Week 5 Discussion

CSE203B Winter25

Agenda

Linear Programming

Duality

Linear Programming (LP)

Definition & General Form

- Objective function and constraint functions are all affine
- General Form:

$$
\begin{aligned}
\min_{x} \quad & c^T x + d \\
\text{s.t.} \quad & Gx \leq h \\
& Ax = b
\end{aligned}
$$

- LP are convex optimization problems

Linear Programming (LP)

Standard Form LP & Inequality Form LP

Standard Form LP

$$\begin{aligned} \min_{x} \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

e.g., HW4 II.1.4

Inequality Form LP

$$\begin{aligned} \min_{x} \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \end{aligned}$$

e.g., HW4 II.1.1

Linear Programming (LP)

How to find the optimal value

Figure out which is the case for the LP

1. **No feasible solutions** -> The optimal value is ∞
2. **Unbounded solutions** -> The optimal value is $-\infty$
3. **Bounded solutions**

Linear Programming (LP)

How to find the optimal value: LP only with equality constraints

$$\min_{x} \quad c^T x$$
$$\text{s.t.} \quad Ax = b$$

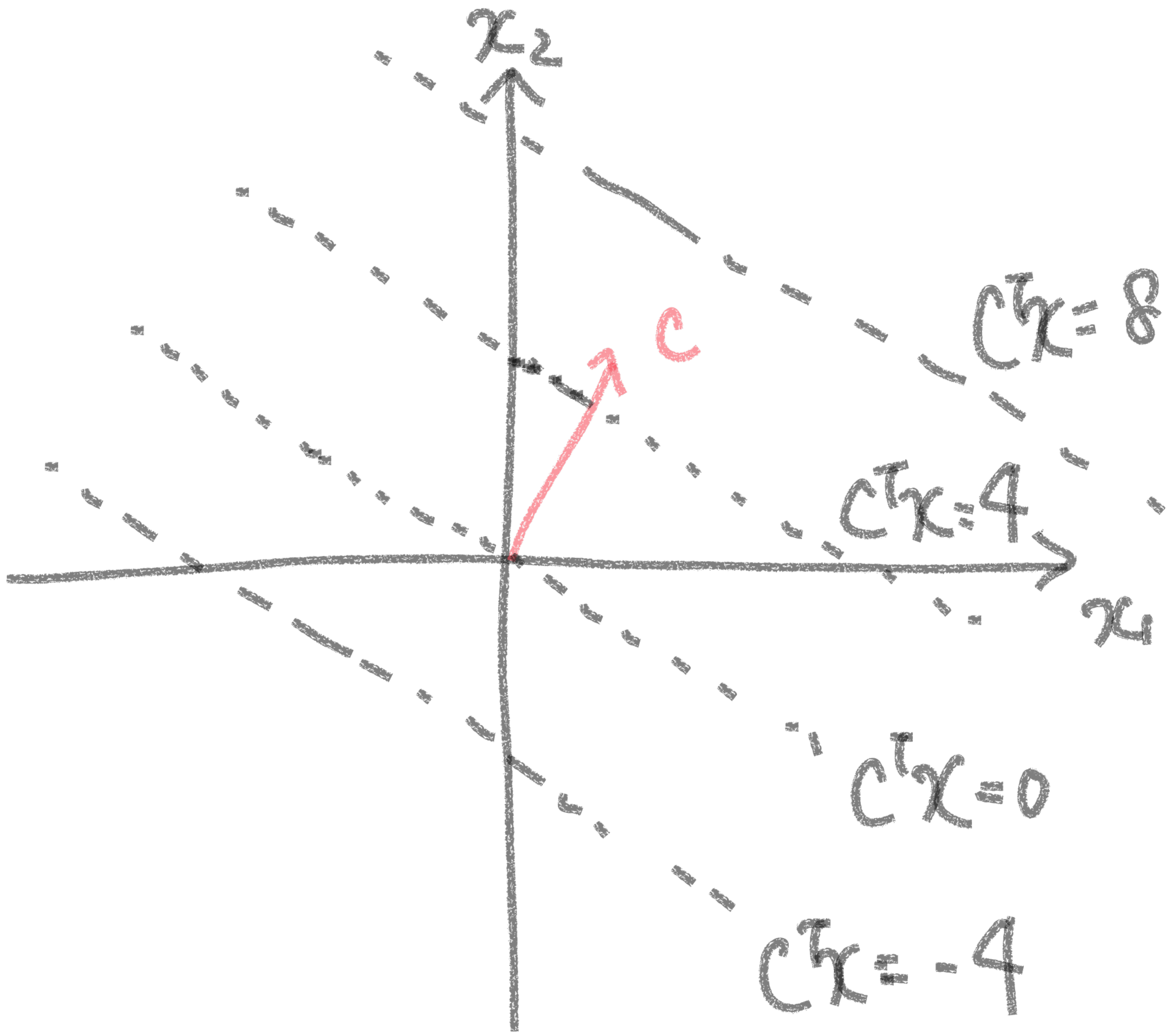
1. **No feasible solutions** -> The optimal value is ∞
2. **Unbounded solutions** -> The optimal value is $-\infty$
3. **Bounded solutions**

Linear Programming (LP)

How to find the optimal value: LP only with equality constraints

$$\min_x \quad c^T x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}^T x = x_1 + 2x_2$$

$$\text{s.t.} \quad Ax = b$$

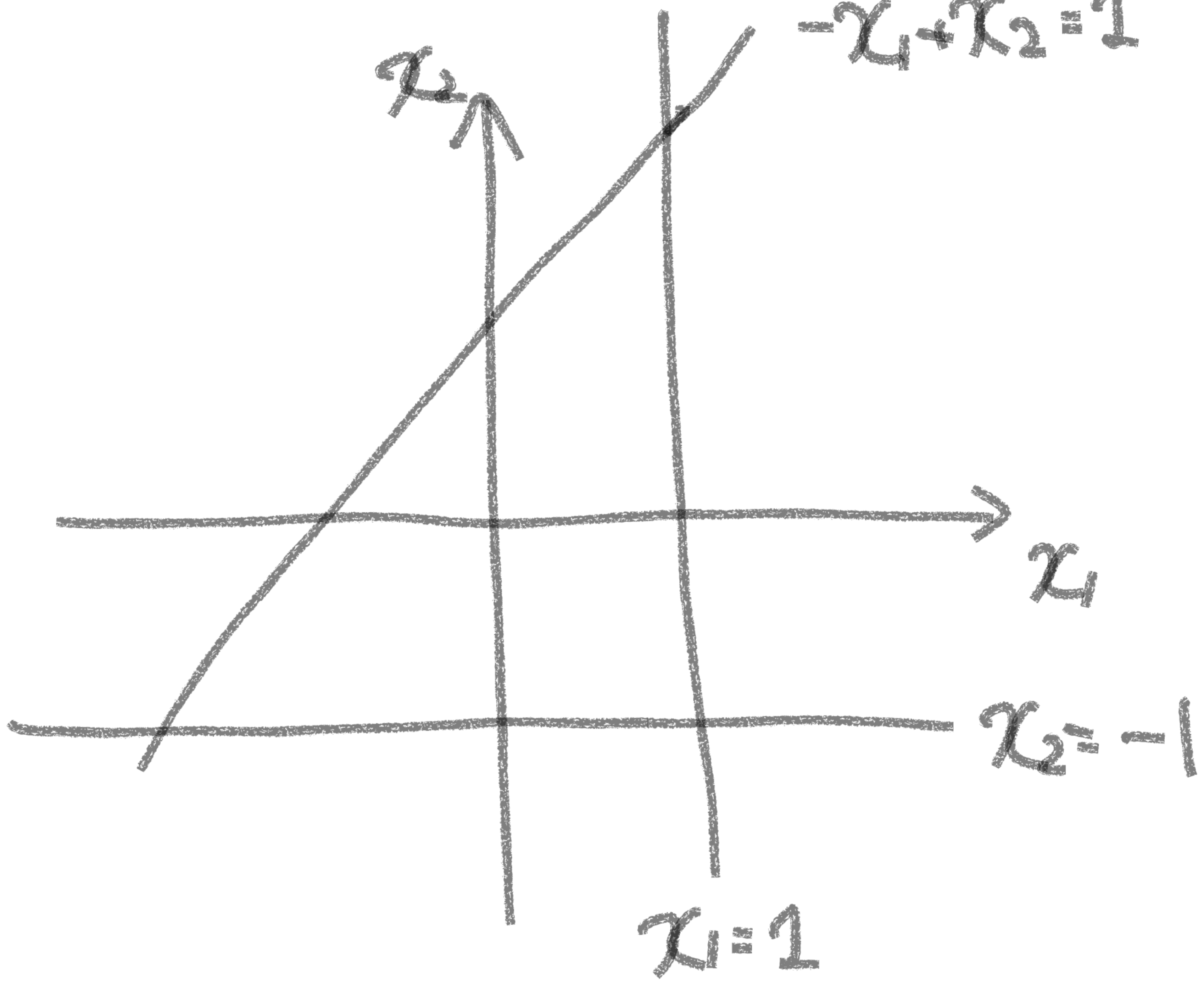


Linear Programming (LP)

How to find the optimal value: LP only with equality constraints

$$\min_{x} \quad x_1 + 2x_2$$
$$\text{s.t.} \quad Ax = b$$

1. **No feasible solutions**



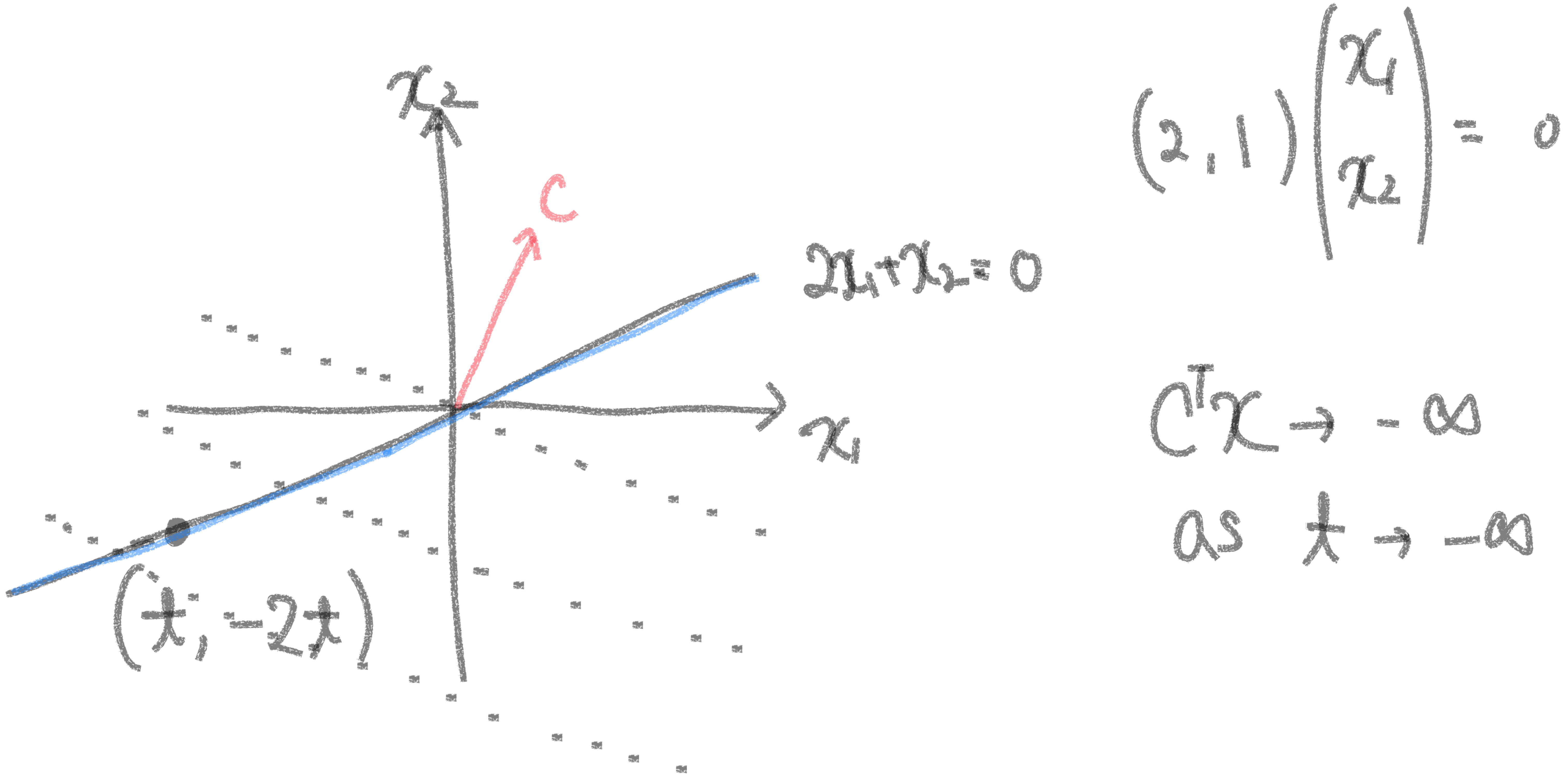
$$\begin{pmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Linear Programming (LP)

How to find the optimal value: LP only with equality constraints

$$\min_x \quad x_1 + 2x_2$$
$$\text{s.t.} \quad Ax = b$$

2. **Unbounded solutions**

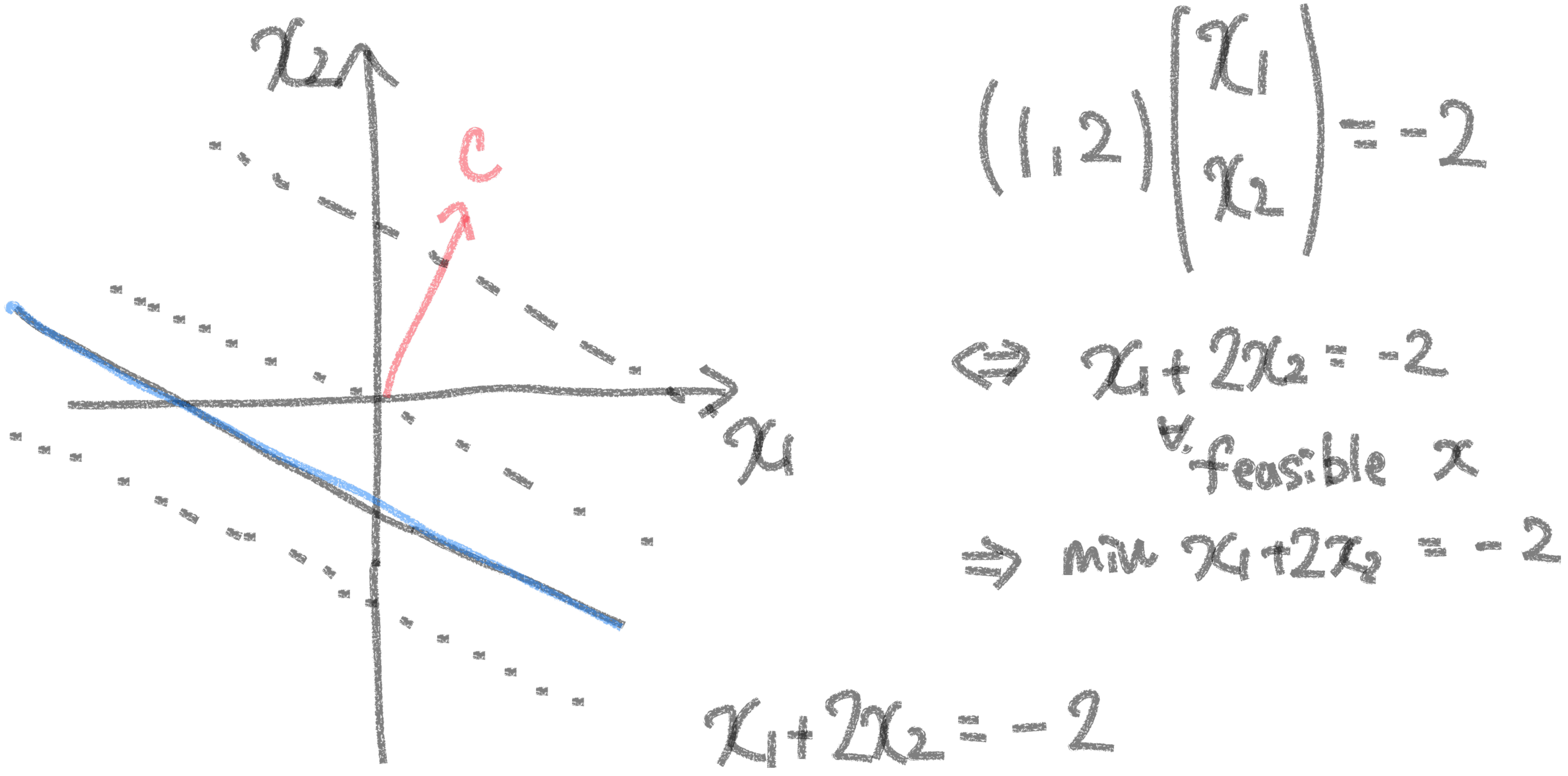


Linear Programming (LP)

How to find the optimal value: LP only with equality constraints

$$\min_x \quad x_1 + 2x_2$$
$$\text{s.t.} \quad Ax = b$$

3. **Bounded solutions**



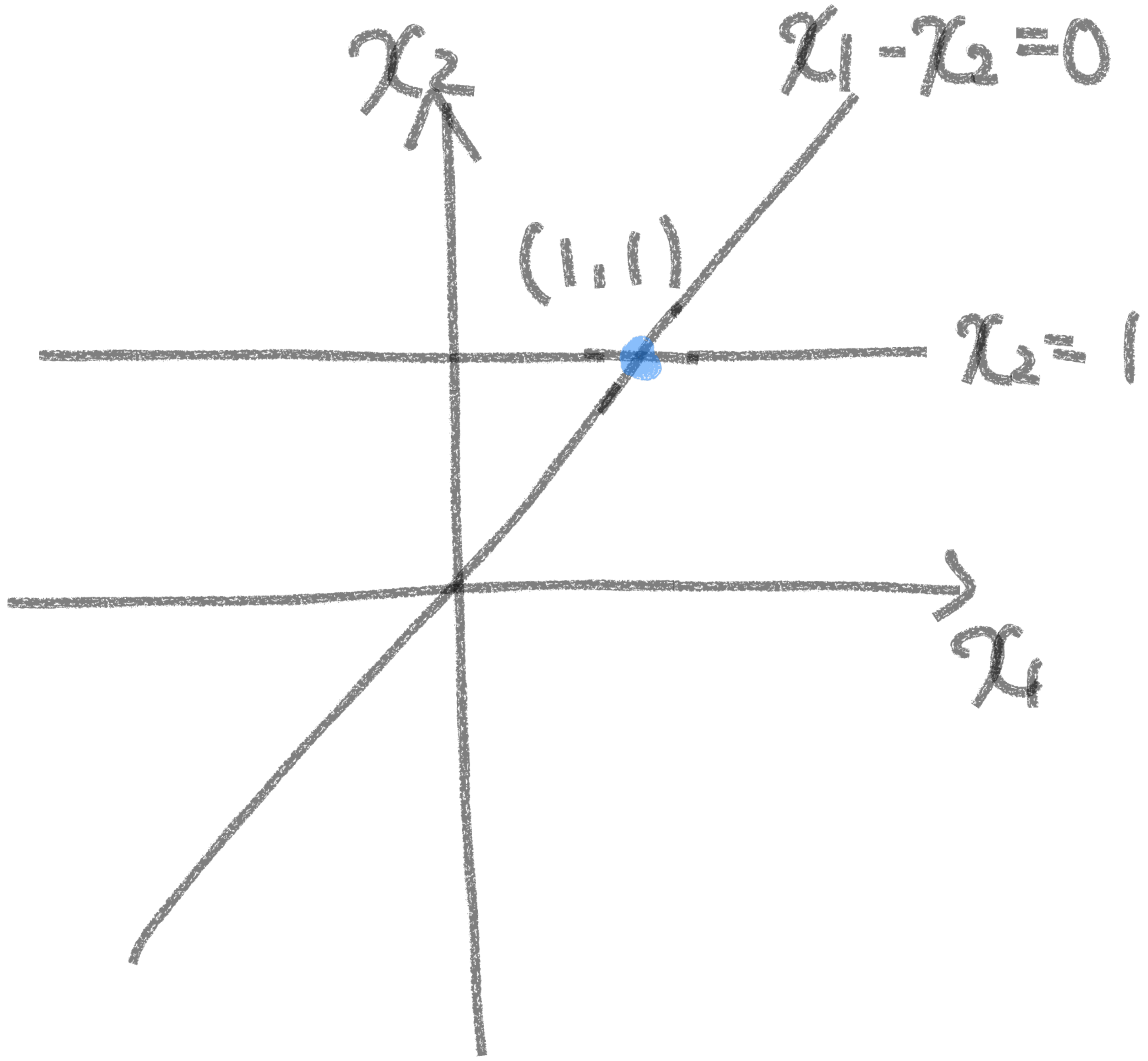
Linear Programming (LP)

How to find the optimal value: LP only with equality constraints

$$\min_{x} \quad x_1 + 2x_2$$

$$\text{s.t.} \quad Ax = b$$

3. **Bounded solutions**



$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\Rightarrow$ domain is $\{(1,1)\}$

$\Rightarrow \min x_1 + 2x_2 = 3$

Linear Programming (LP)

How to find the optimal value: LP only with equality constraints

$$\min_x \quad c^T x$$
$$\text{s.t.} \quad Ax = b$$

1. **No feasible solutions**
   <-> $b \notin \mathrm{R}(A)$
   The optimal value is ∞

2. **Unbounded solutions**
   <-> $b \in \mathrm{R}(A), c \notin \mathrm{R}(A^T)$
   The optimal value is $-\infty$

3. **Bounded solutions**
   <-> $b \in \mathrm{R}(A), c \in \mathrm{R}(A^T)$ (i.e., $c = A^T\lambda$ for some λ)
   The optimal value is $c^T x = (A^T\lambda)^T x = \lambda^T(Ax) = \lambda^T b$

Linear Programming (LP)

How to find the optimal value: LP only with inequality constraints

$$\min_x \quad c^T x$$
$$\text{s.t.} \quad Ax \leq b$$

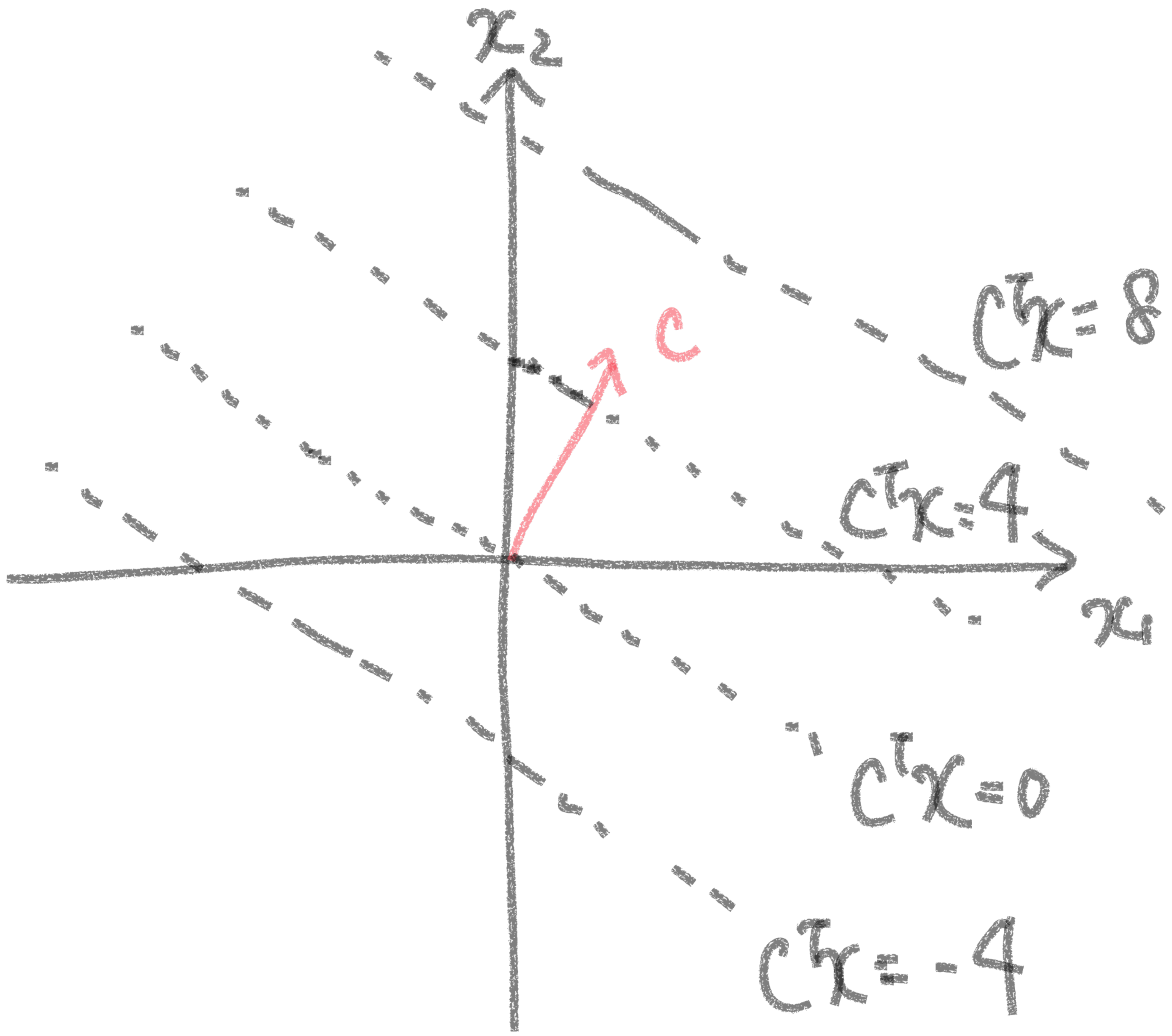
1. **No feasible solutions** -> The optimal value is ∞
2. **Unbounded solutions** -> The optimal value is $-\infty$
3. **Bounded solutions**

Linear Programming (LP)

How to find the optimal value: LP only with inequality constraints

$$\min_{x} \quad c^T x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}^T x = x_1 + 2x_2$$

$$\text{s.t.} \quad Ax \leq b$$

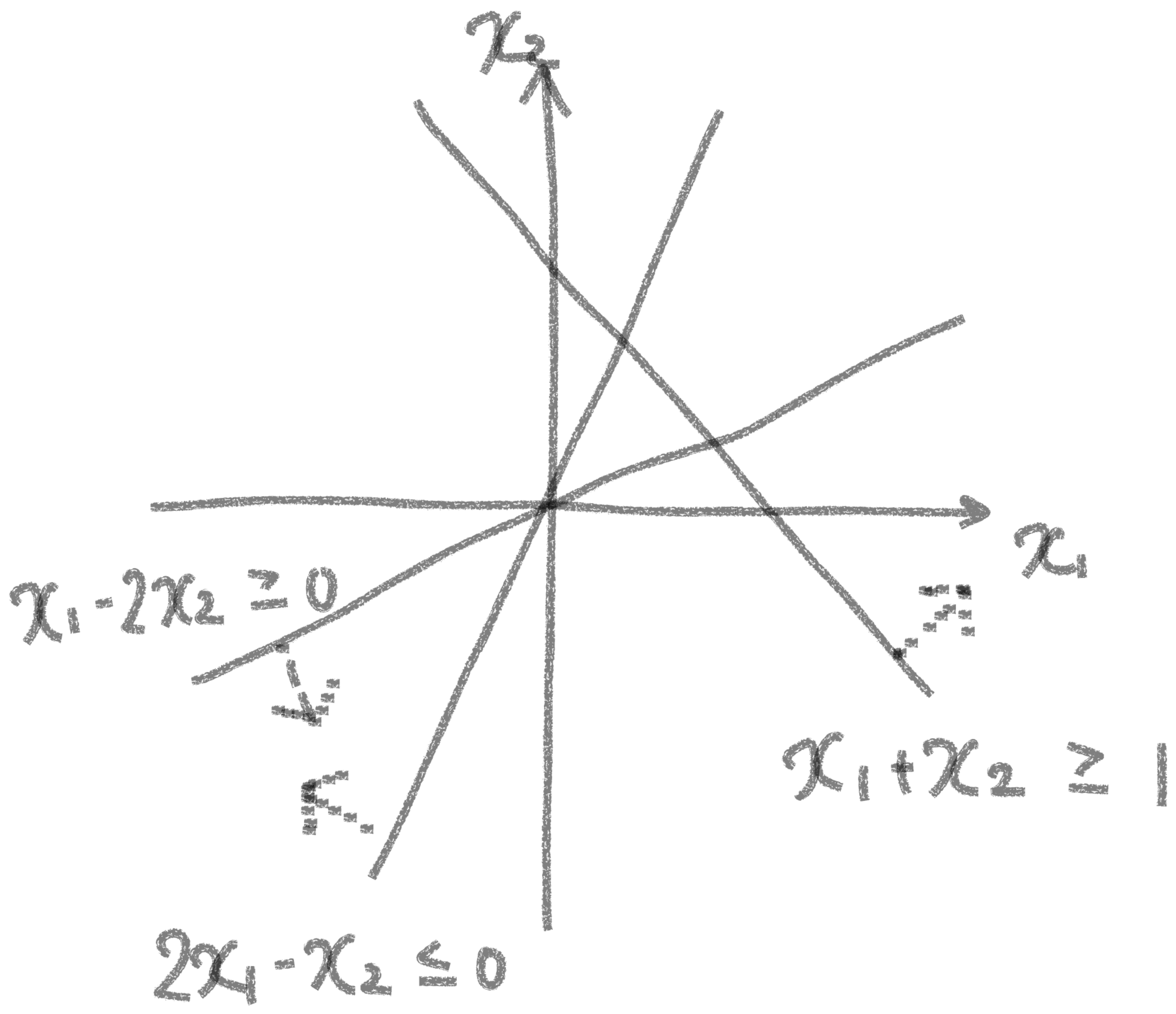


Linear Programming (LP)

How to find the optimal value: LP only with inequality constraints

$$\min_x \quad x_1 + 2x_2$$
$$\text{s.t.} \quad Ax \le b$$

1. **No feasible solutions**

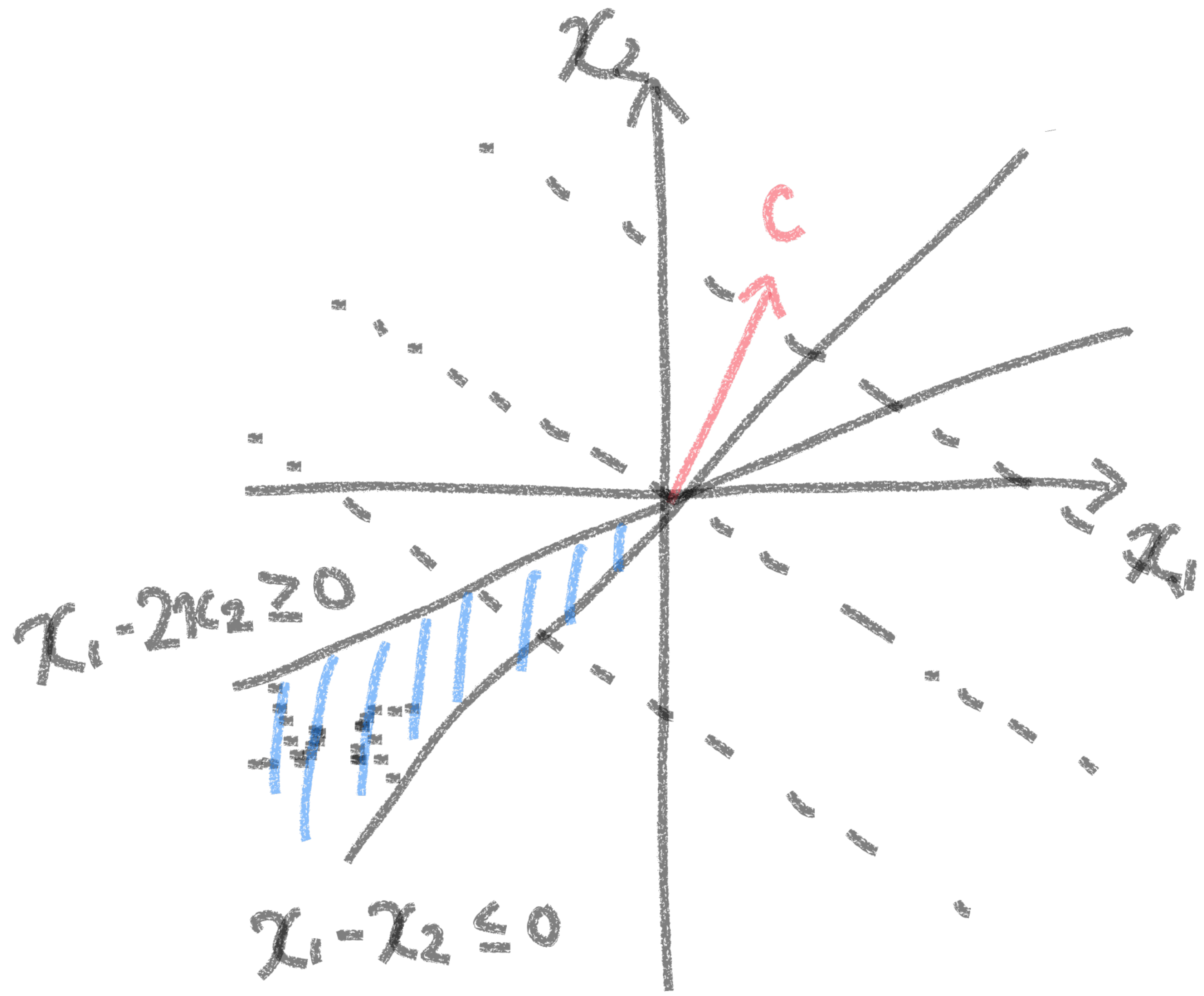


Linear Programming (LP)

How to find the optimal value: LP only with inequality constraints

$$\min_x \quad x_1 + 2x_2$$
$$\text{s.t.} \quad Ax \leq b$$

2. **Unbounded solutions**

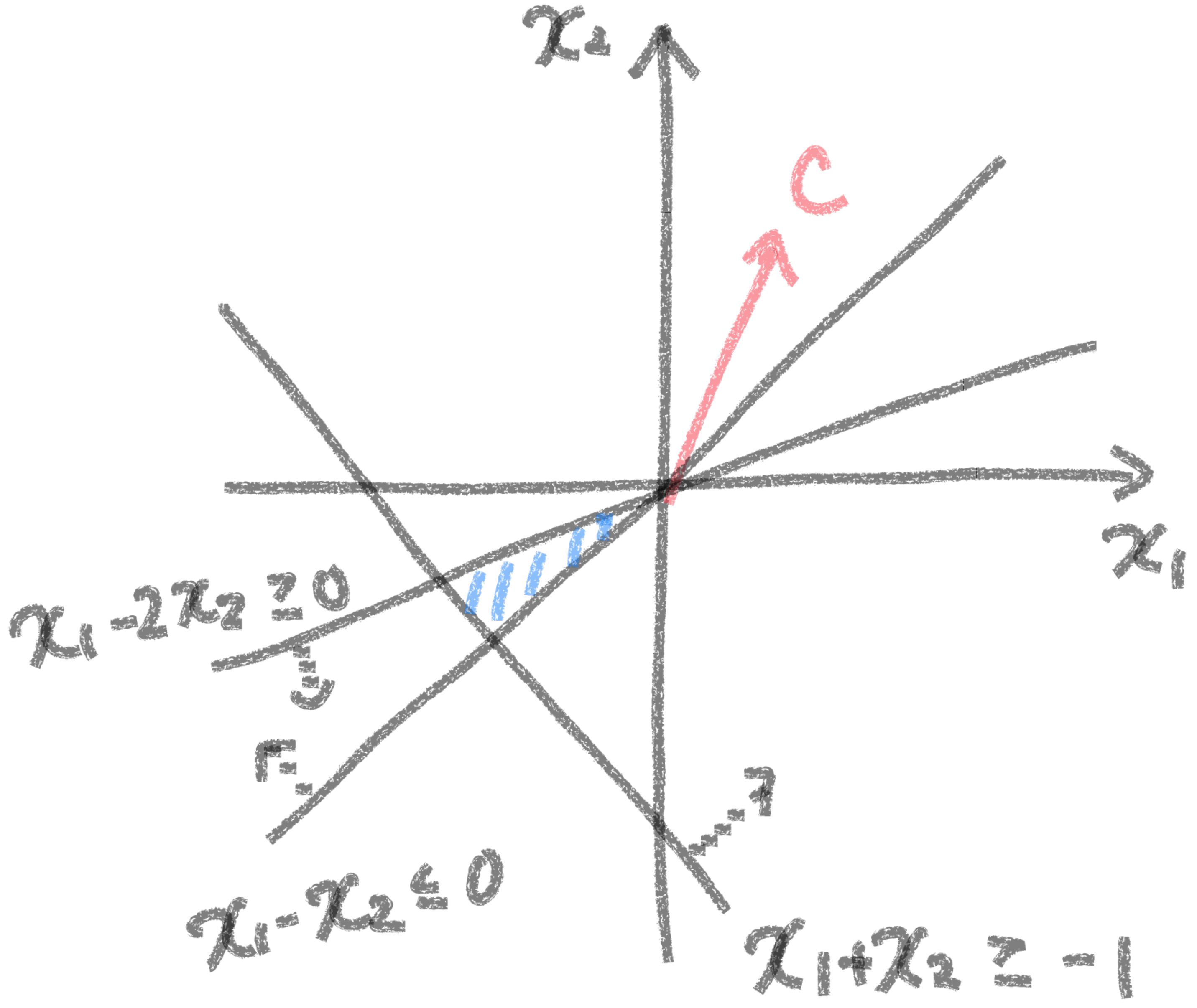


Linear Programming (LP)

How to find the optimal value: LP only with inequality constraints

$$\min_x \quad x_1 + 2x_2$$
$$\text{s.t.} \quad Ax \le b$$

3. **Bounded solutions**

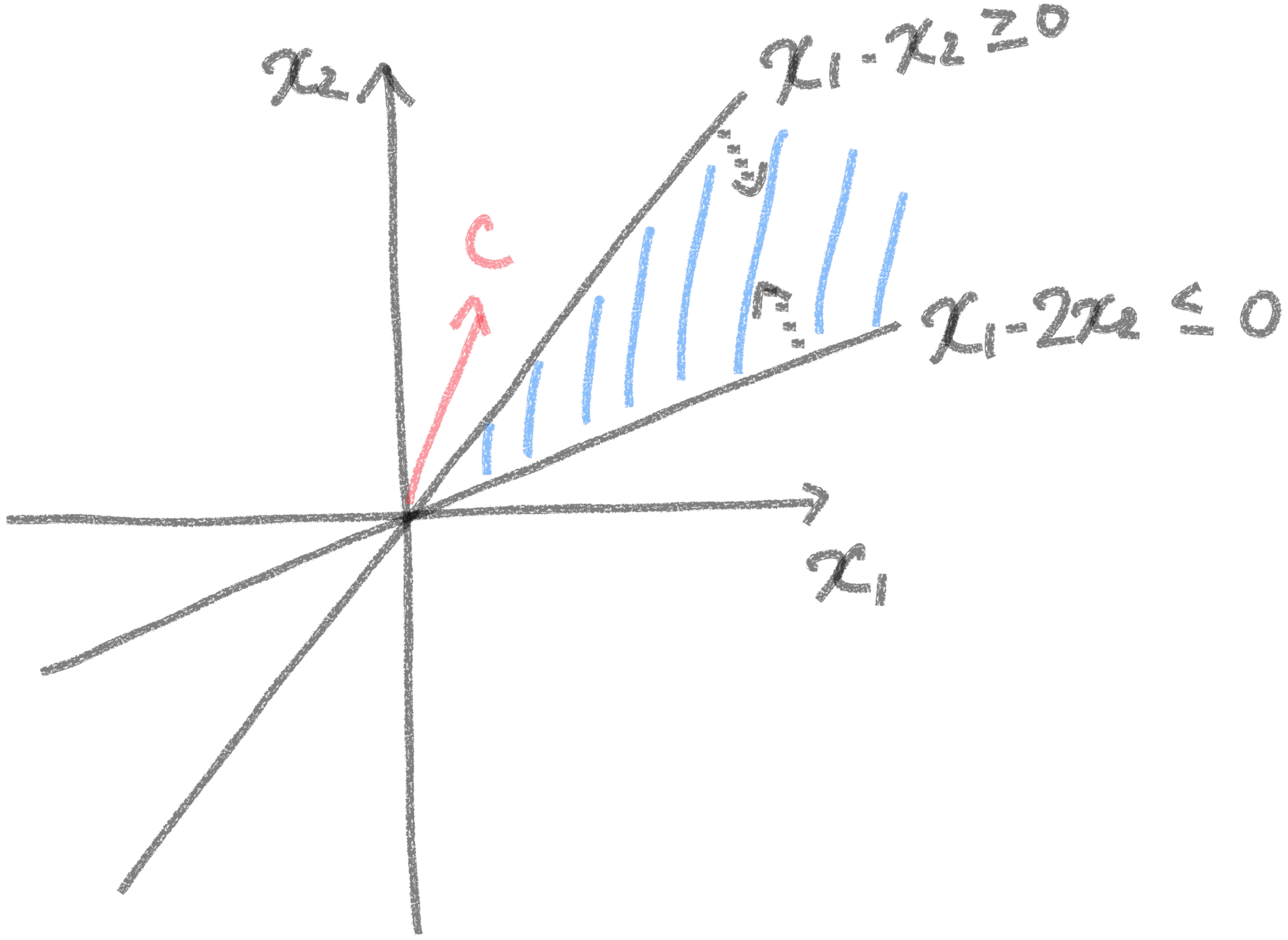


Linear Programming (LP)

How to find the optimal value: LP only with inequality constraints

$$\min_{x} \quad x_1 + 2x_2$$
$$\text{s.t.} \quad Ax \leq b$$

3. **Bounded solutions**



Linear Programming (LP)

Hint on HW4 II.1

Ask the following questions:

1. Are there feasible solutions?
   If not, how can we modify the LP to have ones?

2. Is the problem bounded?
   If not, how can you achieve $-\infty$, and how can you prevent it?

When there are feasible solutions and the problem is bounded, you can find the optimal value:)

Duality

The Lagrangian

- Consider the general primal problem:

$$\begin{aligned} \min_x \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \leq 0 \quad i = 1, \ldots, m \\ & h_i(x) = 0 \quad i = 1, \ldots, p \end{aligned}$$

- The Lagrangian is:

$$L(x, \lambda, v) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} v_i h_i(x)$$

($\lambda \in \mathbb{R}_+^m, v \in \mathbb{R}^p$: Lagrange multipliers)

Duality

The dual function

- The Lagrange dual function is:

$$g(\lambda, v) = \inf_x L(x, \lambda, v) = \inf_x \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} v_i h_i(x) \right)$$

- $g(\lambda, v)$ is **always concave** because it is the pointwise infimum of a family of affine functions of (λ, v)

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Duality

Dual Problem

- The Lagrange dual problem of the primal problem:

$$\begin{aligned} \max_{\lambda, v} \quad & g(\lambda, v) \\ \text{s.t.} \quad & \lambda \geq 0 \end{aligned}$$

- The dual problem is always convex

Duality

Dual of Standard Form LP

$$L(x, \lambda, v) = c^T x - \lambda^T x + v^T(Ax - b)$$
$$= (c - \lambda + A^T v)^T x - v^T b$$

Standard Form LP

$$\min_x \quad c^T x$$
$$\text{s.t.} \quad Ax = b$$
$$x \geq 0$$

$$g(\lambda, v) = \begin{cases} -v^T b & c - \lambda + A^T v = 0 \\ -\infty & \text{o.w.} \end{cases}$$

The dual problem:
$$\max_{\lambda, v} \quad g(\lambda, v)$$
$$\text{s.t.} \quad \lambda \geq 0$$

Duality

Dual of Standard Form LP

Standard Form LP

$$\min_{x} \quad c^T x$$
$$\text{s.t.} \quad Ax = b$$
$$x \geq 0$$

The dual problem: $$\max_{\lambda, v} \quad g(\lambda, v)$$
$$\text{s.t.} \quad \lambda \geq 0$$

$$g(\lambda, v) = \begin{cases} -v^T b & c - \lambda + A^T v = 0 \\ -\infty & \text{o.w.} \end{cases}$$

$$\max_{\lambda, v} \quad -v^T b$$
$$\text{s.t.} \quad \lambda \geq 0$$
$$c - \lambda + A^T v = 0$$

$$\Leftrightarrow$$

$$\max_{v} \quad -v^T b$$
$$\text{s.t.} \quad A^T v + c \geq 0$$

The dual of **Standard Form LP** turns out to be **Inequality Form LP**

Duality

Dual of Inequality Form LP

$$L(x, \lambda) = c^T x + \lambda^T (Ax - b)$$
$$= (c + A^T \lambda)^T x - \lambda^T b$$

Inequality Form LP

$$\min_x \quad c^T x$$
$$\text{s.t.} \quad Ax \leq b$$

$$g(\lambda) = \begin{cases} -\lambda^T b & c + A^T \lambda = 0 \\ -\infty & \text{o.w.} \end{cases}$$

The dual problem:
$$\max_\lambda \quad g(\lambda)$$
$$\text{s.t.} \quad \lambda \geq 0$$

Duality

Dual of Inequality Form LP

Inequality Form LP

$$\begin{aligned} \min_x \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \end{aligned}$$

The dual problem: $\begin{aligned} \max_\lambda \quad & g(\lambda) \\ \text{s.t.} \quad & \lambda \geq 0 \end{aligned}$

$$g(\lambda) = \begin{cases} -\lambda^T b & c + A^T \lambda = 0 \\ -\infty & \text{o.w.} \end{cases}$$

$$\begin{aligned} \max_\lambda \quad & -\lambda^T b \\ \text{s.t.} \quad & \lambda \geq 0 \\ & c + A^T \lambda = 0 \end{aligned}$$

The dual of **Inequality Form LP** turns out to be **Standard Form LP**