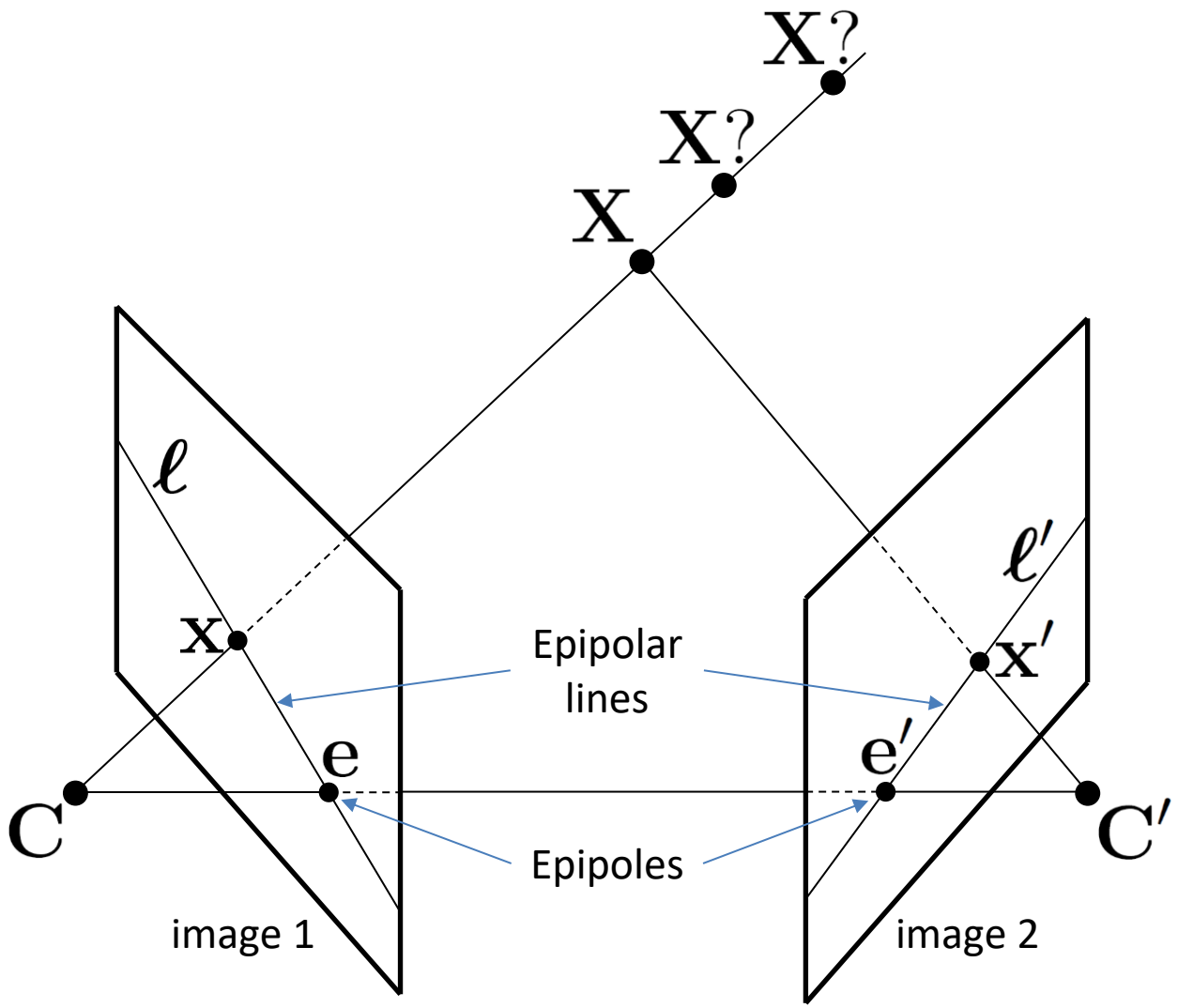
$$\boldsymbol{\ell}' = [\mathbf{v}]_\times \mathtt{M}\mathbf{x} = \mathtt{F}\mathbf{x}, \text{ where the fundamental matrix } \mathtt{F} = [\mathbf{v}]_\times \mathtt{M}$$

3×3 skew symmetric matrices are rank 2, so $\mathtt{F}$ is rank

Fundamental matrix



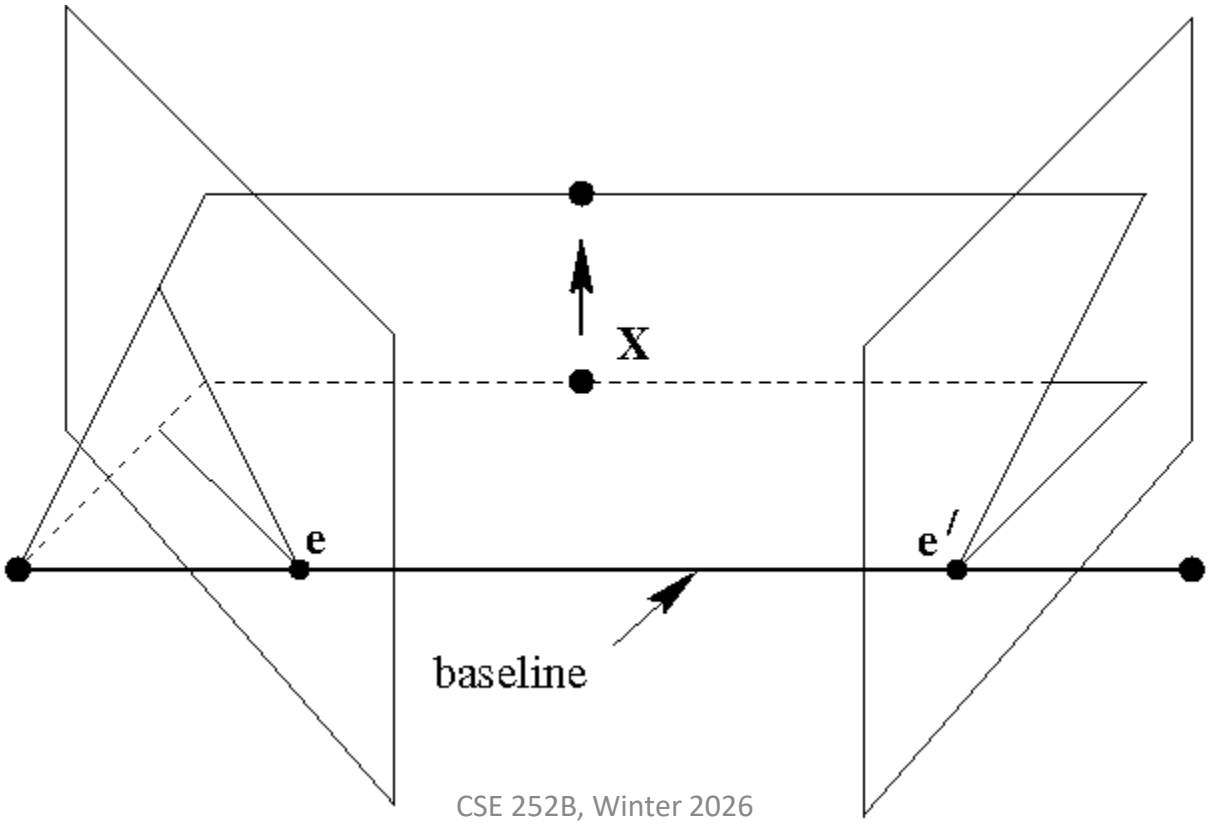
Fundamental matrix



Two view geometry

- All epipolar lines pass through the epipole

F is a rank 2 homogeneous matrix



Fundamental matrix

$$\boldsymbol{\ell}' = \mathtt{F}\mathbf{x} \text{ and } \mathbf{x}'^{\top}\boldsymbol{\ell}' = 0$$

$$\mathbf{x}'^{\top}\mathtt{F}\mathbf{x} = 0 \qquad \text{Epipolar constraint}$$

$\boldsymbol{\ell}' = \mathtt{F}\mathbf{x}$	The epipolar line corresponding to the point $\mathbf{x}$
$\boldsymbol{\ell} = \mathtt{F}^{\top}\mathbf{x}'$	The epipolar line corresponding to the point $\mathbf{x}'$
$\mathtt{F}\mathbf{e} = \mathbf{0}$	The epipole $\mathbf{e}$ is the null space of $\mathtt{F}$
$\mathbf{e}'^{\top}\mathtt{F} = \mathbf{0}^{\top}$	The epipole $\mathbf{e}'$ is the left null space of $\mathtt{F}$
$\mathtt{F}^{\top}\mathbf{e}' = \mathbf{0}$	The epipole $\mathbf{e}'$ is the null space of $\mathtt{F}^{\top}$

The fundamental matrix $\mathtt{F}$ from general cameras $\mathtt{P}$ and $\mathtt{P}'$
Transform the cameras such that first camera projection matrix is $[\mathtt{I} \,|\, \mathbf{0}]$ as follows

$$\mathtt{P}\mathtt{H}_{\mathrm{P}}^{-1} = [\mathtt{P}\mathtt{P}^{+} \,|\, \mathtt{P}\mathbf{C}] = [\mathtt{I} \,|\, \mathbf{0}] \text{ and } \mathtt{P}'\mathtt{H}_{\mathrm{P}}^{-1} = [\mathtt{P}'\mathtt{P}^{+} \,|\, \mathtt{P}'\mathbf{C}] = [\mathtt{P}'\mathtt{P}^{+} \,|\, \mathbf{e}']$$

where $\mathbf{e}' = \mathtt{P}'\mathbf{C}, \mathtt{P}\mathbf{C} = \mathbf{0}$ (i.e., $\mathbf{C}$ is the null space of $\mathtt{P}$), $\mathtt{H}_{\mathrm{P}}^{-1} = [\mathtt{P}^{+} \,|\, \mathbf{C}]$, and $\mathtt{P}^{+} = \mathtt{P}^{\top}(\mathtt{P}\mathtt{P}^{\top})^{-1}$
Then

$$\mathtt{F} = [\mathbf{e}']_{\times}\mathtt{P}'\mathtt{P}^{+}$$

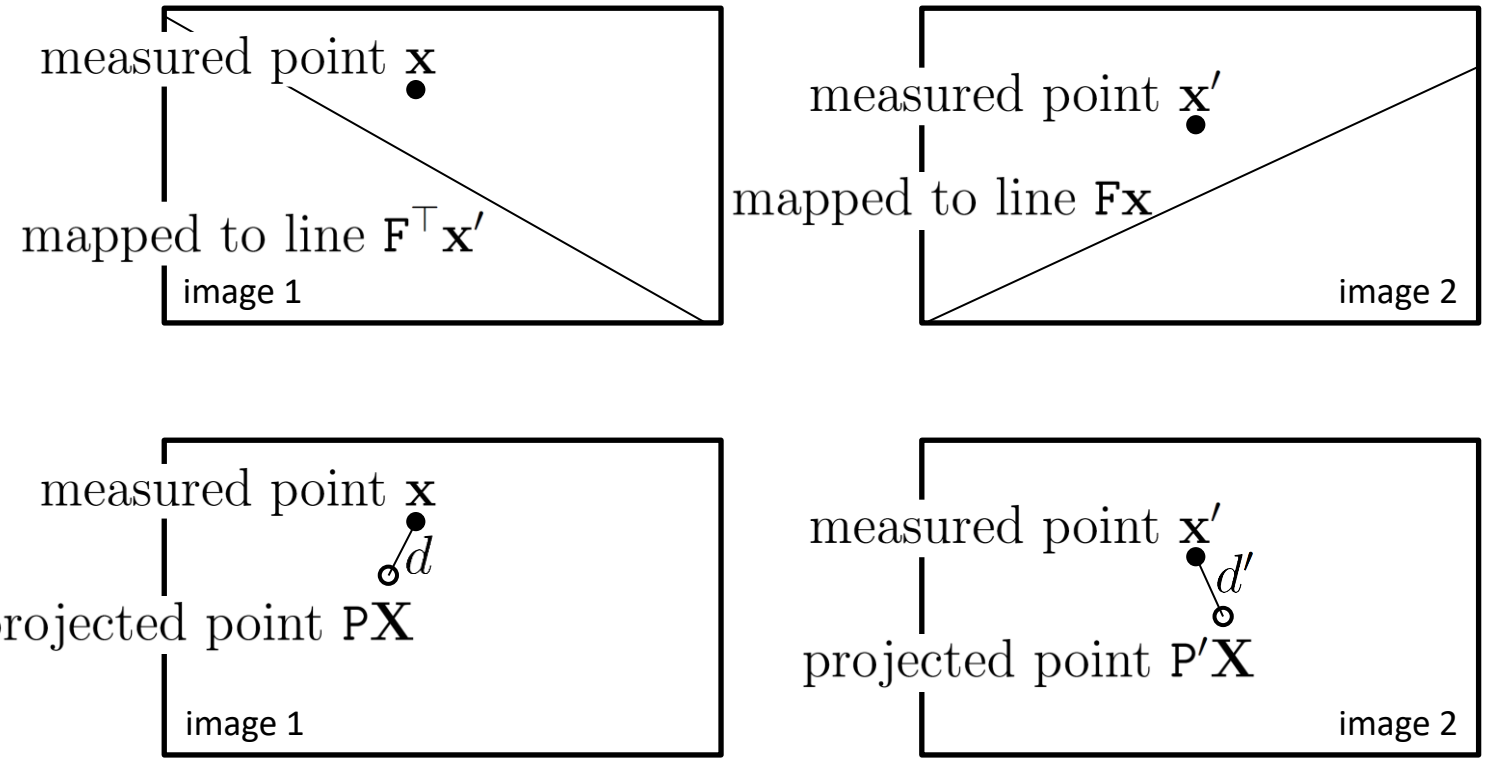
The fundamental matrix $\mathtt{F}$ from canonical cameras $\mathtt{P} = [\mathtt{I} \,|\, \mathbf{0}]$ and $\mathtt{P}' = [\mathtt{M} \,|\, \mathbf{v}]$

$$\mathtt{F} = [\mathbf{v}]_{\times}\mathtt{M}$$

$$\mathtt{F} = [\mathbf{e}']_{\times}\mathtt{M} = \mathtt{M}^{-\top}[\mathbf{e}]_{\times}, \text{ where } \mathbf{e}' = \mathbf{v} \text{ and } \mathbf{e} = \mathtt{M}^{-1}\mathbf{v}$$

Fundamental matrix

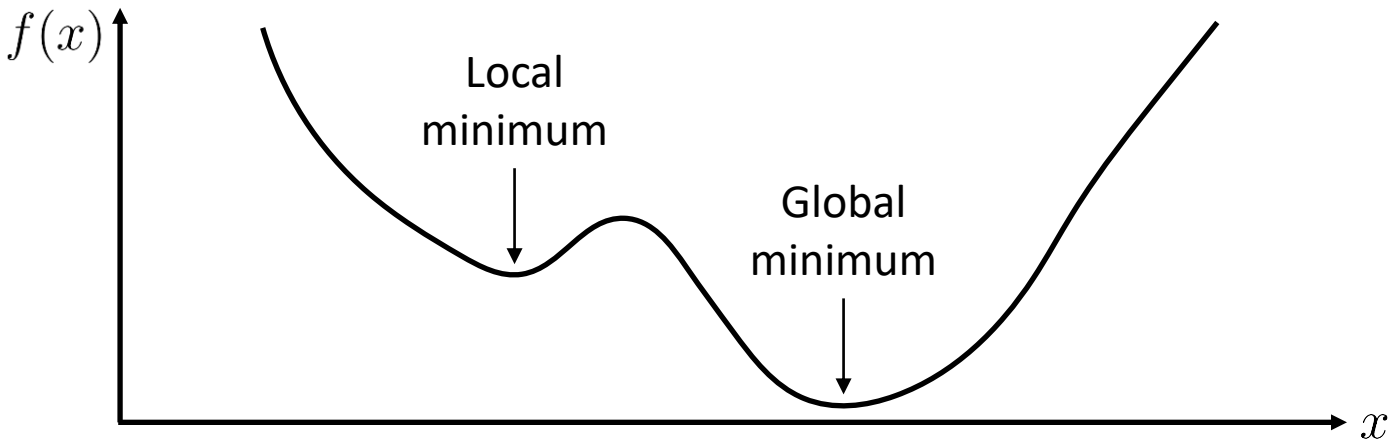
- Error in image point measurements



Estimation of fundamental matrix

$$\mathbf{x}_i = \mathtt{P}\mathbf{X}_i \,\forall\, i \text{ and } \mathbf{x}'_i = \mathtt{P}'\mathbf{X}_i \,\forall\, i, \text{ set } \mathtt{P} = [\mathtt{I} \,|\, \mathbf{0}] \text{ and solve for } \mathtt{P}'$$

- Given point correspondences $\mathbf{x}_i \leftrightarrow \mathbf{x}'_i$
- Minimize geometric error $\sum_i \left(d(\mathbf{x}_i, \mathtt{P}\mathbf{X}_i)^2 + d(\mathbf{x}'_i, \mathtt{P}'\mathbf{X}_i)^2 \right)$
- A nonlinear mapping requires a nonlinear optimization problem solver
  - Use linear estimation for initial estimate (to mitigate converging to local optimum)
  - Iterative process to determine global optimum

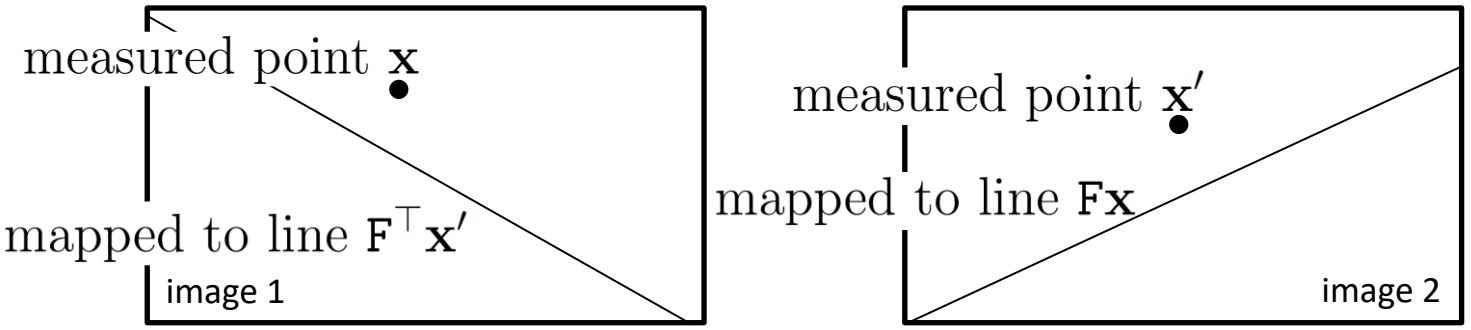


Estimation of fundamental matrix, linear

Linear estimation of fundamental matrix using the direct linear transformation (DLT) algorithm

- Minimize algebraic error

$$\mathbf{x}'^{\top}\mathbf{F}\mathbf{x} = 0 \qquad \text{Epipolar constraint}$$



Linear estimation of fundamental matrix using the direct linear transformation (DLT) algorithm

$$\mathbf{x}'^{\top}\mathtt{F}\mathbf{x} = 0$$

$$\mathbf{x}'^{\top}\begin{bmatrix}\mathbf{f}^{1\top}\\ \mathbf{f}^{2\top}\\ \mathbf{f}^{3\top}\end{bmatrix}\mathbf{x} = 0, \text{ where } \mathtt{F} = \begin{bmatrix}\mathbf{f}^{1\top}\\ \mathbf{f}^{2\top}\\ \mathbf{f}^{3\top}\end{bmatrix}$$

$$\begin{bmatrix}x' & y' & w'\end{bmatrix}\begin{bmatrix}\mathbf{f}^{1\top}\mathbf{x}\\ \mathbf{f}^{2\top}\mathbf{x}\\ \mathbf{f}^{3\top}\mathbf{x}\end{bmatrix} = 0$$

$$x'\mathbf{f}^{1\top}\mathbf{x} + y'\mathbf{f}^{2\top}\mathbf{x} + w'\mathbf{f}^{3\top}\mathbf{x} = 0$$

$$\begin{bmatrix}x'\mathbf{x}^{\top} & y'\mathbf{x}^{\top} & w'\mathbf{x}^{\top}\end{bmatrix}\begin{bmatrix}\mathbf{f}^{1}\\ \mathbf{f}^{2}\\ \mathbf{f}^{3}\end{bmatrix} = 0$$

$$\left(\mathbf{x}'^{\top} \otimes \mathbf{x}^{\top}\right)\mathbf{f} = 0, \text{ where } \otimes \text{ denotes the Kronecker product}$$

$$\text{and } \mathbf{f} = \text{vec}(\mathtt{F}^{\top}) = \begin{bmatrix}\mathbf{f}^{1}\\ \mathbf{f}^{2}\\ \mathbf{f}^{3}\end{bmatrix}$$

Linear estimation of fundamental matrix using the direct linear transformation (DLT) algorithm

Given $n \geq 8$ point correspondences $\mathbf{x}_i \leftrightarrow \mathbf{x}'_i$

$$\mathbf{x}_i'^{\top} \mathtt{F} \mathbf{x}_i = 0 \, \forall \, i, \text{ solve for } \mathtt{F}$$

$$\left(\mathbf{x}_i'^{\top} \otimes \mathbf{x}_i^{\top}\right) \mathbf{f} = 0 \, \forall \, i, \text{ solve for } \mathbf{f} = \mathrm{vec}(\mathtt{F}^{\top})$$

$$\begin{bmatrix} \mathbf{x}_1'^{\top} \otimes \mathbf{x}_1^{\top} \\ \mathbf{x}_2'^{\top} \otimes \mathbf{x}_2^{\top} \\ \vdots \\ \mathbf{x}_n'^{\top} \otimes \mathbf{x}_n^{\top} \end{bmatrix} \mathbf{f} = \mathbf{0}$$

$$\mathtt{A}\mathbf{f} = \mathbf{0}, \text{ where } \mathtt{A} = \begin{bmatrix} \mathbf{x}_1'^{\top} \otimes \mathbf{x}_1^{\top} \\ \mathbf{x}_2'^{\top} \otimes \mathbf{x}_2^{\top} \\ \vdots \\ \mathbf{x}_n'^{\top} \otimes \mathbf{x}_n^{\top} \end{bmatrix}$$

Then, enforce rank 2 constraint (simple and convenient method using SVD)

See textbook for alternative method

$$\mathtt{F} = \mathtt{U}\Sigma\mathtt{V}^{\top}, \text{ where } \Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \sigma_3)$$

$$\mathtt{F} = \mathtt{U}\Sigma'\mathtt{V}^{\top}, \text{ where } \Sigma' = \mathrm{diag}(\sigma_1, \sigma_2, 0)$$

But, data normalization must be used

Linear estimation of fundamental matrix using the direct linear transformation (DLT) algorithm

Given $n \geq 9$ point correspondences $\mathbf{x}_i \leftrightarrow \mathbf{x}'_i$

$$\mathtt{A}\mathbf{f} = \mathbf{0}, \text{ where } \mathtt{A} = \begin{bmatrix} \mathbf{x}'^{\top}_1 \otimes \mathbf{x}^{\top}_1 \\ \mathbf{x}'^{\top}_2 \otimes \mathbf{x}^{\top}_2 \\ \vdots \\ \mathbf{x}'^{\top}_n \otimes \mathbf{x}^{\top}_n \end{bmatrix}$$

In general, due to error in measurements, the (right) null space of the matrix $\mathtt{A}$ will be empty (i.e., $\mathtt{A}$ will be full rank). Instead, solve for the vector $\mathbf{f}$ such that $\|\mathtt{A}\mathbf{f}\|$ is minimized.

$$\mathtt{A} = \mathtt{U}\Sigma\mathtt{V}^{\top}$$

The solution vector $\mathbf{f}$ is the column of $\mathtt{V}$ corresponding to the smallest singular value (i.e., the last column of $\mathtt{V}$ (or last row of $\mathtt{V}^{\top}$)).
Then, enforce rank 2 constraint (simple and conveneint method using SVD) See textbook for alternative method

But, data normalization must be used

Linear estimation of fundamental matrix using the direct linear transformation (DLT) algorithm

Given $n = 8$ point correspondences $\mathbf{x}_i \leftrightarrow \mathbf{x}'_i$

$$\mathtt{A}\mathbf{f} = \mathbf{0}, \text{ where } \mathtt{A} = \begin{bmatrix} \mathbf{x}'^{\top}_1 \otimes \mathbf{x}^{\top}_1 \\ \mathbf{x}'^{\top}_2 \otimes \mathbf{x}^{\top}_2 \\ \mathbf{x}'^{\top}_3 \otimes \mathbf{x}^{\top}_3 \\ \mathbf{x}'^{\top}_4 \otimes \mathbf{x}^{\top}_4 \\ \mathbf{x}'^{\top}_5 \otimes \mathbf{x}^{\top}_5 \\ \mathbf{x}'^{\top}_6 \otimes \mathbf{x}^{\top}_6 \\ \mathbf{x}'^{\top}_7 \otimes \mathbf{x}^{\top}_7 \\ \mathbf{x}'^{\top}_8 \otimes \mathbf{x}^{\top}_8 \end{bmatrix}$$

The (right) null space of the matrix $\mathtt{A}$ is the vector $\mathbf{f}$.

$$\mathtt{A} = \mathtt{U}\Sigma\mathtt{V}^{\top}$$

The solution vector $\mathbf{f}$ is the column of $\mathtt{V}$ not corresponding to a singular value (i.e., the last column of $\mathtt{V}$ (or last row of $\mathtt{V}^{\top}$)).
Then, enforce rank 2 constraint (simple and conveneint method using SVD)

See textbook for alternative method

But, data normalization must be used

Linear estimation of fundamental matrix using the direct linear transformation (DLT) algorithm

Given $n = 7$ point correspondences $\mathbf{x}_i \leftrightarrow \mathbf{x}'_i$

$$\mathbf{x}_i'^{\top} \mathtt{F} \mathbf{x}_i = 0 \, \forall \, i = 1, \ldots, 7, \text{ solve for } \mathtt{F}$$

$$\left(\mathbf{x}_i'^{\top} \otimes \mathbf{x}_i^{\top}\right) \mathbf{f} = 0 \, \forall \, i = 1, \ldots, 7, \text{ solve for } \mathbf{f} = \text{vec}(\mathtt{F}^{\top})$$

$$\begin{bmatrix} \mathbf{x}_1'^{\top} \otimes \mathbf{x}_1^{\top} \\ \mathbf{x}_2'^{\top} \otimes \mathbf{x}_2^{\top} \\ \mathbf{x}_3'^{\top} \otimes \mathbf{x}_3^{\top} \\ \mathbf{x}_4'^{\top} \otimes \mathbf{x}_4^{\top} \\ \mathbf{x}_5'^{\top} \otimes \mathbf{x}_5^{\top} \\ \mathbf{x}_6'^{\top} \otimes \mathbf{x}_6^{\top} \\ \mathbf{x}_7'^{\top} \otimes \mathbf{x}_7^{\top} \end{bmatrix} \begin{bmatrix} f_{1,1} & f_{2,1} \\ f_{1,2} & f_{2,2} \\ f_{1,3} & f_{2,3} \\ f_{1,4} & f_{2,4} \\ f_{1,5} & f_{2,5} \\ f_{1,6} & f_{2,6} \\ f_{1,7} & f_{2,7} \\ f_{1,8} & f_{2,8} \\ f_{1,9} & f_{2,9} \end{bmatrix} = \mathtt{0}$$

$$\mathtt{A} \begin{bmatrix} \mathbf{f}_1 & \mathbf{f}_2 \end{bmatrix} = \mathtt{0}, \text{ where } \mathtt{A} = \begin{bmatrix} \mathbf{x}_1'^{\top} \otimes \mathbf{x}_1^{\top} \\ \mathbf{x}_2'^{\top} \otimes \mathbf{x}_2^{\top} \\ \vdots \\ \mathbf{x}_7'^{\top} \otimes \mathbf{x}_7^{\top} \end{bmatrix},$$

$\mathbf{f}_1 = (f_{1,1}, f_{1,2}, f_{1,3}, f_{1,4}, f_{1,5}, f_{1,6}, f_{1,7}, f_{1,8}, f_{1,9})^{\top}$, and
$\mathbf{f}_2 = (f_{2,1}, f_{2,2}, f_{2,3}, f_{2,4}, f_{2,5}, f_{2,6}, f_{2,7}, f_{2,8}, f_{2,9})^{\top}$

But, data normalization must be used

Linear estimation of fundamental matrix using the direct linear transformation (DLT) algorithm

$$\mathtt{A}\begin{bmatrix}\mathbf{f}_1 & \mathbf{f}_2\end{bmatrix} = \mathtt{0}, \text{ where } \mathtt{A} = \begin{bmatrix}\mathbf{x}_1'^{\top} \otimes \mathbf{x}_1^{\top} \\ \mathbf{x}_2'^{\top} \otimes \mathbf{x}_2^{\top} \\ \mathbf{x}_3'^{\top} \otimes \mathbf{x}_3^{\top} \\ \mathbf{x}_4'^{\top} \otimes \mathbf{x}_4^{\top} \\ \mathbf{x}_5'^{\top} \otimes \mathbf{x}_5^{\top} \\ \mathbf{x}_6'^{\top} \otimes \mathbf{x}_6^{\top} \\ \mathbf{x}_7'^{\top} \otimes \mathbf{x}_7^{\top}\end{bmatrix}$$

The (right) null space of the matrix $\mathtt{A}$ is the matrix $\begin{bmatrix}\mathbf{f}_1 & \mathbf{f}_2\end{bmatrix}$.

$$\mathtt{A} = \mathtt{U}\Sigma\mathtt{V}^{\top}$$

The solution matrix $\begin{bmatrix}\mathbf{f}_1 & \mathbf{f}_2\end{bmatrix}$ are the columns of $\mathtt{V}$ not corresponding to singular values (i.e., the last two columns of $\mathtt{V}$ (or last two rows of $\mathtt{V}^{\top}$)).

But, data normalization must be used

Linear estimation of fundamental matrix using the direct linear transformation (DLT) algorithm

$$\mathbf{f}_1 = (f_{1,1}, f_{1,2}, f_{1,3}, f_{1,4}, f_{1,5}, f_{1,6}, f_{1,7}, f_{1,8}, f_{1,9})^\top, \text{ and}$$
$$\mathbf{f}_2 = (f_{2,1}, f_{2,2}, f_{2,3}, f_{2,4}, f_{2,5}, f_{2,6}, f_{2,7}, f_{2,8}, f_{2,9})^\top$$

$$\mathtt{F}_1 = \begin{bmatrix} f_{1,1} & f_{1,2} & f_{1,3} \\ f_{1,4} & f_{1,5} & f_{1,6} \\ f_{1,7} & f_{1,8} & f_{1,9} \end{bmatrix} \text{ and } \mathtt{F}_2 = \begin{bmatrix} f_{2,1} & f_{2,2} & f_{2,3} \\ f_{2,4} & f_{2,5} & f_{2,6} \\ f_{2,7} & f_{2,8} & f_{2,9} \end{bmatrix}$$

$$\mathtt{F} = \alpha \mathtt{F}_1 + \mathtt{F}_2$$
$$\det(\mathtt{F}) = 0, \text{ solve for } \alpha$$

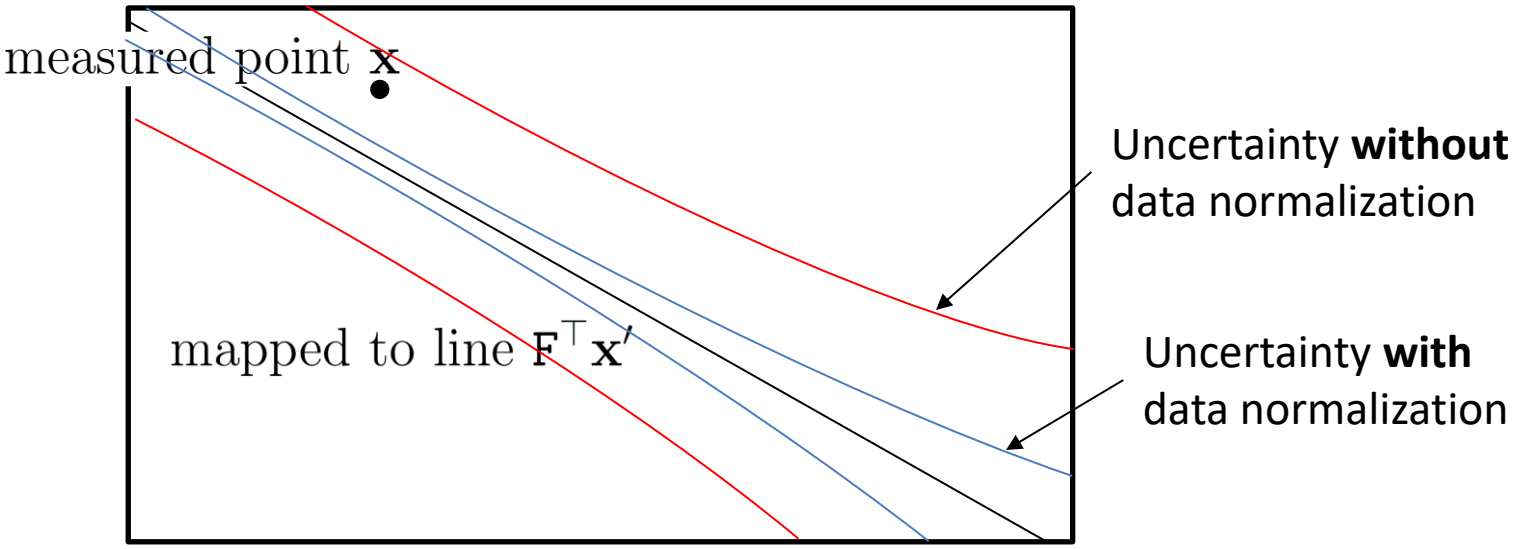
3 solutions α_1, α_2, and α_3 (1 or 3 will be real solutions, i.e., the real roots of the cubic polynomial)
Then substitute α_i to solve for $\mathtt{F}_i$

$$\mathtt{F}_i = \alpha_i \mathtt{F}_1 + \mathtt{F}_2 \qquad (1 \text{ or } 3 \text{ real solutions})$$

But, data normalization must be used

Data normalization

- Data normalization is required to reduce the propagated uncertainty under the nonlinear projection



Data normalizing transformations

- 2D

Determine the similarity transformation $\mathtt{T}$ such that the mean (i.e., centroid) of the transformed points is at the origin and their standard deviation from the origin is $\sqrt{2}$.

$$\mathtt{T} = \begin{bmatrix} s & 0 & -s\mu_{\tilde{x}} \\ 0 & s & -s\mu_{\tilde{y}} \\ 0 & 0 & 1 \end{bmatrix}, \text{ where } s = \sqrt{\frac{2}{\sigma_{\tilde{x}}^2 + \sigma_{\tilde{y}}^2}}$$

where $\mu_{\tilde{x}}$ and $\sigma_{\tilde{x}}^2$, and $\mu_{\tilde{y}}$ and $\sigma_{\tilde{y}}^2$ are the mean and variance of the $\tilde{x}$ and $\tilde{y}$ coordinates, respectively.

Linear estimation of fundamental matrix using the direct linear transformation (DLT) algorithm

- Algorithm

1. Data normalize the points

$$\mathbf{x}_{\mathrm{DN},i} = \mathtt{T}\mathbf{x}_i \,\forall\, i$$
$$\mathbf{x}'_{\mathrm{DN},i} = \mathtt{T}'\mathbf{x}'_i \,\forall\, i$$

2. Estimate the data normalized fundamental matrix $\mathtt{F}_{\mathrm{DN}}$ from the data normalized points

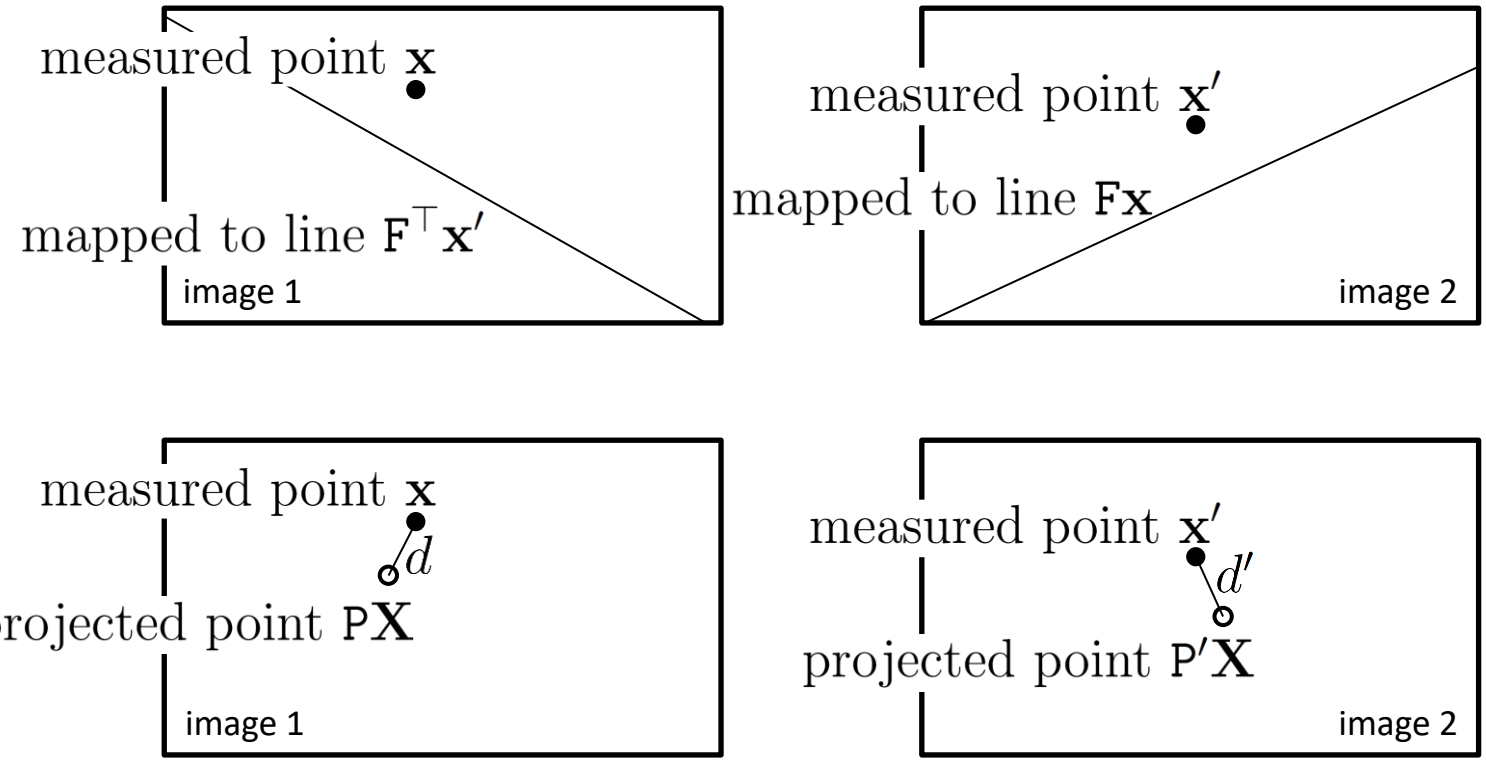
3. Data denormalize the data normalized fundamental matrix

$$\mathbf{x}'^{\top}_{\mathrm{DN},i}\mathtt{F}_{\mathrm{DN}}\mathbf{x}_{\mathrm{DN},i} = 0 \,\forall\, i$$
$$(\mathtt{T}'\mathbf{x}'_i)^{\top}\mathtt{F}_{\mathrm{DN}}\mathtt{T}\mathbf{x}_i = 0 \,\forall\, i$$
$$\mathbf{x}'^{\top}_i\mathtt{T}'^{\top}\mathtt{F}_{\mathrm{DN}}\mathtt{T}\mathbf{x}_i = 0 \,\forall\, i$$
$$\mathbf{x}'^{\top}_i\mathtt{F}\mathbf{x}_i = 0 \,\forall\, i, \text{ where } \mathtt{F} = \mathtt{T}'^{\top}\mathtt{F}_{\mathrm{DN}}\mathtt{T}$$

Sampson correction and Sampson error

Fundamental matrix

- Error in image point measurements



Sampson cost

Sampson error uses a cost function that lies between algebraic and geometric cost functions (in terms of complexity) and gives a close approximation to geometric error.
Measurement vector $\mathbf{X} = (\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}')^\top$
$\mathcal{C}_\mathtt{F}(\mathbf{X})$ is the cost associated with $\mathbf{X}$

$$\mathcal{C}_\mathtt{F}(\mathbf{X}) = 0$$

$$\mathcal{C}_\mathtt{F}(\mathbf{X} + \boldsymbol{\delta}_\mathbf{X}) = \mathcal{C}_\mathtt{F}(\mathbf{X}) + \frac{\mathrm{d}\mathcal{C}_\mathtt{F}(\mathbf{X})}{\partial \mathbf{X}}\boldsymbol{\delta}_\mathbf{X} \qquad \text{first order Taylor expansion}$$

$$\text{if } \hat{\mathbf{X}} = \mathbf{X} + \boldsymbol{\delta}_\mathbf{X}$$

$$\hat{\mathbf{X}} - \mathbf{X} = \boldsymbol{\delta}_\mathbf{X} \text{ and desire } \hat{\mathbf{X}} \text{ such that } \mathcal{C}_\mathtt{F}(\hat{\mathbf{X}}) = 0, \text{ then}$$

$$\mathcal{C}_\mathtt{F}(\mathbf{X}) + \frac{\mathrm{d}\mathcal{C}_\mathtt{F}(\mathbf{X})}{\partial \mathbf{X}}\boldsymbol{\delta}_\mathbf{X} = \mathbf{0}$$

$$\epsilon + \mathtt{J}\boldsymbol{\delta}_\mathbf{X} = 0, \text{ where } \epsilon = \mathcal{C}_\mathtt{F}(\mathbf{X}) \text{ and } \mathtt{J} = \frac{\mathrm{d}\mathcal{C}_\mathtt{F}(\mathbf{X})}{\partial \mathbf{X}}$$

$$\mathtt{J}\boldsymbol{\delta}_\mathbf{X} = -\epsilon, \text{ solve for } \boldsymbol{\delta}_\mathbf{X} \text{ that minimizes } \|\boldsymbol{\delta}_\mathbf{X}\|$$

Sampson corrected and Sampson error

$$\mathtt{J}\boldsymbol{\delta}_{\mathbf{X}} = -\epsilon, \text{ solve for } \boldsymbol{\delta}_{\mathbf{X}} \text{ that minimizes } \|\boldsymbol{\delta}_{\mathbf{X}}\|$$

Use a Lagrange multiplier λ and find extrema of $\boldsymbol{\delta}_{\mathbf{X}}^{\top}\boldsymbol{\delta}_{\mathbf{X}} - 2\lambda(\mathtt{J}\boldsymbol{\delta}_{\mathbf{X}} + \epsilon)$

$$\mathtt{J}\mathtt{J}^{\top}\lambda = -\epsilon, \text{ solve for } \lambda$$

then substitute λ to solve for $\boldsymbol{\delta}_{\mathbf{X}}$

$$\boldsymbol{\delta}_{\mathbf{X}} = \mathtt{J}^{\top}\lambda$$

Sampson corrected	$\hat{\mathbf{X}} = \mathbf{X} + \boldsymbol{\delta}_{\mathbf{X}}$
Squared sampson error	$\|\boldsymbol{\delta}_{\mathbf{X}}\|^2 = \boldsymbol{\delta}_{\mathbf{X}}^{\top}\boldsymbol{\delta}_{\mathbf{X}}$

Sampson corrected and Sampson error

Determine $\mathtt{J}$ and ϵ

$$\mathbf{x}_i'^{\top} \mathtt{F} \mathbf{x}_i = 0 \, \forall \, i$$

$$\mathbf{x}_i'^{\top} \begin{bmatrix} \mathbf{f}^{1\top} \\ \mathbf{f}^{2\top} \\ \mathbf{f}^{3\top} \end{bmatrix} \mathbf{x}_i = 0 \, \forall \, i, \text{ where } \mathtt{F} = \begin{bmatrix} \mathbf{f}^{1\top} \\ \mathbf{f}^{2\top} \\ \mathbf{f}^{3\top} \end{bmatrix}$$

$$\begin{bmatrix} x_i'\mathbf{x}_i^{\top} & y_i'\mathbf{x}_i^{\top} & w_i'\mathbf{x}_i^{\top} \end{bmatrix} \begin{bmatrix} \mathbf{f}^1 \\ \mathbf{f}^2 \\ \mathbf{f}^3 \end{bmatrix} = 0 \, \forall \, i$$

$$\mathtt{A}_i \mathbf{f} = 0 \, \forall \, i, \text{ where } \mathtt{A}_i = \begin{bmatrix} x_i'\mathbf{x}_i^{\top} & y_i'\mathbf{x}_i^{\top} & w_i'\mathbf{x}_i^{\top} \end{bmatrix} \text{ and } \mathbf{f} = \text{vec}(\mathtt{F}^{\top}) = \begin{bmatrix} \mathbf{f}^1 \\ \mathbf{f}^2 \\ \mathbf{f}^3 \end{bmatrix}$$

Sampson corrected and Sampson error

$$\mathtt{A}_i\mathbf{f} = \begin{bmatrix}\tilde{x}'_i\mathbf{x}_i^\top & \tilde{y}'_i\mathbf{x}_i^\top & \mathbf{x}_i^\top\end{bmatrix}\begin{bmatrix}\mathbf{f}^1\\ \mathbf{f}^2\\ \mathbf{f}^3\end{bmatrix} \;\forall\, i$$

$$\mathtt{A}_i\mathbf{f} = \tilde{x}'_i\mathbf{f}^{1\top}\mathbf{x}_i + \tilde{y}'_i\mathbf{f}^{2\top}\mathbf{x}_i + \mathbf{f}^{3\top}\mathbf{x}_i \;\forall\, i$$

$$\mathbf{X}_i = (\tilde{x}_i, \tilde{y}_i, \tilde{x}'_i, \tilde{y}'_i)^\top$$

$$\epsilon_i = \mathcal{C}_\mathtt{F}(\mathbf{X}_i) = \mathtt{A}_i\mathbf{f}$$

$$\epsilon_i = \tilde{x}_i\tilde{x}'_i f_{11} + \tilde{x}_i\tilde{y}'_i f_{21} + \tilde{x}_i f_{31} + \tilde{y}_i\tilde{x}'_i f_{12} + \tilde{y}_i\tilde{y}'_i f_{22} + \tilde{y}_i f_{32} + \tilde{x}'_i f_{13} + \tilde{y}'_i f_{23} + f_{33}$$

$$\mathtt{J}_i = \frac{\mathrm{d}\mathcal{C}_\mathtt{F}(\mathbf{X}_i)}{\partial \mathbf{X}_i} = \frac{\mathrm{d}(\mathtt{A}_i\mathbf{f})}{\partial(\tilde{x}_i, \tilde{y}_i, \tilde{x}'_i, \tilde{y}'_i)}$$

$$\mathtt{J}_i = \begin{bmatrix}\tilde{x}'_i f_{11} + \tilde{y}'_i f_{21} + f_{31} & \tilde{x}'_i f_{12} + \tilde{y}'_i f_{22} + f_{32} & \tilde{x}_i f_{11} + \tilde{y}_i f_{12} + f_{13} & \tilde{x}_i f_{21} + \tilde{y}_i f_{22} + f_{23}\end{bmatrix}$$

$\mathtt{J}_i\mathtt{J}_i^\top\lambda_i = -\epsilon_i$, solve for $\lambda_i = -\frac{\epsilon_i}{\mathtt{J}_i\mathtt{J}_i^\top}$, then $\boldsymbol{\delta}_{\mathbf{X}_i} = \mathtt{J}_i^\top\lambda_i$

Sampson corrected points $\quad \hat{\mathbf{X}}_i = (\hat{\tilde{x}}_i, \hat{\tilde{y}}_i, \hat{\tilde{x}}'_i, \hat{\tilde{y}}'_i)^\top = \mathbf{X}_i + \boldsymbol{\delta}_{\mathbf{X}_i}$

Squared Sampson error $\quad \|\boldsymbol{\delta}_{\mathbf{X}_i}\|^2 = \boldsymbol{\delta}_{\mathbf{X}_i}^\top\boldsymbol{\delta}_{\mathbf{X}_i} = \epsilon_i(\mathtt{J}_i\mathtt{J}_i^\top)^{-1}\epsilon_i = \dfrac{\epsilon_i^2}{\mathtt{J}_i\mathtt{J}_i^\top}$

Outlier rejection

Outlier rejection

- Even the presence of a single outlier may result in an inaccurate estimate
- Linear estimate
  - Minimizes sum of squared error and, in general, outliers have large error relative to inliers
  - Resulting estimate may not be near global minimum
- Nonlinear estimate
  - Cost function minimizes sum of squared error, so may not converge to accurate solution
    - Alternatively, use a robust cost function

Random sample consensus (RANSAC) and M-estimator sample consensus (MSAC)

Objective is to determine the consensus set with the minimum cost

tol is the tolerance for establishing datum/model compatibility
τ_{cost} is the upper bound on the cost of an acceptable consensus set
maxTrials is the maximum number of attempts to find a consensus set

$\text{consensus}_{\text{cost}} = \infty$
for (trials $= 0$; trials $<$ maxTrials && $\text{consensus}_{\text{cost}} > \tau_{\text{cost}}$; ++trials)
    Select a random sample of unique data points
    Calculate the model using the random sample
    Calculate the error for each data point using model
    Calculate the cost
    if cost $< \text{consensus}_{\text{cost}}$
        $\text{consensus}_{\text{cost}} = \text{cost}$
        $\text{consensus}_{\text{model}} = \text{model}$
Calculate the error for each data point using $\text{consensus}_{\text{model}}$
Calculate the set of inliers (i.e., data points with error $\leq$ tol)

If multiple solutions, try each one, but it is still a single trial

Outlier rejection

- RANSAC cost

count is the total number of data points
tol is the tolerance for establishing datum/model compatibility

```
cost = 0
for (n = 0; n < count; ++n)
    cost += error[n] ≤ tol ? 0 : 1
```

- MSAC cost

count is the total number of data points
tol is the tolerance for establishing datum/model compatibility

```
cost = 0
for (n = 0; n < count; ++n)
    cost += error[n] ≤ tol ? error[n] : tol
```

RANSAC and MSAC

- Maximum number of trials
  - Adaptive maximum number of trials
- The tolerance for establishing datum/model compatibility

RANSAC and MSAC

- Maximum number of trials

The smaller the sample size, the smaller maximum number of trials. Use minimum solution!

Sample size	Proportion of outliers ϵ						
s	5%	10%	20%	25%	30%	40%	50%
2	2	3	5	6	7	11	17
3	3	4	7	9	11	19	35
4	3	5	9	13	17	34	72
5	4	6	12	17	26	57	146
6	4	7	16	24	37	97	293
7	4	8	20	33	54	163	588
8	5	9	26	44	78	272	1177

Table 4.3. *The number N of samples required to ensure, with a probability $p = 0.99$, that at least one sample has no outliers for a given size of sample, s, and proportion of outliers, ϵ.*

Adaptive maximum number of trials

s is the sample size

p is the assumed probability that at least one of the random samples does not contain any outliers

tol is the tolerance for establishing datum/model compatibility

τ_{cost} is the upper bound on the cost of an acceptable consensus set

$\text{maxTrials} = \infty$

$\text{consensus}_{\text{cost}} = \infty$

for (trials $= 0$; trials $<$ maxTrials && $\text{consensus}_{\text{cost}} > \tau_{\text{cost}}$; ++trials)

- Select a random sample of unique data points
- Calculate the model using the random sample
- Calculate the error for each data point using model
- Calculate the cost
- if cost $< \text{consensus}_{\text{cost}}$
  - $\text{consensus}_{\text{cost}} = \text{cost}$
  - $\text{consensus}_{\text{model}} = \text{model}$
  - Calculate the number of inliers
  - $w = \dfrac{\text{number of inliers}}{\text{total number of data points}}$
  - $\text{maxTrials} = \dfrac{\log(1-p)}{\log(1-w^s)}$

If multiple solutions, try each one, but it is still a single trial

Calculate the error for each data point using $\text{consensus}_{\text{model}}$

Calculate the set of inliers (i.e., data points with error $\leq$ tol)

RANSAC and MSAC

- The tolerance for establishing datum/model compatibility

Probability α that a data point is an inlier (usually α is chosen to be 0.95)
Variance σ^2 of the measurement error (if unkonwn, assumed to be 1)
Codimension m
Tolerance is square distance $t^2 = F_m^{-1}(\alpha)\sigma^2$, where $F_m^{-1}(\alpha)$ is the inverse chi-squared cumulative distribution function with m degress of freedom at the probability α

- Codimension

If w is a subspace of a finite-dimensional vector space v, then $\text{codim}(w) = \dim(v) - \dim(w)$
The fundamental matrix is a variety of dimension 3 in $(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}') \in \mathbb{R}^4$, so has codimension 1

Use the square Sampson error

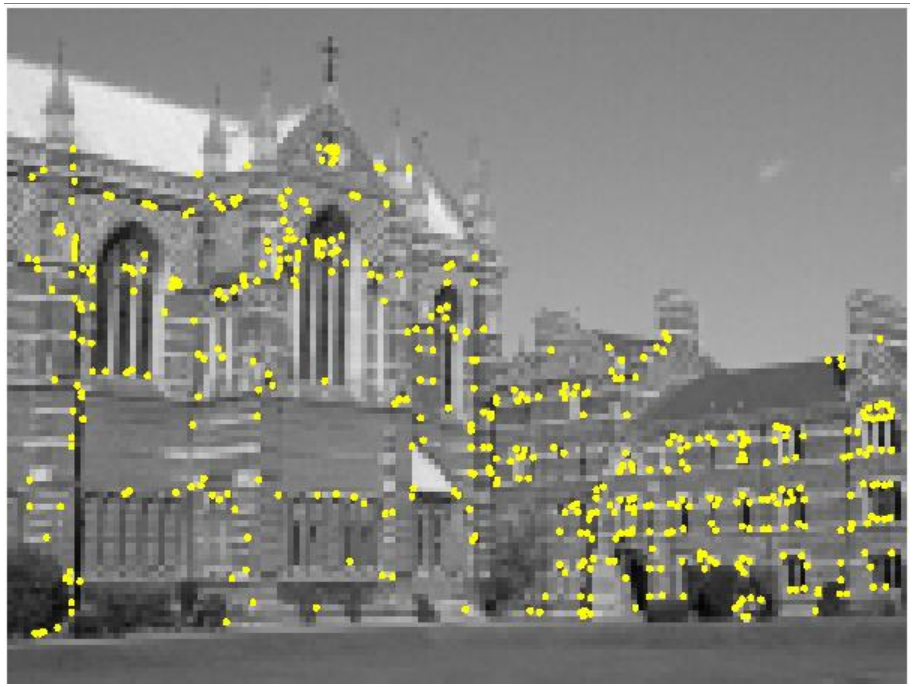
Feature detection and matching

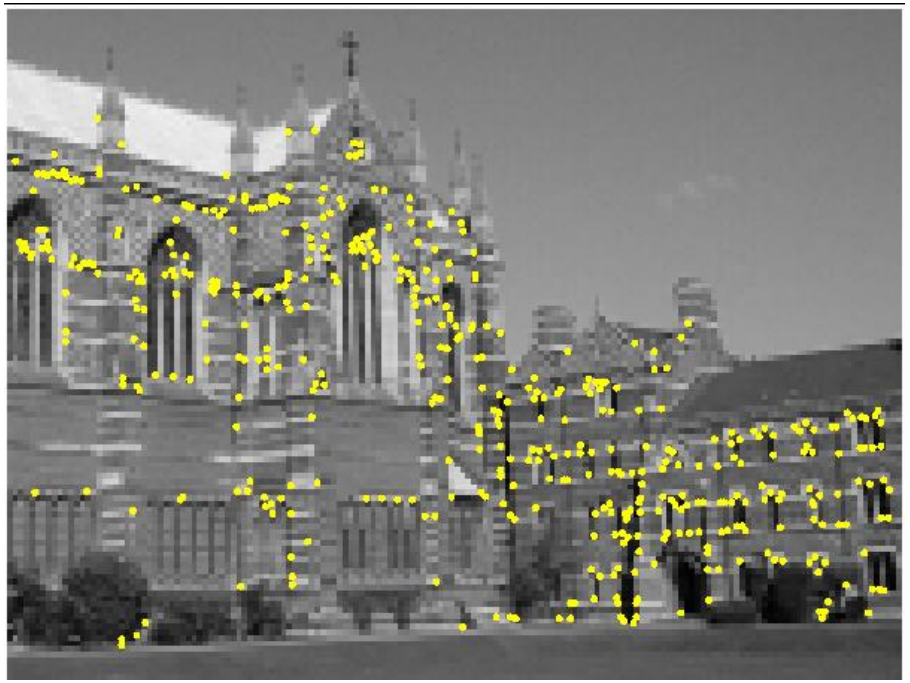




Input Images

Feature detection and matching

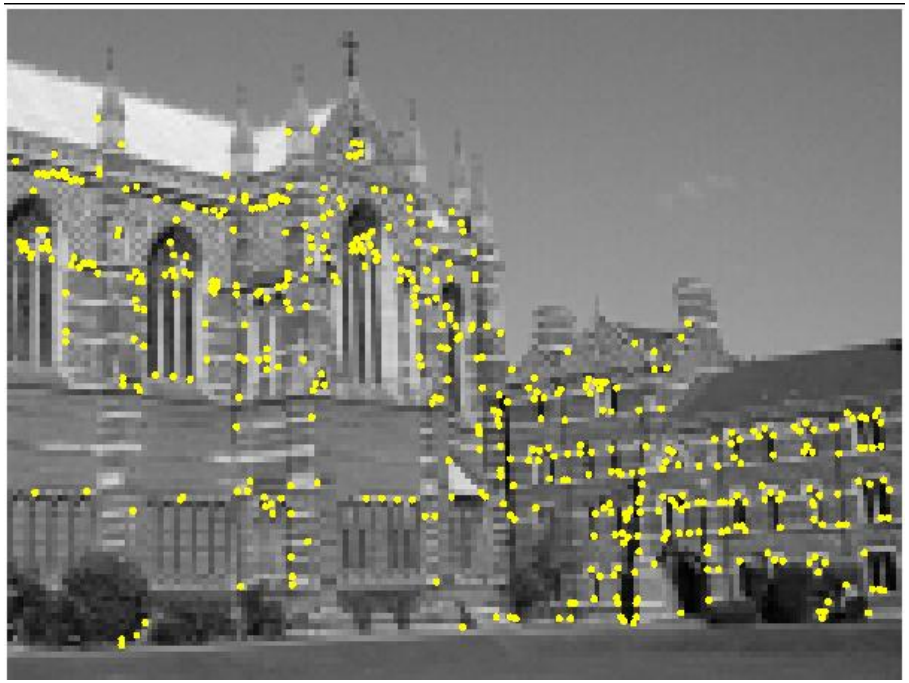




Detected Corners

Feature detection and matching

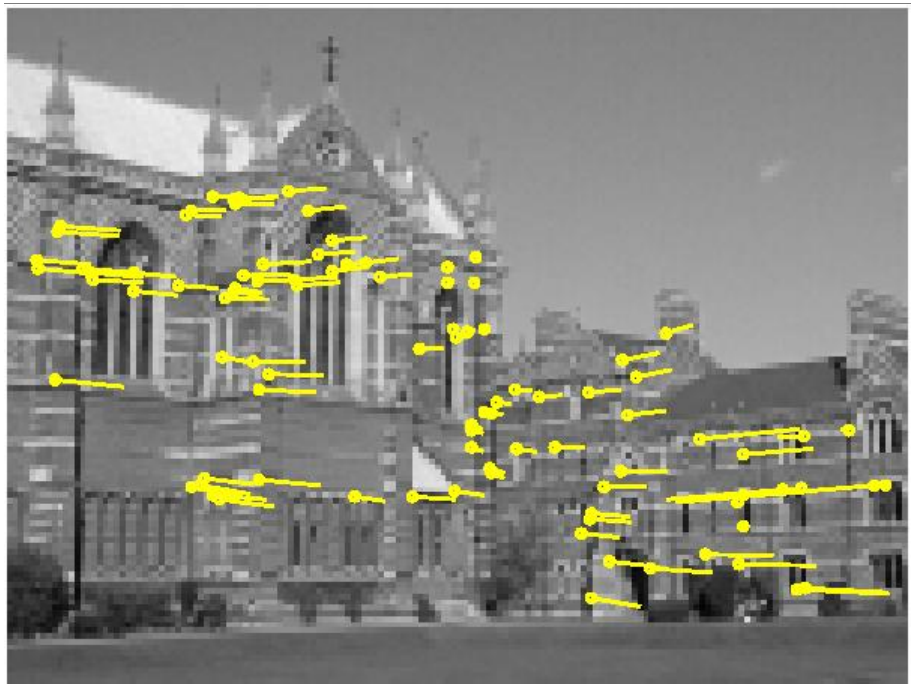


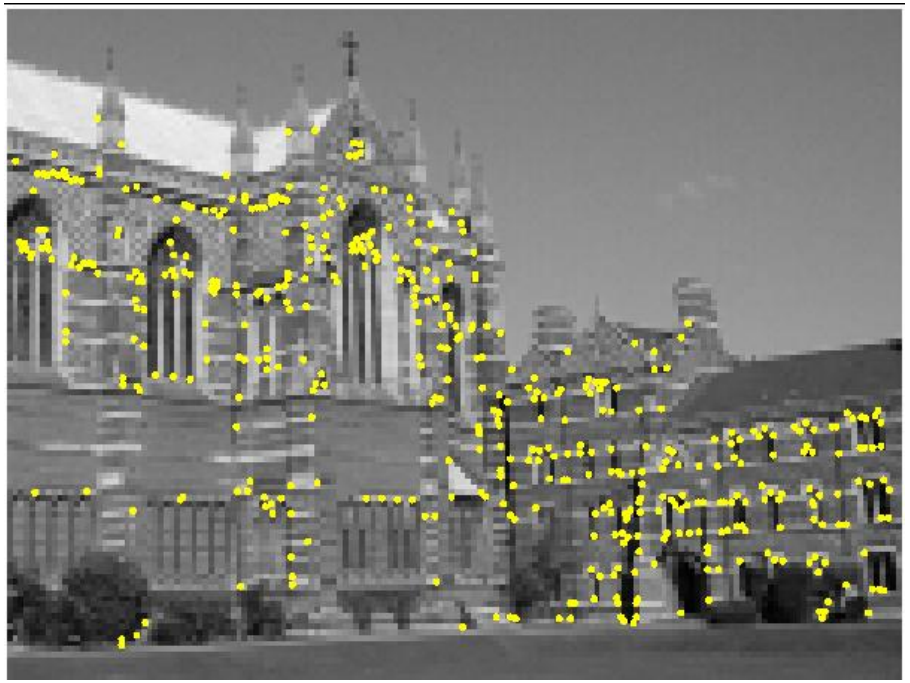


Simple Matching

Expect some false matches

Feature detection and matching





Simple Matching
Including Outlier Rejection

No geometric outliers
But still some false matches

Retrieve camera projection matrices from fundamental matrix

Camera projection matrices must be compatible with the fundamental matrix.
General formula

$$\mathtt{P} = [\mathtt{I} \,|\, \mathbf{0}] \text{ and } \mathtt{P}' = \left[[\mathbf{e}']_\times \mathtt{F} + \mathbf{e}'\mathbf{v}^\top \,|\, \lambda \mathbf{e}'\right] \text{ up to 3D projective transformation}$$

where $\mathtt{F}^\top \mathbf{e}' = \mathbf{0}$ (i.e., $\mathbf{e}'$ is the null space of $\mathtt{F}^\top$), $\mathbf{v}$ is any 3-vector, and λ is any nonzero scalar.
If $\mathbf{v} = 0$ and $\lambda = 1$, then

Rank 2

$$\mathtt{P} = [\mathtt{I} \,|\, \mathbf{0}] \text{ and } \mathtt{P}' = \left[[\mathbf{e}']_\times \mathtt{F} \,|\, \mathbf{e}'\right] \text{ up to 3D projective transformation}$$

Up to 3D projective transformation

$$\mathbf{x} = \mathtt{P}\mathbf{X} \qquad \mathbf{x}' = \mathtt{P}'\mathbf{X}$$
$$\mathbf{x} = \mathtt{P}\mathtt{H}^{-1}\mathtt{H}\mathbf{X} \qquad \mathbf{x}' = \mathtt{P}'\mathtt{H}^{-1}\mathtt{H}\mathbf{X}, \text{ where } \mathtt{H} \text{ is any 3D projective transformation}$$

Both pairs of camera projection matrices, $\mathtt{P}$ and $\mathtt{P}'$, and $\mathtt{P}\mathtt{H}^{-1}$ and $\mathtt{P}'\mathtt{H}^{-1}$ have the same fundamental matrix $\mathtt{F}$

Retrieve camera projection matrices from fundamental matrix

$$\mathtt{P} = [\mathtt{I} \,|\, \mathbf{0}] \text{ and } \mathtt{P}' = [\mathtt{M} \,|\, \mathbf{v}] \text{ up to 3D projective transformation}$$

$\mathtt{M}$ of full rank (i.e., rank 3) and as far away from rank 2 as possible is obtained as follows

$\mathtt{F} = \mathtt{U}\mathtt{D}\mathtt{V}^\top$ singular value decomposition, where $\mathtt{U} = [\mathbf{u}_1 \,|\, \mathbf{u}_2 \,|\, \mathbf{u}_3]$ and $\mathtt{D} = \mathrm{diag}(s, t, 0)$

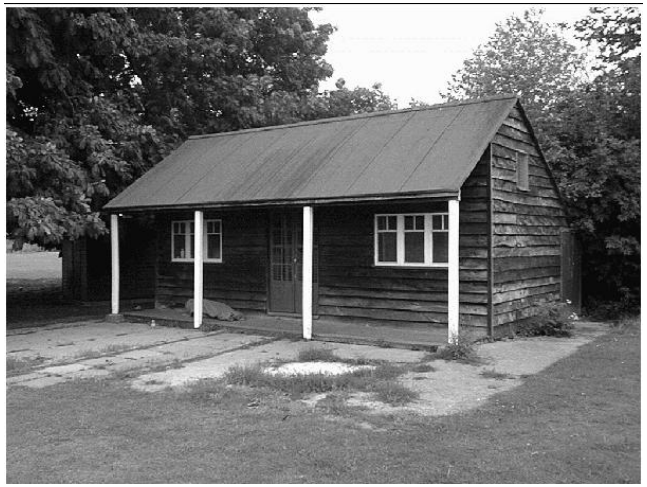
$\mathtt{F} = \mathtt{U}\mathtt{W}\mathtt{Z}\mathtt{D}'\mathtt{V}^\top$, where $\mathtt{D} = \mathtt{W}\mathtt{Z}\mathtt{D}'$ and $\mathtt{W}\mathtt{Z} = \mathrm{diag}(1, 1, 0)$

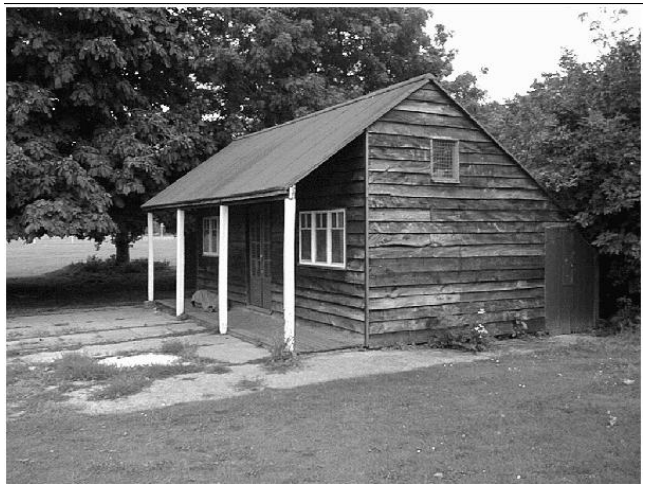
$$\mathtt{W} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathtt{Z} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } \mathtt{D}' = \mathrm{diag}(s, t, (s+t)/2)$$

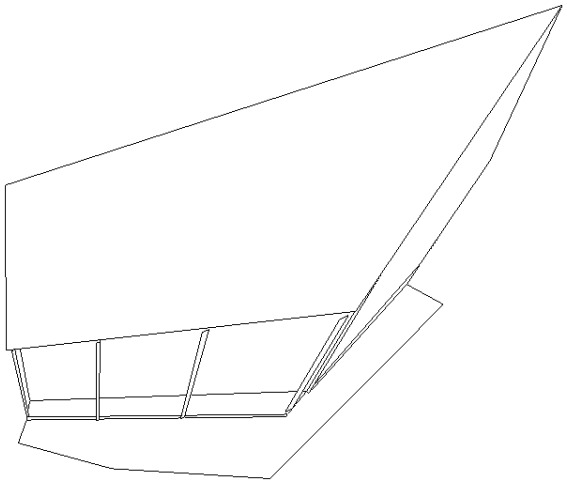
$\mathtt{F} = (\mathtt{U}\mathtt{W}\mathtt{U}^\top)(\mathtt{U}\mathtt{Z}\mathtt{D}'\mathtt{V}^\top)$

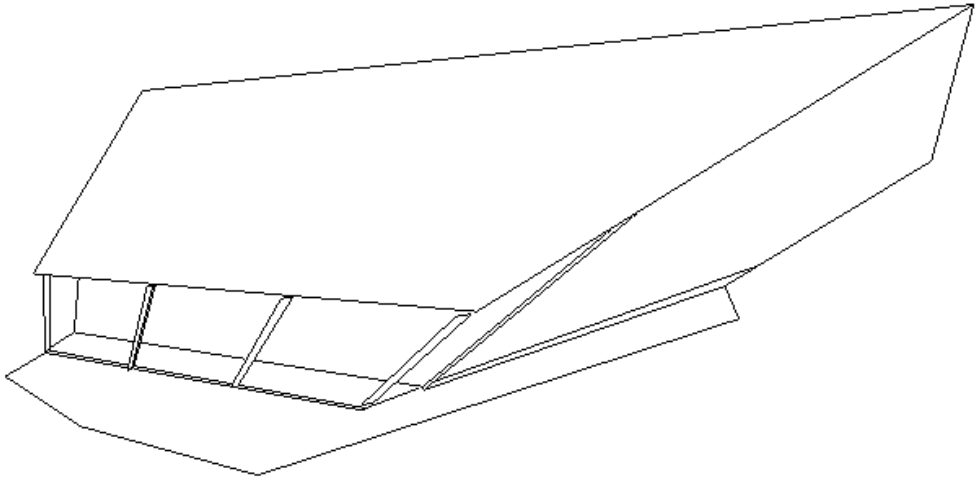
$\mathtt{F} = \mathtt{S}\mathtt{M}$ and $\mathtt{P}' = [\mathtt{M} \,|\, \mathbf{v}]$, where $\mathtt{S} = \mathtt{U}\mathtt{W}\mathtt{U}^\top = -[\mathbf{v}]_\times = -[\mathbf{u}_3]_\times$ and $\mathtt{M} = \mathtt{U}\mathtt{Z}\mathtt{D}'\mathtt{V}^\top$ and $\mathbf{v} = \mathbf{u}_3 = \mathbf{e}'$

Up to 3D projective transformation









Next lecture

- Estimation of fundamental matrix, nonlinear
- Reading
  - Sections 10.1, 9.5, 11.4, 12.5, 10.5