CSE 203B Week 4 Discussion

Enming Guo

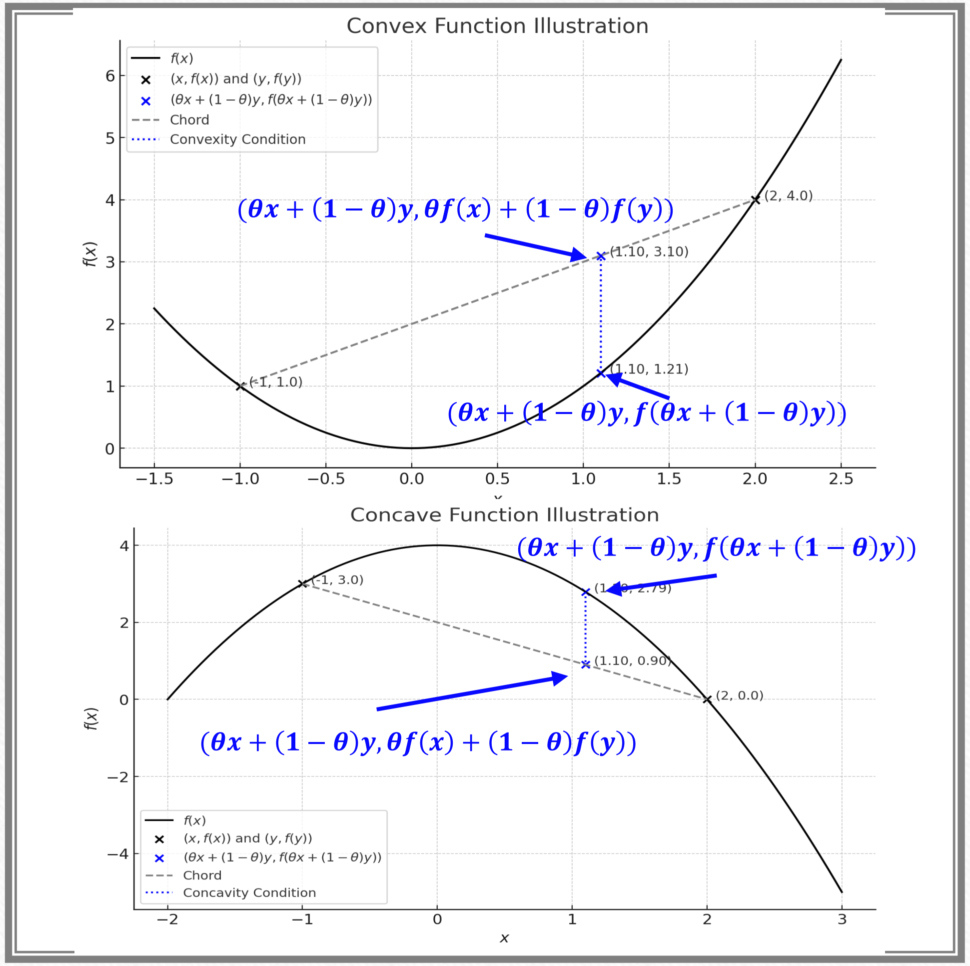
2025.01.29

Reminders

- Course website: https://cseweb.ucsd.edu//classes/wi25/cse203B-a/
- HW3 due on Feb 6 (Thursday) 11:50 PM
- ≤4 Members for Exercises; Individually for Assignments
- Late policy for homework: Piazza note @19

Contents

- Recap Convex Functions
  * Definitions
  * 1^{st} & 2^{nd} Order Conditions
  * Convexity Preserving Operations
- Conjugate Functions
  * Definitions
  * Convexity Proof
  * Steps to find the conjugate functions
  * Examples



Convex Functions

- A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is **convex** if:

Its domain ($\boldsymbol{dom\ f}$) is convex. [A convex domain means that for any two points x, y in the domain, the line segment joining x and y lies entirely within the domain.]

The following inequality holds for all $\boldsymbol{x, y\ \epsilon}$ $\boldsymbol{dom\ f\ and\ \theta\ \in [0, 1]}$:

$$\boldsymbol{f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)}$$

- Concave function:

for all $\boldsymbol{x, y\ \in dom\ f\ and\ \theta\ \in [0, 1]}$

$$\boldsymbol{f(\theta x + (1-\theta)y) \geq \theta f(x) + (1-\theta) f(y)}$$

- A function f is concave if $-f$ is convex

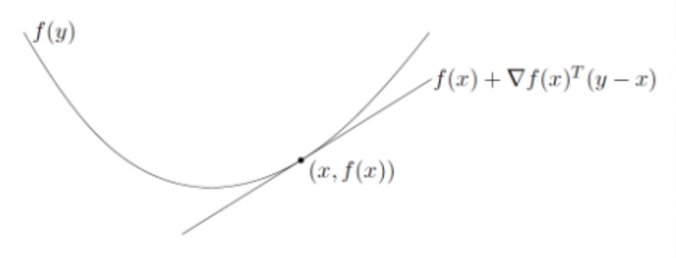
Reduce Multivariate function to 1-dimensional $g(t)$

A function $f: R^n \to R$ is convex if and only if the function $g: R \to R$,

$$g(t) = f(x + tv), dom\ g = \{t | x + tv \in dom\ f\}$$

is a convex on its domain for $\forall x \in dom\ f, v \in R^n$.

Convex Functions ---- First Order Conditions



Suppose f is differentiable, i.e. $dom\ f$ is open and ∇f exists for all $x \in dom\ f$, then f is convex if and only if $dom\ f$ is convex, and ,

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \ \ \forall x, y \in dom\ f$$

$\nabla f(x)$: *The gradient of f at x*
$\nabla f(x)^T (y - x)$: The first order of taylor approximation of $f(y)$ around x

Concave function:

$$f(y) \leq f(x) + \nabla f(x)^T (y - x) \ \ \forall x, y \in dom\ f$$

Convex Functions ---- Second Order Conditions

Suppose f is twice differentiable, i.e, $dom\ f$ is open and the Hessian $\nabla^2 f$ exists for all $x \in dom\ f$, then f is convex if and only if $dom\ f$ is convex, and

$$\nabla^2 f(x) \succcurlyeq 0, \forall x \in dom\ f$$

Hessian Matrix is positive semidefinite

- All eigenvalues ≥ 0
- $x^T \nabla^2 f(x) x \geq 0, \forall x$

Concave function: Hessian matrix is negative semidefinite

$$\nabla^2 f(x) \preccurlyeq 0, \forall x \in dom\ f$$

- All eigenvalues ≤ 0
- $x^T \nabla^2 f(x) x \leq 0, \forall x$

Linear Algebra ---- whether a matrix is positive definite or negative definite

positive definite

Necessary Condition: Diagonal Elements all > 0

Leading Principal Minors Determinants should be all > 0

$$A = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

negative definite

Necessary Condition: Diagonal Elements all < 0

Odd Term Leading Principal Minors Determinants should be all < 0
Even Term Leading Principal Minors Determinants should be all > 0

$$A = \begin{bmatrix} -4 & -2 & -1 \\ -2 & -3 & -1 \\ -1 & -1 & -2 \end{bmatrix}$$

Convex Functions ---- Second Order Conditions

Prove that the following function is convex using the **Second Order Condition**:

$$f(x,y) = \frac{1}{xy} \quad dom\, f = \{(x,y) \in R^2_{++}\}$$

Convex Functions ---- Operations that preserve convexity

- **Non-negative Weighted Sums**

  if $f_1(x)$ and $f_2(x)$ are convex functions, their weighted sum is also convex if the weights are non-negative

$$f(x) = w_1 f_1(x) + w_2 f_2(x), w_1, w_2 \geq 0$$

- **Composition of Functions**

  $f(x) = h(g(x))$, where $g(x)$ is the inner function and $h(y)$ is the outer function.

  Conditions for Convexity:

  1.If $h(y)$ is convex and non-decreasing, and $g(x)$ is convex, then $f(x)$ is convex.

  2.If $h(y)$ is convex and non-decreasing, and $g(x)$ is concave, then $f(x)$ is concave.

  Conditions for Concavity:

  1.If $h(y)$ is concave and non-decreasing, and $g(x)$ is concave, then $f(x)$ is concave.

  2.If $h(y)$ is concave and non-decreasing, and $g(x)$ is convex, then $f(x)$ is convex.

- **Pointwise Maximum and Supremum**

$$f(x) = max\{f_1(x), f_2(x)\}$$

$$f(x) = sup_y g(x,y)\ [g(x,y)\ is\ convex\ in\ x\ for\ each\ fixed\ y]$$

- **Partial Minimization**

$$f(x) = inf_y g(x,y)$$

Convex Functions ---- Proof of the convexity or concavity

- By definition
- Reduce multivariate function to 1-dimensional $g(t)$
- First order conditions
- Second order conditions
- Operations that preserve convexity

…

Conjugate Functions

Given a function $f: \mathbb{R}^n \to \mathbb{R}$ that is not necessarily convex, the conjugate of f is defined as

$$f^*(y) = sup_x(y^T x - f(x))$$

Definition: $f^*(y)$ represents the supremum of the linear function $y^T x - f(x)$ and over all x in the domain of f.

- $f^*(y)$ is a convex function, even if f may not be convex.
- For any point y in domain of f^*, $f^*(y)$ is finite and bounded.

Conjugate Functions ---- Convexity Proof

The conjugate of f is defined as:

$$f^*(y) = sup_x\left(y^T x - f(x)\right), where\ x \in \mathbb{R}^n$$

- $y^T x - f(x)$ is an affine function of y, meaning it is linear plus a constant shift.
- The supremum is taken over all x, which makes $f^*(y)$ the maximum value of this affine function.

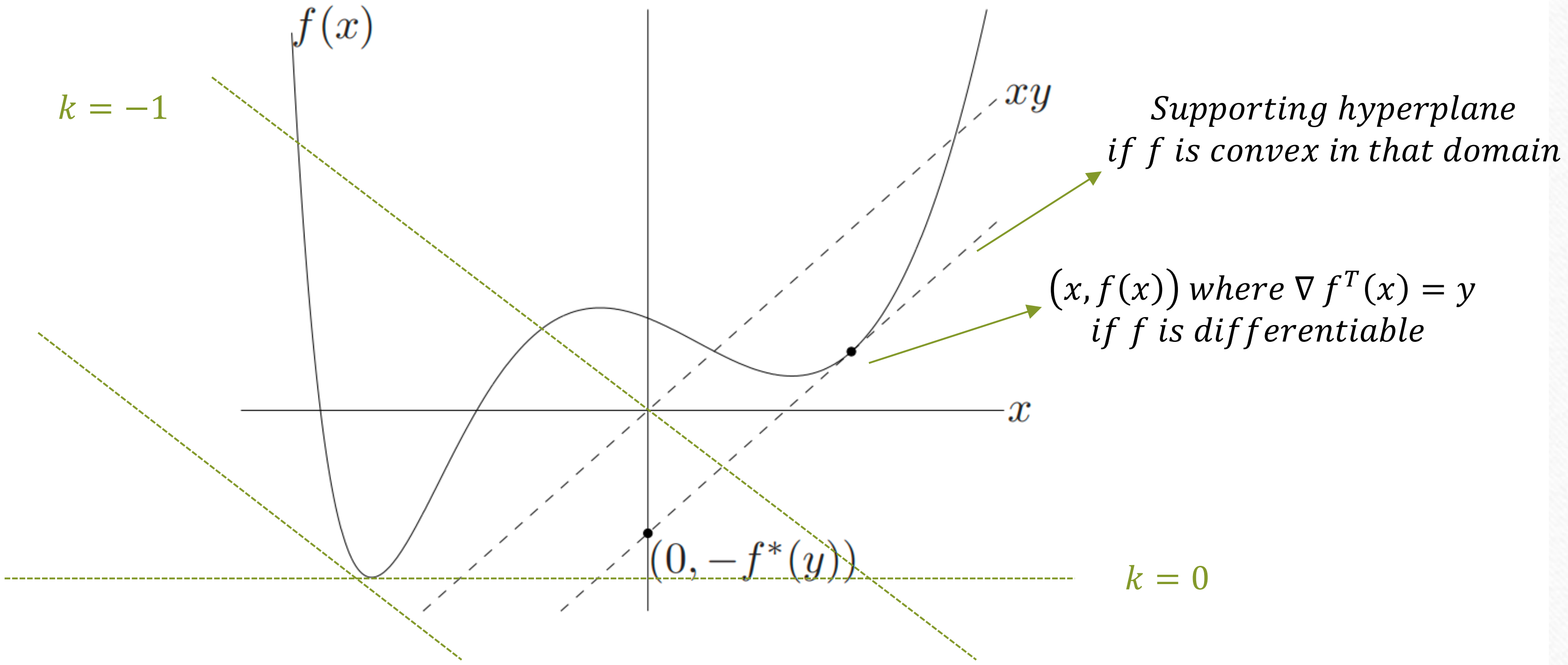


Figure 3.8 A function $f : \mathbf{R} \to \mathbf{R}$, and a value $y \in \mathbf{R}$. The conjugate function $f^*(y)$ is the maximum gap between the linear function yx and $f(x)$, as shown by the dashed line in the figure. If f is differentiable, this occurs at a point x where $f'(x) = y$.

Conjugate Functions ---- Steps to Find the Conjugate Function

The conjugate of f is defined as:

$$f^*(y) = sup_x\left(y^T x - f(x)\right), where\ x \in \mathbb{R}^n$$

(1) Break Down into Cases

- Identify scenarios where the supremum is unbounded:
  If $y^T x - f(x)$ grows indefinitely, $f^*(y)$ is not finite for such y. These values of y cannot belong to $dom\ f^*$, making this scenario infeasible.
- Determine cases where $f^*(y)$ is a piecewise function:
  For feasible y, where $f^*(y)$ is finite, it may involve separate cases based on the structure of $f(x)$, requiring piecewise expressions for $f^*(y)$.

(2) Find the Supremum for Feasible Cases

- In regions where $f^*(y)$ is finite, the supremum condition is met when the gradient of $y^T x - f(x)$ with respect to x equals zero:

$$\nabla_x\left(y^T x - f(x)\right)=0$$

(3) Construct final conjugate function

- Combine the results for all feasible regions (including any piecewise definitions) to form the complete expression for $f^*(y)$.

Conjugate Functions ---- Solving Conjugate Function Problems

$$f(x) = ax + b$$

$$f^*(y) = sup_x(yx - ax\ - b), where\ x \in \mathbb{R}$$

Case 1: $y < a$

$$f^*(y) = \infty$$

Case 2: $y = a$

$$f^*(y) = -b$$

Case 3: $y > a$

$$f^*(y) = \infty$$

Conjugate Functions ---- Solving Conjugate Function Problems

$$f(x) = e^x$$
$$f^*(y) = sup_x(yx - e^x), where\ x \in \mathbb{R}$$

Case 1: $y < 0$

$$f^*(y) = \infty$$

Case 2: $y = 0$

$$f^*(y) = 0$$

Case 3: $y > 0$

$$g(x) = yx - e^x$$
$$g'(x) = y - e^x$$
$$g'(x) = 0 \rightarrow y = e^x \rightarrow \mathrm{x} = ln(y)$$
$$f^*(y) = yln(y) - y$$

Conjugate Functions ---- Solving Conjugate Function Problems

$$f(x) = -logx$$
$$f^*(y) = sup_x(yx + logx), where\ x \in \mathbb{R}_{++}$$

Case 1: $y \geq 0$

$$f^*(y) = \infty$$

Case 2: $y < 0$

$$g(x) = yx + logx$$
$$g'(x) = y + \frac{1}{x}$$
$$g'(x) = 0 \rightarrow y = \frac{-1}{x} \rightarrow \text{x} = \frac{-1}{y}$$
$$f^*(y) = -1 + log(\frac{-1}{y})$$
$$\Rightarrow f^*(y) = -1 - log(-y)$$

Conjugate Functions ---- Solving Conjugate Function Problems

$$f(x) = \frac{1}{2}x^T Qx, x \in R^n, Q \in S_{++}^n$$