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Polyhedron (Qualification oriented expression)

(Textbook pp.31) “the solution set of a finite number of linear equalities and inequalities”:

\[ P = \{ x \mid Ax \leq b, Cx = d \} \]

\( (Cx = d \text{ is equivalent to } Cx \leq d, -Cx \geq -d) \)

=intersection of halfspaces
Convex Hull

(Textbook pp.34) Convex hull of a finite set \( \{v_1,\ldots,v_k \in \mathbb{R}^n\} \) is \( \text{conv}(v_1,\ldots,v_k) = \{\theta_1v_1+\ldots+\theta_kv_k \mid \theta \geq 0; \ 1^T\theta = 1\} \) (each entry of \( 1 \) is 1)
Simplex (Enumeration oriented expression)

(Textbook pp.46).

A k-simplex $S$ is the convex hull of $k+1$ affinely independent vectors:

$$S = \text{conv}(v_0, ..., v_k) = \{\theta_0 v_0 + ... + \theta_k v_k \mid \theta \geq 0; \ 1^T \theta = 1\}$$

(each entry of $1$ is 1)

$$= \{U\theta \mid U = [v_0... v_k]; \ \theta \geq 0; \ 1^T \theta = 1\}$$
Qualification oriented expression (polyhedron) \arrow{\rightarrow}\text{Enumeration oriented expression (simplex)}

Problem: Given a bounded* set of points \( S := \{ x \in \mathbb{R}^{n^+} | Ax \leq b \} \), derive an equivalent expression in the form \( E = \{ U\theta | \theta \geq 0; 1^T\theta = 1 \} \)

The purple region is convex but unbounded, which cannot be turned into a simplex.

*The term “bounded” might not be accurate
Qualification oriented expression (polyhedron) —> Enumeration oriented expression (simplex)

(1) Find $\tilde{A}$ s.t. $R := \{x | \tilde{A}x \leq r\}$ is equivalent to $S := \{x \in \mathbb{R}^{n^+} \mid Ax \leq b\}$

(2) Find all $[C \mid d]$’s that are submatrices of $[\tilde{A} \mid r]$: $C$ has $n$ rows; rank($C$) = $n$ (Intuition: each vertex is intersection of exactly $n$ hyperplanes)

(3) Construct $U$ from points in $S$ that satisfy the submatrices: for each $[C \mid d]$, solve $Cu = d$; if $u \in S$, $u$ is a column of $U$. 
Example

$S = \{ x | Ax \leq b, \ x \geq 0 \}$, where $A = [\sqrt{3}, \sqrt{3}, \sqrt{3}]$, $b = [\sqrt{3}]$,

Intuition: tetrahedron

$[\tilde{A} \mid r] = [-1, 0, 0 \mid 0]$

$[0, -1, 0 \mid 0]$

$[0, 0, -1 \mid 0]$

$[\sqrt{3}, \sqrt{3}, \sqrt{3} \mid \sqrt{3}]$
Solve all submatrices \([C | d]\) of \([\bar{A} | r]\) that have 3 rows and \(\text{rank}(C) = 3\)

\[
\begin{bmatrix}
-1, 0, 0 & 0 \\
0, -1, 0 & 0 \\
0, 0, -1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\sqrt{3}, \sqrt{3}, \sqrt{3} \\
\sqrt{3}, \sqrt{3}, \sqrt{3} \\
\sqrt{3}, \sqrt{3}, \sqrt{3} \\
\end{bmatrix}
\]
gives \([0, 0, 0]\) gives \([0, 0, 1]\) gives \([0, 1, 0]\) gives \([1, 0, 0]\)
Tetrahedron

\[ U = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ E = \{ U \theta \mid \theta \geq 0; \ 1^T \theta = 1 \} \]

\[ = S = \{ x \mid Ax \leq b, x \geq 0 \}, \text{ where } A = [\sqrt{3}, \sqrt{3}, \sqrt{3}], \ b = [\sqrt{3}] \]
II.2.2. Derive the dual cone of the set \( \{ U\theta | \theta \in \mathbb{R}_+^3 \} \). (4 pts)

**Solution** Let \( C = \{ U\theta | \theta \in \mathbb{R}_+^3 \} \), the dual cone is the set

\[
C^* = \{ y | y^T x \geq 0, \forall x \in C \}
= \{ y | y^T U\theta \geq 0, \forall \theta \in \mathbb{R}_+^3 \}
= \{ y | \theta^T (U^T y) \geq 0, \forall \theta \in \mathbb{R}_+^3 \}
= \{ y | U^T y \geq 0, y \in \mathbb{R}_+^4 \}
\]

Therefore, \( C^* = \{ y | U^T y \geq 0, y \in \mathbb{R}_+^4 \} \).
Support Vector Machine, hard & soft margin

Winter 24 HW2 II.2.1, II.2.2, II.2.3: SVM hard margin

II.2.4: SVM soft margin
Enumeration oriented expression (simplex) —> Qualification oriented expression (polyhedron)

Textbook pp. 33

To describe the simplex (2.7) as a polyhedron, i.e., in the form (2.6), we proceed as follows. By definition, \( x \in C \) if and only if \( x = \theta_0 v_0 + \theta_1 v_1 + \cdots + \theta_k v_k \) for some \( \theta \succeq 0 \) with \( 1^T \theta = 1 \). Equivalently, if we define \( y = (\theta_1, \ldots, \theta_k) \) and

\[
B = \begin{bmatrix} v_1 - v_0 & \cdots & v_k - v_0 \end{bmatrix} \in \mathbb{R}^{n \times k},
\]

we can say that \( x \in C \) if and only if

\[
x = v_0 + B y
\]

for some \( y \succeq 0 \) with \( 1^T y \leq 1 \). Now we note that affine independence of the points \( v_0, \ldots, v_k \) implies that the matrix \( B \) has rank \( k \). Therefore there exists a nonsingular matrix \( A = (A_1, A_2) \in \mathbb{R}^{n \times n} \) such that

\[
A B = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} B = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.
\]
Enumeration oriented expression (simplex)  
—> Qualification oriented expression (polyhedron)

\[
\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

Multiplying (2.8) on the left with \( A \), we obtain

\[ A_1 x = A_1 v_0 + y, \quad A_2 x = A_2 v_0. \]

From this we see that \( x \in C \) if and only if \( A_2 x = A_2 v_0 \), and the vector \( y = A_1 x - A_1 v_0 \) satisfies \( y \geq 0 \) and \( 1^T y \leq 1 \). In other words we have \( x \in C \) if and only if

\[ A_2 x = A_2 v_0, \quad A_1 x \geq A_1 v_0, \quad 1^T A_1 x \leq 1 + 1^T A_1 v_0, \]

which is a set of linear equalities and inequalities in \( x \), and so describes a polyhedron.
Citation

1. Textbook
2. CSE 203B Winter 23

Acknowledgement:

1. Google Slides