CSE 203B W25 Homework 4

Due Time : 11:50pm, Thursday Feb. 13, 2024 Submit to Gradescope

In this homework, we work on exercises from the textbook. Problems 4.1, 4.8, and 4.11 are related to LP. Problems 4.21, and 4.39 are related to QCQP, and SDP. Problems 5.4, 5.5, 5.6, 5.8, and 5.9 are examples and applications of duality. Also, we practice using the convex optimization tools on a linear programming problem, and a semidefinite programming problem.

Total points: 50. Exercises are graded by completion, and assignments are graded by correctness.

I. Exercises from textbook chapters 4 & 5 (10 pts, 1pt for each problem)

4.1, 4.8, 4.11, 4.21, 4.39, 5.4, 5.5, 5.6, 5.8, 5.9.

II. Assignments (35 pts)

II.1 Linear Programming: You are free to use any software packages. (15 pts)

Given

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

$$b^T = \begin{bmatrix} 4 & 3 & 4 & 1 & 2 \end{bmatrix},$$

$$c^T = \begin{bmatrix} -1 & -2 & -3 & -4 \end{bmatrix},$$

and $n = 4$, perform steps A, B, and C for problems II.1.1, II.1.2, II.1.3, and II.1.4.

A. Solve the following linear programming problems twice, once using the primal formulation and once using the dual formulation.

B. Check the feasibility of the solution. If a solution is not found, explain why a solution is not available and suggest how to mitigate the issue if you are the project leader.

C. Compare the primal and dual solutions. If the primal and dual formulation solutions are different, explain the difference.

II.1.1. minimize $f_0(x) = c^T x$ subject to $Ax \le b$, $x \in R^n$.

A. Solve in primal formulation.

The minimum value is -4, which can be achieved at the feasible point $x^T = [0\ 0\ 0\ 1]$. This can be verified by checking:

$$Ax = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \le \begin{bmatrix} 4 \\ 3 \\ 4 \\ 1 \\ 2 \end{bmatrix} = b$$

And the objective value:

$$c^T x = \begin{bmatrix}-1 & -2 & -3 & -4\end{bmatrix} \begin{bmatrix}0\\0\\0\\1\end{bmatrix} = -4$$

Solve in the dual formulation:

Refers to the formula (5.22) of the textbook Chapter 5.2.1, pp. 225.

The dual problem is equivalent to $\max -b^T \lambda$ s.t. $A^T\lambda + c = 0$, $\lambda \geq 0$.

However, this is not the definition of but is equivalent to the dual problem. The dual problem is defined as

$$\max g(\lambda) \text{ s.t. } \lambda \geq 0,$$

where $g(\lambda) = \begin{cases} -b^T\lambda & \text{if } A^T\lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases}$ is the Lagrange dual function.

The textbook Chapter 5.2.1, pp. 224-225 shows the derivation.

Now check if the constraints $A^T\lambda + c = 0$, $\lambda \geq 0$ are feasible. Now assume one of the two constraints holds, which is $A^T\lambda + c = 0$

We can use the Python library sympy's rref() to achieve Gaussian elimination for $[A^T | -c]$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0.5 & | & 1 \\ 0 & 1 & 0 & 0 & -0.5 & | & 0 \\ 0 & 0 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & -0.5 & | & 0 \end{bmatrix}$$

As we can see, the $\lambda \geq 0$ holds for all values.

The dual problem is:

$$\max -b^T\lambda \text{ subject to } A^T\lambda + c = 0,\ \lambda \geq 0$$

This gives us:

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 1 & 1 & 1 \\ 4 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\\\lambda_4\\\lambda_5\end{bmatrix} = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$$

The solution is $\lambda^T = [1\ 0\ 0\ 0\ 0]$, giving:

$$-b^T\lambda = -\begin{bmatrix}4 & 3 & 4 & 1 & 2\end{bmatrix} \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} = -4$$

B. Both primal and dual solutions are feasible. The primal solution provides a minimum value of -4 at $x^T = [0\ 0\ 0\ 1]$, while the dual solution provides a maximum value of -4 at $\lambda^T = [1\ 0\ 0\ 0\ 0]$. These solutions satisfy all constraints and complementary slackness conditions.

C. The primal and dual optimal values are equal in magnitude but opposite in sign, which is expected as the dual maximizes $-b^T\lambda$ while the primal minimizes $c^T x$. This demonstrates strong duality, as the optimal values match (considering the sign change in the dual formulation). The solutions are optimal as they satisfy both feasibility and complementary slackness conditions.

II.1.2. minimize $f_0(x) = c^T x$ subject to $Ax = b$, $x \in R^n$.

$[A|b] \overset{RREF}{\longrightarrow} I_{5\times5}$

$[A] \overset{RREF}{\longrightarrow} I_{4\times4}$

Rubric: Rank of [A] < Rank of $[A|b]$, the primal is infeasible. The dual will be unbounded. The student needs to compare and explain why this is the case. They need to point out that the original constraints are infeasible. Then the student is expected to relax as the least equality constraints to inequality. Relaxation can be achieved by changing = to ≤, such as $a_5^T x - b_5 = 0$ to $a_5^T x - b_5 \leq 0$. However, I think it is also acceptable to change = to ≥. If just relaxing one constraint does not help, then the multiple could be relaxed. You need to give one example of such a relaxation and the answer derived from it.

II.1.3. minimize $f_0(x) = c^T x$ subject to $Ax \leq b$, $x \in R^n_+$.

A.

For the primal formulation, The primal solution remains $x^T = [0; 0; 0; 1]$, giving:

$$c^T x = \begin{bmatrix} -1 & -2 & -3 & -4 \end{bmatrix} . \begin{bmatrix} 0\ 0\ 0\ 1 \end{bmatrix} = -4$$

For the dual formulation: The dual solution remains $\lambda^T = [1; 0; 0; 0; 0]$, giving:

$$-b^T \lambda = -\begin{bmatrix} 4 & 3 & 4 & 1 & 2 \end{bmatrix} . \begin{bmatrix} 1\ 0\ 0\ 0\ 0 \end{bmatrix} = -4$$

B. Both primal and dual solutions are feasible. The primal solution provides a minimum value of -4 at $x^T = [0\ 0\ 0\ 1]$, while the dual solution provides a maximum value of -4 at $\lambda^T = [1\ 0\ 0\ 0\ 0]$. These solutions satisfy all constraints and complementary slackness conditions.

C. The primal and dual optimal values are equal in magnitude but opposite in sign, which is expected as the dual maximizes $-b^T\lambda$ while the primal minimizes $c^T x$. This demonstrates strong duality, as the optimal values match (considering the sign change in the dual formulation). The solutions are optimal as they satisfy both feasibility and complementary slackness conditions.

II.1.4. minimize $f_0(x) = c^T x$ subject to $Ax = b$, $x \in R^n_+$.

$[A|b] \overset{RREF}{\longrightarrow} I_{5\times5}$

$[A] \overset{RREF}{\longrightarrow} I_{4\times4}$

Rubric: Rank of [A] < Rank of $[A|b]$, the primal is infeasible. The dual will be unbounded. The student needs to compare and explain why this is the case. They need to point out that the original constraints are infeasible. Then the student is expected to relax as the least equality constraints to inequality. Relaxation can be achieved by changing = to ≤, such as $a_5^T x - b_5 = 0$ to $a_5^T x - b_5 \leq 0$. However, I think it is also acceptable to change = to ≥. If just relaxing one constraint does not help, then the multiple could be relaxed. You need to give one example of such a relaxation and the answer derived from it.

II.2 Eigenvalue Optimization Problem (20 pts) Eigenvalues of certain matrices are related to physical phenomenon in our world. For example, in mechanical structures, we have something called stiffness matrices, whose maximum eigenvalue corresponds to the highest frequency in the mechanical system. Hence, sometimes it is of utmost importance, that if the matrix is parametrized, we want the parameters in the matrix such that, the maximum eigenvalue of the matrix is minimized.

We will try to solve this problem for a certain class of parametrized matrices. Given symmetric matrices $A_i \in S_n$, Let $F = \sum_i \alpha_i A_i$. We have to chose $\alpha_i \in \mathbf{R}$ such that $\alpha_i \geq 0$ and $\sum_i \alpha_i = 1$. (20 pts)

(i) Let us order the eigenvalues of F from large to small, i.e. $\lambda_0 \geq \lambda_1 ... \geq \lambda_{n-1}$. Prove that for some $t \in R$, if the matrix $t\mathbf{I} - F \succeq 0$, then $t \geq \lambda_0$. (Here **I** is the identity matrix)

Answer: The matrix F would be a symmetric matrix since its linear combination of symmetric matrices A_i. Since F is a symmetric matrix, we can decompose it as following

$$F = Q\Lambda Q^T$$

where Q is an orthogonal matrix whose column form an orthonormal basis with eigenvectors of A and Λ is a diagonal matrix of its eigenvalues.

$$tI - F = tQQ^T - Q\Lambda Q^T = Q(tI - \Lambda)Q^T$$

Now since $tI - \Lambda$ would be a diagonal matrix containing eigenvalues from the matrix $tI - F$.

If $tI - F \succeq 0$, all its eigenvalues would also be positive. Observe that if λ_i are eigenvalues of F, then $t - \lambda_i$ are eigenvalues of $tI - F$ and hence

$$t \geq \lambda_0$$

Hence, if $t\mathbf{I} - F \succeq 0$ then $t \geq \lambda_0$ which is the largest eigenvalue of matrix F.

(ii) Using the above result, the problem of minimizing the maximum eigenvalue becomes

$$\begin{aligned} &\min_{t,\alpha_i} t \\ \text{s.t.} \quad &\sum_i \alpha_i A_i - tI \preceq 0, \\ &\sum_i \alpha_i = 1, \\ &\alpha_i \geq 0 \end{aligned} \tag{1}$$

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Using the above formulation, derive the KKT conditions and the dual for this problem.

Answer:

For the lagrangian, we associate a dual variable $Z \in S_+$

$$L(t, \alpha_i, Z, \mu, \lambda) = t + tr(\sum_i (\alpha_i A_i Z) - tZ) + \mu * (\sum_i \alpha_i - 1) + \lambda^T \alpha$$

The KKT conditions for the above case are:

Primal Constraints:

$$\sum_i \alpha_i A_i - tI \preceq 0$$

$$\sum_i \alpha_i = 1$$

$$\alpha_i \geq 0$$

Dual Constraints

$$\lambda_i \leq 0$$

$$Z \succeq 0$$

Complementary Slackness

$$\lambda_i \alpha_i = 0$$

$$tr(\sum_i \alpha_i A_i Z - tZ) = 0$$

Gradient should be 0 wrt t and α_i

$$\frac{\partial L}{\partial t} = 1 - tr(Z) = 0$$

$$\frac{\partial L}{\partial \alpha_i} = tr(A_i Z) + \mu + \lambda_i = 0$$

Now the dual function would be

$$g(Z, \mu, \lambda) = \min_{\alpha_i, t} L(t, \alpha_i, Z, \mu, \lambda)$$

$$g(Z, \mu, \lambda) = \{ \begin{array}{ll} -\mu & \text{if } tr(Z) = 1, tr(A_i Z) + \mu + \lambda_i = 0, Z \succeq 0, \lambda_i \leq 0 \\ -\infty & \text{otherwise} \end{array}$$

Hence the dual problem would be

$$\max_{\mu, \lambda, Z} -\mu$$

$$\text{s.t. } tr(Z) = 1$$

$$tr(A_i Z) + \mu + \lambda_i = 0 \forall i \in [1, n]$$

$$\lambda_i \leq 0$$

$$Z \succeq 0$$

(iii) Now, let's try a concrete example, you may use any convex optimization package to solve the problem.

Given

$$A_0 = \begin{bmatrix} -300 & 5 & -5 \\ 5 & 4 & -1 \\ -5 & -1 & 4 \end{bmatrix}, A_1 = \begin{bmatrix} -4 & -5 & 5 \\ -5 & 4 & 5 \\ 5 & 5 & -4 \end{bmatrix}, A_2 = \begin{bmatrix} 4 & -5 & -5 \\ -5 & 2 & -3 \\ -5 & -3 & 2 \end{bmatrix} \tag{2}$$

,

minimize the maximum eigenvalue for $F = \alpha_0 * A_0 + \alpha_1 * A_1 + \alpha_2 * A_2$, such that $\alpha_i \geq 0$ and $\sum_i \alpha_i = 1$.

Report the value of maximum eigenvalue of F after the optimization and also the values of α_i which should be used to get that value.

Answers:

$$\lambda_0 = 3.131603520240919$$

$$\alpha_0 = 0.19594756$$

$$\alpha_1 = 0.27584813$$

$$\alpha_2 = 0.52820431$$