Policy of the Exam: Here is the policy of the exam:
1. This is an open-book take-home exam. Internet search is permitted. However, you are required to work by yourself. Consultation or discussion with any other parties is not allowed.
2. You are not required to typeset your solutions. We do expect your writing to be legible and your final answers clearly indicated. Also, please allow sufficient time to upload your solutions.
3. You are allowed to check your answers with programs in Matlab, CVX, Mathematica, Maple, NumPy, etc. Be aware that these programs may not produce the intermediate steps needed to receive credit.
4. If something is unclear, state the assumptions that seem most natural to you and proceed under those assumptions. Out of fairness, we will not be answering questions about the technical content of the exam on Piazza or by email. The solution will then be graded based on the reasonable assumptions made.

Part I: True or False: Explain your answer with one sentence (30 pts)

I.1 (convex set): Set \( \{x| \log x \leq 1, x \in \mathbb{R}^+\} \) is convex.

T/F:

\textbf{True}: Assume that the log is \( e \)-based, then \( \log x \leq 1 \Rightarrow x \in (0,e] \). So, \( \forall x_1, x_2 \in (0,e], \alpha x_1 + (1-\alpha)x_2 \in (0,e] \) such that \( \alpha \in [0,1] \). Hence, the set is convex.

I.2 (dual cone): Given cone \( K = \{x|Ax = 0, A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^n\} \), its dual cone is \( K^* = \{A^Ty| y \in \mathbb{R}^m\} \).

T/F:

\textbf{False}: Given cone \( K = \{x|Ax = 0, A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^n\} \), its dual cone is \( K^* = \{A^Ty| y \in \mathbb{R}^m\} \).

I.3 (dual cone): The dual cone of \( K = \{M| M \in S^+\} \) is itself.

T/F:
True: The dual cone of $K = \{ M \mid M \in S^n_+ \}$ is $K^* = \{ M \mid M \in S^n_+ \}$.

I.4 (Convex Function): Given two convex functions $f(x)$ and $g(x)$, where $x \in R^n$, its product $f(x) \times g(x)$ is a convex function.

T/F:

False: Consider the counter example $f(x) = x(x - 1)$ and $g(x) = x - 2$. $f(x)$ and $g(x)$ are convex but $f(x) \times g(x)$ is not convex.

I.5 (Convex Function): Function $g(x) = \min_y f(x, y)$ is a convex function, if $f(x, y)$ is a convex function in terms of variable $x \in R$ and in terms of $y \in R$ but not in terms of $(x, y) \in R^2$.

T/F:

False: Consider the counter example $f(x, y) = -xy$. $f(x, y)$ is convex in terms of $x$ and in terms of $y$ but not in terms of $(x, y)$. But $g(x) = \min_y f(x, y)$ is not convex.

I.6 (Conjugate Function): Given a convex function $f(x)$ and its conjugate function $f^*(y)$, then we have $f(x) + f^*(y) \geq 0$.

T/F:

False: Refers to the textbook’s example 3.21 in section 3.3.1, on pp. 91.

The affine function $f : R \rightarrow R$ such $f(x) = 5x + 10$’s conjugate function

$f^*(y) = \begin{cases} -10 & \text{if } y = 1 \\ \infty & \text{otherwise} \end{cases}$

Then $f(-2) + f^*(5) = 0 - 10 = -10 < 0$.

Note: The domain of the function $f$ must be in the form $R^n$, but cannot be restricted to a specific interval such as a closed interval. That is because the concept of conjugate function is defined on function in the form $R^n \rightarrow R$.

I.7 (Convex Function): Given two convex functions $f(x)$, $g(x)$, where $x \in R^n$, then function $p(y) = \sup_{x} (y \times g(x) - f(x))$ is a concave function, where $y \in R$.

T/F:

False. Consider the counterexample where $g(x) = x$, $f(x) = x^2$. Then $p(y) = \sup_{x} (yx - x^2)$. Note that for any fixed $y_0$, we can treat the function

$h(x, y_0) = y_0 x - x^2$ as a univariate function with respect to $x$. From the first order condition, we know that $\nabla_x h(x, y_0) = y_0 - 2x = 0 \Rightarrow x = \frac{y_0}{2}$ is a potential maximizer, which we can confirm with the second order condition: $\nabla^2_x h(x, y_0) = -2 < 0$. Therefore, $p(y) = \sup_{x} (yx - x^2) = y(\frac{y}{2}) - (\frac{y}{2})^2 = \frac{1}{4}y^2$, which is convex but not linear, thus not concave.

I.8 (Geometric Programming): For the constraint $x_1^{0.1} x_2^{-2} - x_1^{0.2} x_2^{-0.1} \leq 1$, where $x_1, x_2 \in R_+^+$, the feasible set may not be convex. However, after geometric programming transformation, i.e. $x_1 = e^{y_1}$, $x_2 = e^{y_2}$, we convert the feasible set
into a convex set.

T/F:

**False.** Intuitively, \( X = \{(x_1, x_2) \in \mathbb{R}_+^2 | x_1^{0.1}x_2^{-2} - x_1^{0.2}x_2^{-0.1} \leq 1 \} \) is not convex, due to the minus sign. However, it is hard to come up with an example. Fortunately, whether \( X \) is convex is not of this question's interest. What is of interest is if \( Y = \{(y_1, y_2) | e^{0.1y_1} - 2y_2 - e^{0.2y_1} - 0.1y_2 \leq 1 \} \) is convex. It turns out not. Borrowing the example from a student submission, \( (a_1, a_2) = (10, -0.5), (b_1, b_2) = (-10, -0.5) \in Y \). However, consider the convex combination of \( (a_1, a_2) \) and \( (b_1, b_2) \) when \( \lambda = 0.5 \in [0, 1] \):

\[
(c_1, c_2) = 0.5(a_1, a_2) + 0.5(b_1, b_2) = (0, -0.5).
\]

Since \( e^{0.1c_1} - 2c_2 - e^{0.2c_1} - 0.1c_2 = e^1 - e^{0.05} \approx 1.6670 > 1, (c_1, c_2) \notin Y \). Thus, \( Y \) is not convex.

I.9 (Duality): In Example 5.9 (textbook page 257), if the linear program formulation (5.68) has a bounded solution, then formulation (5.69) and formulation (5.71) derive the same solution (i.e. value of the objective function) as formulation (5.68).

T/F:

**True.** At the end of the textbook’s section 5.7.3’s example 5.9, on page 258, it suggests "The problems (5.69) and (5.71) are closely related, in fact, equivalent”.

I.10 (Min Max Problem): Given an arbitrary but bounded function \( f(x, y) \), where \( x, y \in \mathbb{R}^n \), we can claim the inequality \( \max_y f(\tilde{x}, y) \geq \min_x f(x, \tilde{y}) \) for arbitrary \( \tilde{x}, \tilde{y} \in \mathbb{R}^n \).

T/F:

**True.** \( \max_y f(\tilde{x}, y) \geq f(\tilde{x}, \tilde{y}) \geq \min_x f(x, \tilde{y}) \).

Part II: Problem Solving: Show your process

Problem 1. Conjugate Function: Find the conjugate function of the following functions. (20 pts)

1.1 \( f(x) = x^2 - 3x + 2 \), where \( x \in R \).

\[
f^*(y) = \sup_x (yx - f(x)) = \sup_x yx - x^2 + 3x - 2
\]

Consider \( g(x, y) = yx - f(x) = yx - x^2 + 3x - 2 = -x^2 + (y + 3)x - 2 \)

Note that this is bounded over all \( R \), since it is a downward-facing parabola - i.e., for any value of \( y \), this can never be \( +\infty \). Also note that it will always have a maximum. Thus, our domain for the conjugate function is \( R \).

\[
\nabla_x g(x, y) = -2x + (y + 3) = 0 \Rightarrow x = \frac{y + 3}{2}
\]
Since $\nabla_x g(x, y) = -2 < 0$, the function $f^*(y)$ achieves its global maximum at $x = \frac{y+3}{2}$ with value:

$$-\left(\frac{y+3}{2}\right)^2 + \frac{(y+3)^2}{2} - 2 = \frac{(y+3)^2}{4} - 2$$

Therefore, the conjugate function for $f(x)$ is

$$f^*(y) = \frac{(y+3)^2}{4} - 2 = \frac{y^2 + 6y + 1}{4}, \quad y \in \mathbb{R}$$

1.2 $f(p, q) = p \times \log(p/q)$, where $p, q \in \mathbb{R}^+$. 

Consider $y = \left(\begin{array}{c} u \\ v \end{array}\right)$, then

$$f^*(u, v) = \sup_{p, q} \left( up + vq - f(p, q) \right) = \sup_{p, q} \left( up + vq - p \log \frac{p}{q} \right)$$

Let $g(p, q) = up + vq - p \log \frac{p}{q}$.

**Case 1:** If $v \geq 0$, let $q \to +\infty$, $g(p, q) \to +\infty$. So $g(p, q)$ is unbounded above for $v \geq 0$. That is, we can always find a value of $p \& q$ such that the expression will become unbounded (or tend to $+\infty$). Therefore, this cannot be in the domain of the conjugate function.

**Case 2:** For $v < 0$,

$$\nabla_p g(p, q) = u - \log \frac{p}{q} - 1$$

$$\nabla_q g(p, q) = v + \frac{p}{q}$$

In order to get the extremum, $\nabla_p g(p, q) = 0$ and $\nabla_q g(p, q) = 0$. This gives us

$$u = \log \frac{p}{q} + 1, \quad v = -\frac{p}{q}$$

It can be checked using the Hessian whether this will be maxima or minima, for confirmation.

**Case 2.1:** If $u = \log(-v) + 1$, then $g(p, q) = 0$ and thus $f^*(u, v) = 0$.

**Case 2.2:** If $u < \log(-v) + 1$, then $g(p, q) < p \log(-v) + p + vq - p \log \frac{p}{q} = q \left[ \frac{p}{q} \log(-v) + \frac{p}{q} + v - \frac{p}{q} \log \frac{p}{q} \right]$. Let $\frac{p}{q} = x$. Then, we can write the above expression in terms of $x$. Let that be
Thus, we have $x > 0$ and

$$h(x) = x \log(-v) + x + v - x \log x$$

And

$$h'(x) = \log(-v) - \log x; \quad h''(x) = -\frac{1}{x} < 0$$

So $h'(x) = 0$ gives us the maximum of $h(x) = h(-v) = 0$.

This implies that, in this case, $g(p, q) < 0$.

So, if $\frac{p}{q} = -v$, then $g(p, q) = q[-vu + v + v \log(-v)] \to 0$ where $q \to 0$.

Thus $f^*(u, v) = 0$.

**Case 2.3:** If $u > \log(-v) + 1$,

then in this case as well $f^*(u, v)$ can be made $+\infty$ by some particular choice of $p$ & $q$.

Let $\frac{p}{q} = -v$, then $g(p, q) = q[-vu + v + v \log(-v)]$.

Now, $-vu + v + v \log(-v) > -v \log(-v) - v + v \log(-v) = 0$.

So when $q \to +\infty$, $g(p, q) \to +\infty$. Thus $f^*(u, v) = +\infty$.

Therefore, we can see that $f^*(u, v)$ is bounded only in one particular case and its closed form solution can be written as:

$$f^*(y) = f^*(u, v) = \begin{cases} 0 & u \leq \log(-v) + 1, \quad v < 0 \\ +\infty & \text{otherwise} \end{cases}$$

Problem 2. Linear Programming. (20 pts)

Given

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 \\ 2 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3 & -1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & -1 & -1 & 0 \end{bmatrix},$$

$$b^T = [1 \ 5 \ 2 \ 3 \ 2],$$

$$c^T = [2 \ 1 \ 2 \ 3 \ 1 \ 2 \ 1],$$

and $n = 7$, perform steps A, B, and C for problems 2.1, 2.2, 2.3, and 2.4.

A. Solve the following linear programming problems twice, once using the primal formulation and once using the dual formulation.

B. Check the feasibility of the solution. If a solution is not found, explain why a solution is not available and suggest how to mitigate the issue if you are the project leader. (For this exam, there is no need to solve the mitigated problem unless you feel the explanation is not convincing enough.)
C. Compare the primal and dual solutions. If the primal and dual formulation solutions are different, explain the difference.

2.1. minimize $f_0(x) = c^T x$ subject to $Ax \leq b$, $x \in R^n$.

Rubrics:
- 5 pts: Correct
- -1 pt: Incorrect/Missing primal problem solution
- -1 pt: Incorrect/Missing dual problem solution
- -0.5 pt: Minor mistake

Solution:
Primal formulation:
$$\min c^T x$$
$$\text{s.t } Ax \leq b$$

Dual formulation:
$$\max g(\lambda)$$
$$\text{s.t } A^T \lambda + c = 0$$
$$\lambda \geq 0$$

where,
$$g(\lambda) = \begin{cases} -b^T \lambda & \text{if } A^T \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases}$$

A. The minimum value of the primal problem is $-\infty$, which can be achieved by one feasible point $x = [0 \ t \ 0 \ 0 \ 0 \ 0 \ 0] \text{ where } t \leq 0$.
Then, $\lim_{t \to -\infty} c^T x = \lim_{t \to -\infty} t = -\infty$.

To check the feasibility of the dual problem we consider $[A^T - c]$, since $A^T \lambda + c = 0 \Rightarrow A^T \lambda = -c$.
Reducing $[A^T - c]$ to RREF, we get
$$[A^T - c] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From the first five equations we get $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$. But the sixth equation is not satisfied. Therefore, the dual problem is infeasible and
\( g(\lambda) = -\infty \)

B. The primal problem is feasible but its value is unbounded below. The dual problem is infeasible and its value is also unbounded below.

C. The primal and dual optimal values are the same, which is \(-\infty\). They are the same since the primal problem is strictly feasible.

2.2. \text{minimize } f_0(x) = c^T x \text{ subject to } Ax = b, \ x \in \mathbb{R}^n.

Rubrics:
- 5 pts: Correct
- -1 pt: Incorrect/Missing primal problem solution
- -1 pt: Incorrect/Missing dual problem solution
- -0.5 pt: Minor mistake

Solution:
Primal formulation:
\[
\begin{align*}
\min c^T x \\
s.t \ Ax &= b
\end{align*}
\]

Dual formulation:
\[
\begin{align*}
\max g(\nu) \\
s.t \ A^T \nu + c &= 0
\end{align*}
\]
where,
\[
g(\nu) = \begin{cases} 
-\nu^T b & \text{if } A^T \nu + c = 0 \\
-\infty & \text{otherwise}
\end{cases}
\]

A. To find the solution to the primal problem, we find the RREF of \([A|b]\)
\[
[A|b] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & -\frac{2}{3} & -\frac{4}{3} & -\frac{11}{3} \\
0 & 1 & 0 & 0 & 0 & \frac{2}{3} & 2 & -\frac{14}{3} \\
0 & 0 & 1 & 0 & 0 & \frac{3}{7} & \frac{4}{7} & -\frac{2}{7} \\
0 & 0 & 0 & 1 & 0 & -\frac{1}{3} & -\frac{5}{3} & \frac{22}{3} \\
0 & 0 & 0 & 0 & 1 & 1 & \frac{4}{3} & -3
\end{bmatrix}
\]

Let \(x_6 = t\) and \(x_7 = 0\), for any \(t \leq 0\), then we see that,
\[
x = \begin{bmatrix}
\frac{2t}{3} + \frac{11}{3}, & -\frac{2t}{3} - \frac{14}{3}, & -\frac{t}{3} - \frac{7}{3}, & \frac{t}{4} + \frac{22}{3}, & -\frac{t}{4} - 3, & t, & 0
\end{bmatrix}
\]
satisfies \(Ax = b\). This form results in \(c^T x = 2t + 17\), and observe that, \(\lim_{t \to -\infty} c^T x = \lim_{t \to -\infty} t = -\infty\).
Therefore, the primal problem is again feasible but unbounded.
The dual problem is infeasible and \( g(\nu) = -\infty \) (can be shown as in 2.1)

B. The primal problem is feasible but its value is unbounded below. The dual problem is infeasible and its value is also unbounded below.

C. The primal and dual optimal values are the same, which is \(-\infty\). They are the same since the primal problem is strictly feasible.

2.3. minimize \( f_0(x) = c^T x \) subject to \( Ax \leq b \), \( x \in \mathbb{R}^n_+ \).

Rubrics:
- 5 pts: Correct
- -1 pt: Incorrect/Missing primal problem solution
- -1 pt: Incorrect/Missing dual problem solution
- -0.5 pt: Minor mistake

Solution:
Primal formulation:
\[
\begin{align*}
\text{min} & \quad c^T x \\
\text{s.t} & \quad Ax \leq b \\
& \quad x \geq 0
\end{align*}
\]

Dual formulation:
\[
\begin{align*}
\text{max} & \quad g(\lambda) \\
\text{s.t} & \quad A^\top \lambda + c \geq 0 \\
& \quad \lambda \geq 0
\end{align*}
\]

where,
\[
g(\lambda) = \begin{cases} 
-\lambda^\top b & \text{if } A^\top \lambda + c \geq 0 \\
-\infty & \text{otherwise}
\end{cases}
\]

A. The primal’s optimal value is 0.
Since, \( c \geq 0 \), it must be true that \( \forall x \in \mathbb{R}^n_+ \), \( c^\top x \geq 0 \).
And since, \( c^\top x = 0 \) at the feasible point 0, the minimum possible value of \( c^\top x \) is 0.

The dual’s optimal value is also 0.
Since, \( b \geq 0 \), it must be true that \( \forall \lambda \in \mathbb{R}^n_+ \), \( \lambda^\top b \geq 0 \), i.e. \( -\lambda^\top b \leq 0 \)
And since, \( -\lambda^\top b = 0 \) at the feasible point 0, the maximum possible value of \( -\lambda^\top b \) is 0.
B. Both the primal and dual problems are feasible.

C. The primal and dual optimal values are the same, which is 0.

2.4. minimize $f_0(x) = c^T x$ subject to $Ax = b, x \in R^+_n$.

Rubrics:
• 5 pts: Correct
• -1 pt: Incorrect/Missing primal problem solution
• -1 pt: Incorrect/Missing dual problem solution
• -1 pt: Incorrect/Missing suggestions on mitigating infeasibility.
• -0.5 pt: Minor mistake

Solution:
Primal formulation:
\[
\begin{align*}
\min & \quad c^T x \\
\text{s.t} \quad & Ax = b \\
& x \geq 0
\end{align*}
\]

Dual formulation:
\[
\begin{align*}
\max & \quad g(\nu) \\
\text{s.t} \quad & A^T \nu + c \geq 0
\end{align*}
\]

where,
\[
g(\nu) = \begin{cases} 
-\nu^T b & \text{if } A^T \nu + c \geq 0 \\
-\infty & \text{otherwise}
\end{cases}
\]

A. The primal problem is infeasible. Consider the RREF of \([A|b]\) from 2.2. Since $x \geq 0$, equality constraints 2, 3, and 5 will never be satisfied.

The dual problem is feasible but unbounded above. Consider $\nu = [t \quad -t \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$, for any $t \geq 0$. This form satisfies $A^T \nu + c \geq 0$. Therefore, the dual is feasible. But, $\lim_{t \to \infty} -\nu^T b = \lim_{t \to \infty} 4t = \infty$.

B. The primal problem is not feasible. We can make it feasible by relaxing the constraints 2, 3 and 5 (change them to inequalities with $\leq$).

C. Since the primal problem is infeasible, its solution can be assumed as $\infty$ which is the same as the dual’s solution.

Problem 3. KKT Conditions. (30 pts)
Derive examples to demonstrate either analytically or numerically on problems the KKT conditions as posted in slide #29 of lecture note for Chapter 5. Perform steps A, B, and C on problems 3.1 and 3.2.

Hint: Start with a simple but nontrivial case. For example, use a low dimensional space, i.e. $x \in R$ or $x \in R^2$. Fixing the objective function, e.g., $f_0(x) = x^2$, $x \in R$, would be a good starting point.

A. State the optimization problem including the objective functions and constraint(s). Prove that the problem is convex.
B. Describe the KKT conditions of the problem formulation in step A.
C. Describe your assignment on the variables and Lagrange multipliers. Show that the assignment fits the specific requirements of the problem.

3.1. Demonstrate an example with an assignment on the variables and Lagrange multipliers that satisfy all the other conditions but not the primal constraints.

Solution:

A. Consider the standard form constrained optimization problem:

$$\begin{align*}
\text{minimize} & \quad f_0(x) = x^2 \\
\text{subject to} & \quad f_1(x) = x + 3 \leq 0, \\
& \quad h_1(x) = x + 8 = 0
\end{align*}$$

To prove convexity, it is sufficient to show that $f_0, f_1$ are convex and $h_1$ is affine. We can show that $f_0, f_1$ are convex because $f_0''(x) = 2 \geq 0$ and $f_1''(x) = 0 \geq 0$ and $h_1$ is clearly affine, so our problem is convex.

B. Applying the KKT conditions to our specific problem we have:

1. $x + 3 \leq 0$ and $x + 8 = 0$ (primal feasibility)
2. $\lambda \geq 0$ (dual feasibility)
3. $\lambda(x + 3) = 0$ (complementary slackness)
4. $2x + \lambda + \nu = 0$ (stationarity)

C. One example can be $x = 0, \lambda = 0, \nu = 0$

3.2. Demonstrate an example with an assignment on the variables and Lagrange multipliers that satisfy all the other conditions but not the complementary slackness constraints.

Solution:

(A)-(B): Same as 3.1

(C) One example can be $x = -8, \lambda = 6, \nu = 10$