Week 6 Discussion
CSE203B Winter24

Tatsuki Koga 2024/2/16
Agenda

Linear Programming
Duality
How to Solve Convex Optimization Problems
   Introduction to CVXPY
Linear Programming (LP)
Definition & General Form

• Objective function and constraint functions are all affine

• General Form:
  \[
  \min_x c^T x + d \\
  \text{s.t.} \quad Gx \leq h \quad A x = b
  \]

• LP are convex optimization problems
Linear Programming (LP)
Standard Form LP & Inequality Form LP

**Standard Form LP**

\[
\begin{align*}
\text{min} \quad & c^T x \\
\text{s.t.} \quad & Ax = b \\
& x \geq 0
\end{align*}
\]

e.g., HW4 II.1.4

**Inequality Form LP**

\[
\begin{align*}
\text{min} \quad & c^T x \\
\text{s.t.} \quad & Ax \leq b
\end{align*}
\]

e.g., HW4 II.1.1
Linear Programming (LP)
How to find the optimal value

Figure out which is the case for the LP

1. **No feasible solutions** -> The optimal value is $\infty$

2. **Unbounded solutions** -> The optimal value is $-\infty$

3. **Bounded solutions**
Linear Programming (LP)
How to find the optimal value: LP only with equality constraints

\[
\min_{x} c^T x \quad \text{s.t.} \quad Ax = b
\]

1. **No feasible solutions** -> The optimal value is \( \infty \)
2. **Unbounded solutions** -> The optimal value is \( -\infty \)
3. **Bounded solutions**
Linear Programming (LP)
How to find the optimal value: LP only with equality constraints

\[
\begin{align*}
\min_{x} & \quad c^T x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}^T x = x_1 + 2x_2 \\
\text{s.t.} & \quad Ax = b
\end{align*}
\]
Linear Programming (LP)
How to find the optimal value: LP only with equality constraints

\[
\begin{align*}
\min_{x} & \quad x_1 + 2x_2 \\
\text{s.t.} & \quad Ax = b
\end{align*}
\]

1. No feasible solutions
Linear Programming (LP)

How to find the optimal value: LP only with equality constraints

\[
\begin{align*}
\min_x & \quad x_1 + 2x_2 \\
\text{s.t.} & \quad Ax = b
\end{align*}
\]

2. Unbounded solutions

\[\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0\]

\[C^T x \to -\infty \quad \text{as} \quad t \to -\infty\]
Linear Programming (LP)

How to find the optimal value: LP only with equality constraints

\[
\begin{align*}
\min_x & \quad x_1 + 2x_2 \\
\text{s.t.} & \quad Ax = b
\end{align*}
\]

3. Bounded solutions

\[
(1, 2) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2
\]

\[
\Rightarrow x_1 + 2x_2 = -2
\]

\[
\Rightarrow \min x_1 + 2x_2 = -2
\]
Linear Programming (LP)
How to find the optimal value: LP only with equality constraints

\[
\begin{align*}
\min_x & \quad x_1 + 2x_2 \\
\text{s.t.} & \quad Ax = b
\end{align*}
\]

3. Bounded solutions

\[
\begin{pmatrix}
1 & -1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= \begin{pmatrix}
0 \\
1
\end{pmatrix}
\Rightarrow \text{domain is } \{(1,1)\}
\Rightarrow \min x_1 + 2x_2 = 3
Linear Programming (LP)

How to find the optimal value: LP only with equality constraints

\[
\min_x c^T x \quad \text{s.t.} \quad Ax = b
\]

1. **No feasible solutions**
   \[ \iff b \not\in \mathbb{R}(A) \]
   The optimal value is \(\infty\)

2. **Unbounded solutions**
   \[ \iff b \in \mathbb{R}(A), \ c \not\in \mathbb{R}(A^T) \]
   The optimal value is \(-\infty\)

3. **Bounded solutions**
   \[ \iff b \in \mathbb{R}(A), \ c \in \mathbb{R}(A^T) \ (\text{i.e., } c = A^T \lambda \text{ for some } \lambda) \]
   The optimal value is \(c^T x = (A^T \lambda)^T x = \lambda^T (Ax) = \lambda^T b\)
Linear Programming (LP)
How to find the optimal value: LP only with inequality constraints

\[ \min_x c^T x \quad \text{s.t.} \quad Ax \leq b \]

1. **No feasible solutions** -> The optimal value is \( \infty \)
2. **Unbounded solutions** -> The optimal value is \(-\infty\)
3. **Bounded solutions**
Linear Programming (LP)
How to find the optimal value: LP only with inequality constraints

\[
\begin{align*}
\min_x & \quad c^T x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}^T x = x_1 + 2x_2 \\
\text{s.t.} & \quad Ax \leq b
\end{align*}
\]
Linear Programming (LP)
How to find the optimal value: LP only with inequality constraints

\[
\begin{align*}
\min_{x} & \quad x_1 + 2x_2 \\
\text{s.t.} \quad & Ax \leq b
\end{align*}
\]

1. No feasible solutions
Linear Programming (LP)
How to find the optimal value: LP only with inequality constraints

\[
\min_x \quad x_1 + 2x_2 \\
\text{s.t.} \quad Ax \leq b
\]

2. Unbounded solutions
Linear Programming (LP)
How to find the optimal value: LP only with inequality constraints

\[
\begin{align*}
\min_x & \quad x_1 + 2x_2 \\
\text{s.t.} & \quad Ax \leq b
\end{align*}
\]

3. Bounded solutions
Linear Programming (LP)
How to find the optimal value: LP only with inequality constraints

\[
\min_x \quad x_1 + 2x_2 \\
\text{s.t.} \quad Ax \leq b
\]

3. Bounded solutions
Linear Programming (LP)

Hint on HW4 II.1

Ask the following questions:

1. Are there feasible solutions?  
   If not, how can we modify the LP to have ones?

2. Is the problem bounded?  
   If not, how can you achieve $-\infty$, and how can you prevent it?

When there are feasible solutions and the problem is bounded, you can find the optimal value:)
Duality

The Lagrangian

- Consider the general primal problem:

\[
\begin{align*}
\min_{x} & \quad f_0(x) \\
\text{s.t.} & \quad f_i(x) \leq 0 \quad i = 1, \ldots, m \\
& \quad h_i(x) = 0 \quad i = 1, \ldots, p
\end{align*}
\]

- The Lagrangian is:

\[
L(x, \lambda, v) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} v_i h_i(x)
\]

\((\lambda \in \mathbb{R}_+^m, v \in \mathbb{R}^p: \text{Lagrange multipliers})\)
Duality

The dual function

• The Lagrange dual function is:

\[
g(\lambda, v) = \inf_{x} L(x, \lambda, v) = \inf_{x} \left( f_{0}(x) + \sum_{i=1}^{m} \lambda_{i}f_{i}(x) + \sum_{i=1}^{p} v_{i}h_{i}(x) \right)\]

• \( g(\lambda, v) \) is **always concave** because it is the pointwise infimum of a family of affine functions of \((\lambda, v)\).
Duality

Dual Problem

• The Lagrange dual problem of the primal problem:

\[
\max_{\lambda, v} \quad g(\lambda, v) \\
\text{s.t.} \quad \lambda \geq 0
\]

• The dual problem is always convex
Duality
Dual of Standard Form LP

\[ L(x, \lambda, v) = c^T x - \lambda^T x + v^T (A x - b) \]
\[ = (c - \lambda + A^T v)^T x - v^T b \]

Standard Form LP

\[
\begin{align*}
\min_{x} & \quad c^T x \\
\text{s.t.} & \quad A x = b \\
& \quad x \geq 0
\end{align*}
\]

\[ g(\lambda, v) = \begin{cases} 
-v^T b & c - \lambda + A^T v = 0 \\
-\infty & \text{o.w.}
\end{cases} \]

The dual problem:

\[
\begin{align*}
\max_{\lambda, v} & \quad g(\lambda, v) \\
\text{s.t.} & \quad \lambda \geq 0
\end{align*}
\]
Duality

Dual of Standard Form LP

The dual problem:

\[
\begin{align*}
    \max_{\lambda, v} & \quad g(\lambda, v) \\
    \text{s.t.} & \quad \lambda \geq 0
\end{align*}
\]

\[
g(\lambda, v) = \begin{cases} 
- v^T b & \text{if } c - \lambda + A^T v = 0 \\
- \infty & \text{otherwise}
\end{cases}
\]

\[
\begin{align*}
    \min_x & \quad c^T x \\
    \text{s.t.} & \quad A x = b \\
    & \quad x \geq 0
\end{align*}
\]

\[
\begin{align*}
    \max_{\lambda, v} & \quad - v^T b \\
    \text{s.t.} & \quad \lambda \geq 0
\end{align*}
\]

\[
\begin{align*}
    c - \lambda + A^T v & = 0 \\
    A^T v + c & \geq 0
\end{align*}
\]

The dual of **Standard Form LP** turns out to be **Inequality Form LP**
Duality

Dual of Inequality Form LP

\[ L(x, \lambda) = c^T x + \lambda^T (Ax - b) \]
\[ = (c + A^T \lambda)^T x - \lambda^T b \]

Inequality Form LP

\[
\begin{align*}
\min_x & \quad c^T x \\
\text{s.t.} & \quad Ax \leq b
\end{align*}
\]

\[
\begin{align*}
g(\lambda) = \begin{cases} 
-\lambda^T b & c + A^T \lambda = 0 \\
-\infty & \text{o.w.}
\end{cases}
\end{align*}
\]

The dual problem:

\[
\begin{align*}
\max_{\lambda} & \quad g(\lambda) \\
\text{s.t.} & \quad \lambda \geq 0
\end{align*}
\]
Duality

Dual of Inequality Form LP

\[
\begin{align*}
\text{min} \quad & c^T x \\
\text{s.t.} \quad & Ax \leq b
\end{align*}
\]

The dual problem:

\[
\begin{align*}
\max \quad & g(\lambda) \\
\text{s.t.} \quad & \lambda \geq 0
\end{align*}
\]

\(g(\lambda) = \begin{cases} \ -\lambda^T b & c + A^T \lambda = 0 \\ \ -\infty & \text{o.w.} \end{cases}\)

\[
\begin{align*}
\max \quad & -\lambda^T b \\
\text{s.t.} \quad & \lambda \geq 0 \\
\text{s.t.} \quad & c + A^T \lambda = 0
\end{align*}
\]

The dual of Inequality Form LP turns out to be Standard Form LP
How to Solve Convex Optimization Problems
Introduction to CVXPY

• CVXPY is a Python library for solving convex optimization problems

• Today, we try to solve the LP we used as an example
  https://colab.research.google.com/drive/1HJb70wEGViQaE87mctZm2cDm3Np2CBCL?usp=sharing
Any Question?