In this homework, we work on exercises from text book including level sets of convex, concave, quasi-convex, quasi-concave functions (3.1, 3.2), second-order conditions for convexity on affine sets (3.9), Kullback-Leibler divergence (3.13), saddle points of convex-concave functions (3.14) determination of convex, concave, quasi-convex, quasi-concave functions (3.16), conjugate functions (3.36), and gradient and Hessian of conjugate functions (3.40). Extra assignments are given on Kullback-Leibler divergence, and softmax functions.

I. Exercises from textbook chapter 3 (8 pts)


II. Assignments (42 pts)

II. 1. Kullback Leibler Divergence.

Let us define Kullback Leibler (KL) divergence as a function

\[ D_{kl}(u, v) = \sum_i u_i \log(u_i/v_i) - u_i + v_i \]

with the constraints \(1^T u = 1, 1^T v = 1\) where \(u, v \in \mathbb{R}^n_+\). Derive the following that honors the constraints. (hint: Convert the feasible set from qualification to enumeration expression) (17 pts)

II.1.1. Design a numerical example to show the value of the function. For this example, let us set \(n = 5\). (4 pts)

[Rubrics]

• 4 pts: Correct
• -2 pts: Incorrect example

[Solution] Consider the following example where,

\( u = [0.1, 0.2, 0.3, 0.1, 0.3] \)

\( v = [0.2, 0.2, 0.2, 0.2, 0.2] \)

then,

\[ D_{kl}(u, v) = 0.105 \]

II.1.2. Derive the first order derivative of the KL divergence, i.e. \( \nabla_{u,v} D_{KL}(u, v) \). (4 pts)

[Rubrics]
[Solution] (Incorporating the constraints) Suppose \( u, v \in \mathbb{R}^n \). Then let
\[
\begin{align*}
    u &= [u_1 \quad u_2 \quad ... \quad u_{n-1} \quad 1 - \sum_{k=1}^{n-1} u_k], \\
    v &= [v_1 \quad v_2 \quad ... \quad v_{n-1} \quad 1 - \sum_{k=1}^{n-1} v_k]
\end{align*}
\]
where \( u_n = 1 - \sum_{k=1}^{n-1} u_k \) and \( v_n = 1 - \sum_{k=1}^{n-1} v_k \) since \( 1^T u = 1 \) and \( 1^T v = 1 \)

Then \( \nabla_{u,v} D_{kl}(u,v) \in \mathbb{R}^{2(n-1)} \) and
\[
    D_{kl}(u,v) = (1 - \sum_{k=1}^{n-1} u_k) \log \left( \frac{1 - \sum_{k=1}^{n-1} u_k}{1 - \sum_{k=1}^{n-1} v_k} \right) + \sum_{i=1}^{n-1} u_i \log(u_i/v_i)
\]

The first-order derivative of KL divergence will then be given by, \( \nabla_{u,v} D_{kl}(u,v) \in \mathbb{R}^{2(n-1)}, \)

\[
\nabla_{u,v} D_{kl}(u,v) = \begin{bmatrix} \nabla_u D_{kl}(u,v) \\ \nabla_v D_{kl}(u,v) \end{bmatrix}
\]

where,

\[
\nabla_u D_{kl}(u,v) = \begin{bmatrix}
\log \frac{u_1(1-\sum_{k=1}^{n-1} u_k)}{v_1(1-\sum_{k=1}^{n-1} u_k)} \\
\vdots \\
\log \frac{u_{n-1}(1-\sum_{k=1}^{n-1} u_k)}{v_{n-1}(1-\sum_{k=1}^{n-1} v_k)}
\end{bmatrix},
\]

\[
\nabla_v D_{kl}(u,v) = \begin{bmatrix}
\frac{1-\sum_{k=1}^{n-1} u_k}{1-\sum_{k=1}^{n-1} v_k} - \frac{u_1}{v_1} \\
\vdots \\
\frac{1-\sum_{k=1}^{n-1} u_k}{1-\sum_{k=1}^{n-1} v_k} - \frac{u_{n-1}}{v_{n-1}}
\end{bmatrix}
\]

[alternative accepted solution] (Ignoring the constraints)

\( \nabla_{u,v} D_{kl}(u,v) \in \mathbb{R}^{2n}, \)

\[
\nabla_{u,v} D_{kl}(u,v) = \begin{bmatrix} \nabla_u D_{kl}(u,v) \\ \nabla_v D_{kl}(u,v) \end{bmatrix}
\]

where,

\[
\nabla_u D_{kl}(u,v) = \begin{bmatrix}
\log \frac{u_1}{v_1} \\
\vdots \\
\log \frac{u_n}{v_n}
\end{bmatrix}, \quad \nabla_v D_{kl}(u,v) = \begin{bmatrix}
1 - \frac{u_1}{v_1} \\
\vdots \\
1 - \frac{u_n}{v_n}
\end{bmatrix}
\]
II.1.3. Derive the Hessian matrix of the KL divergence, i.e. $\nabla_{u,v}^2 D_{KL}(u,v)$. (4 pts)

[Rubrics]

- 4 pts: Correct
- 0.5 pts: Minor mistake
- 1 pt: Partially correct
- 0 pts: Incorrect

[Solution] (Incorporating the constraints)

$$
\nabla_{u,v}^2 D_{kl}(u,v) = \begin{bmatrix}
\nabla^2_u D_{kl}(u,v) & \nabla_u \nabla_v D_{kl}(u,v) \\
\nabla_v \nabla_u D_{kl}(u,v) & \nabla_v^2 D_{kl}(u,v)
\end{bmatrix} \in \mathbb{R}^{2(n-1) \times 2(n-1)}
$$

where,

$$
\nabla^2_u D_{kl}(u,v) = \begin{bmatrix}
\frac{1}{u_1} + \frac{1}{1-\sum_{k=1}^{n-1} u_k} & \frac{1}{u_2} + \frac{1}{1-\sum_{k=1}^{n-1} u_k} & \ldots & \frac{1}{u_{n-1}} + \frac{1}{1-\sum_{k=1}^{n-1} u_k} \\
\frac{1}{1-\sum_{k=1}^{n-1} u_k} & \frac{1}{u_2} + \frac{1}{1-\sum_{k=1}^{n-1} u_k} & \ldots & \frac{1}{u_{n-1}} + \frac{1}{1-\sum_{k=1}^{n-1} u_k} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{1-\sum_{k=1}^{n-1} u_k} & \frac{1}{1-\sum_{k=1}^{n-1} u_k} & \ldots & \frac{1}{u_{n-1}} + \frac{1}{1-\sum_{k=1}^{n-1} u_k}
\end{bmatrix}
$$

$$
\nabla_v \nabla_u D_{kl}(u,v) = \nabla_u \nabla_v D_{kl}(u,v) = \begin{bmatrix}
-\frac{1}{v_1} + \frac{1}{1-\sum_{k=1}^{n-1} v_k} & -\frac{1}{v_2} + \frac{1}{1-\sum_{k=1}^{n-1} v_k} & \ldots & -\frac{1}{v_{n-1}} + \frac{1}{1-\sum_{k=1}^{n-1} v_k} \\
-\frac{1}{1-\sum_{k=1}^{n-1} v_k} & -\frac{1}{v_2} + \frac{1}{1-\sum_{k=1}^{n-1} v_k} & \ldots & -\frac{1}{v_{n-1}} + \frac{1}{1-\sum_{k=1}^{n-1} v_k} \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{1}{1-\sum_{k=1}^{n-1} v_k} & -\frac{1}{1-\sum_{k=1}^{n-1} v_k} & \ldots & -\frac{1}{v_{n-1}} + \frac{1}{1-\sum_{k=1}^{n-1} v_k}
\end{bmatrix}
$$

$$
\nabla_v^2 D_{kl}(u,v) = \begin{bmatrix}
\frac{1-\sum_{k=1}^{n-1} u_k}{u_1^2} + \frac{u_2}{v_1^2} & \frac{1-\sum_{k=1}^{n-1} u_k}{u_2^2} + \frac{u_3}{v_2^2} & \ldots & \frac{1-\sum_{k=1}^{n-1} u_k}{u_{n-1}^2} + \frac{u_n}{v_{n-1}^2} \\
\frac{1-\sum_{k=1}^{n-1} u_k}{u_2^2} + \frac{u_3}{v_1^2} & \frac{1-\sum_{k=1}^{n-1} u_k}{u_3^2} + \frac{u_4}{v_2^2} & \ldots & \frac{1-\sum_{k=1}^{n-1} u_k}{u_n^2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1-\sum_{k=1}^{n-1} u_k}{u_{n-1}^2} + \frac{u_n}{v_{n-1}^2} & \frac{1-\sum_{k=1}^{n-1} u_k}{u_n^2} + \frac{u_{n-1}}{v_{n-1}^2}
\end{bmatrix}
$$

[alternative accepted solution] (Ignoring the constraints)

$$
\nabla_{u,v}^2 D_{kl}(u,v) = \begin{bmatrix}
\nabla^2_u D_{kl}(u,v) & \nabla_v \nabla_u D_{kl}(u,v) \\
\nabla_u \nabla_v D_{kl}(u,v) & \nabla_v^2 D_{kl}(u,v)
\end{bmatrix} \in \mathbb{R}^{2n \times 2n}
$$

where,

$$
\nabla^2_u D_{kl}(u,v) = \begin{bmatrix}
\frac{e_1}{u_1} & \ldots & \frac{e_n}{u_n}
\end{bmatrix}
$$

$$
\nabla_v \nabla_u D_{kl}(u,v) = \nabla_u \nabla_v D_{kl}(u,v) = \begin{bmatrix}
-\frac{e_1}{v_1} & \ldots & -\frac{e_n}{v_n}
\end{bmatrix}
$$

$$
\nabla_v^2 D_{kl}(u,v) = \begin{bmatrix}
\frac{u_1}{e_1} e_1 & \ldots & \frac{u_n}{e_n} e_n
\end{bmatrix}
$$

and $e_i \in \mathbb{R}^n$ is the elementary vector with a 1 at $i$ and 0 everywhere else.
II.1.4. Is KL divergence a convex function? Show your proof if it is convex. (5 pts)

[Rubrics]
• 5 pts: Correct
• -0.5 pts: Minor mistake in proving convexity
• -1 pts: Partially correct proof
• -2 pts: Incomplete proof
• -3 pts: Incorrect proof
• -4 pts: Incorrect proof and answer

[Solution] (Boyd. pg. 90)

It can be shown that KL divergence is convex since it is a combination of convex functions over operations that preserve convexity.

Consider the negative logarithm function \( f(x) = -\log x \) which we know is convex on \( \mathbb{R}^+ \). The perspective of \( f(x) \),

\[
g(x, t) = -t \log(x/t) = t \log(t/x),
\]

also called relative entropy, is convex on \( \mathbb{R}^2^+ \), because perspective preserves convexity.

The relative entropy of two vectors \( u, v \), \( \sum_{i=1}^n u_i \log(u_i/v_i) \) is also convex since it is a sum of relative entropies of \( u_i, v_i \).

KL Divergence, \( D_{KL}(u, v) = \sum_{i=1}^n u_i \log(u_i/v_i) - u_i + v_i \) is the sum of relative entropy of two vectors and a linear function of \( (u, v) \). Therefore, it is convex.

[alternative accepted solution] (Ignoring the constraints)

Since the domain of KL divergence is convex, to see if KL divergence is a convex function, we can check if its Hessian is positive semidefinite, i.e., if \( x^T \nabla^2_{u,v} D_{kl}(u, v)x \geq 0 \ \forall x \in \mathbb{R}^2n \).

Let \( x = [a_1 \ldots a_n \ b_1 \ldots b_n]^T \ \forall a_i, b_i \in \mathbb{R} \)

Then,

\[
x^T \nabla^2_{u,v} D_{kl}(u, v)x = [a_1 \ldots a_n \ b_1 \ldots b_n]\begin{bmatrix}
\frac{e_1}{u_1} & \cdots & \frac{e_n}{u_n} & -\frac{e_1}{v_1} & \cdots & -\frac{e_n}{v_n} \\
\frac{e_2}{u_2} & \cdots & \frac{e_n}{u_n} & -\frac{e_2}{v_2} & \cdots & -\frac{e_n}{v_n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{e_n}{u_n} & \cdots & \frac{e_n}{u_n} & -\frac{e_n}{u_n} & \cdots & -\frac{e_n}{v_n}
\end{bmatrix}\begin{bmatrix}
a_1 \\
a_n \\
b_1 \\
b_n
\end{bmatrix}
\]
\[ x^T \nabla^2_{u,v} D_{kl}(u,v)x = \begin{bmatrix} \frac{a_1}{u_1} - \frac{b_1}{v_1} & \cdots & \frac{a_n}{u_n} - \frac{b_n}{v_n} & \frac{b_1u_1}{v_1^2} - \frac{a_1}{v_1} & \cdots & \frac{b_nu_n}{v_n^2} - \frac{a_n}{v_n} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \\ b_1 \\ \vdots \\ b_n \end{bmatrix} \]

\[ x^T \nabla^2_{u,v} D_{kl}(u,v)x = \sum_{i=1}^n \frac{a_i^2}{u_i} - \frac{a_i b_i}{v_i} + \frac{b_i^2 u_i}{v_i^2} - \frac{a_i b_i}{v_i} \]

\[ x^T \nabla^2_{u,v} D_{kl}(u,v)x = \sum_{i=1}^n \frac{a_i^2}{u_i} - \frac{2a_i b_i}{v_i} + \frac{b_i^2 u_i}{v_i^2} \]

\[ x^T \nabla^2_{u,v} D_{kl}(u,v)x = \sum_{i=1}^n u_i \left( \frac{a_i^2}{u_i^2} - \frac{2a_i b_i}{u_i v_i} + \frac{b_i^2}{v_i^2} \right) \]

\[ x^T \nabla^2_{u,v} D_{kl}(u,v)x = \sum_{i=1}^n u_i \left( \frac{a_i}{u_i} - \frac{b_i}{v_i} \right)^2 \geq 0 \quad \therefore u \in \mathbb{R}_{++}^n \]

Therefore, the Hessian is positive semi-definite and hence, KL divergence is a convex function.

II.2.1 \( f_1(x) = \|x\|_p, p = 100 \), where \( x \in \mathbb{R}_{++}^n \). (6pts)

[Rubrics]

- (1) +1 pt Mentions \( f^*(y) = 0 \) if \( \|y\|_* \leq 1 \) or \( \|y\|_{100} \leq 1 \)
- (2) +2 pts: Proof of (1), which could be the textbook example 3.26’s way, setting gradient of \( f^*(y) \) wrt x to 0, etc.
- (3) +1 pt: Mentions \( f^*(y) = \infty \) otherwise.
- (4) +2 pts: Proof of (3), which could be the textbook example 3.26’s way, setting gradient of \( f^*(y) \) wrt x to 0, etc.
- (5) +1 pt: Attempts to solve the problem, but does not meet any other requirement.
- (6) +1 pt: Extra credit if the solution provides more analytic information than the textbook example 3.26 does. This problem is not the same as textbook example 3.26.
- (7) Positive point adjustment if there is significant effort but the result is wrong.

[Solution]

Result: \( f^*(y) = \begin{cases} 0, & \text{in case 1: } \|y\|_q \leq 1, \text{ where } \frac{1}{p} + \frac{1}{q} = 1 \\ \infty, & \text{otherwise: } \|y\|_q > 1 \end{cases} \)

Note: \( \forall x \in \mathbb{R}_{++}^n, p \in [1, \infty), \|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p} \), and its dual norm \( \|x\|_* := \max_{\|z\|_p \leq 1} z^T x = \|y\|_q \), where \( \frac{1}{p} + \frac{1}{q} = 1 \). (There still needs a further investigation into whether \( p \) must be rational).
(Refers to textbook pp. 93).

Proof. By definition, the conjugate function is \( y \in \mathbb{R}^n, f^*(y) = \sup_{z \in \mathbb{R}^n} y^T z - f(z) \)
Then \( \forall x \in \mathbb{R}^n, y^T x - f(x) \leq \sup_{z \in \mathbb{R}^n} y^T z - f(z) = f^*(y) \)

Case 1: \( f^*(y) = 1, y \geq 0 \) and \( ||y||_s \leq 1. \)

By Appendix Claim 1, \( \forall x \in \mathbb{R}^n, y^T x \leq ||x||_p ||y||_s, \) so \( y^T x - f(x) = y^T x - ||x||_p ||y||_s \leq 0. \)
Thus, \( \forall x \in \mathbb{R}, y^T x - f(x) = y^T x - ||x||_p \leq 0, \) which means that 0 is an upper bound of \( \{y^T x - f(x) | x \in \mathbb{R}^n \}. \)

However, to finish the proof, we still need to show that 0 is the least upper bound of \( \{y^T x - f(x) | x \in \mathbb{R}^n \}. \)
It turns out to be true, since \( 0 \in \{y^T x - f(x) | x \in \mathbb{R}^n \} \): when \( x = 0 \in \mathbb{R}^n, 0 = y^T x - f(x) \)
Thus, \( f^*(y) = \sup \{y^T x - f(x) | x \in \mathbb{R}^n \} = 0 \) (1)

Case 2: \( \forall z \in \mathbb{R}^n_+, ||z||_p \leq 1, y^T z \leq 1. \)
Let \( z \in \mathbb{R}^n_+ \) be arbitrary.
Then \( y^T z - f_1(z) = y^T z - ||z||_p \leq ||z||_p (y^T (\frac{z}{||z||_p}) - 1) \)
Since \( \frac{z}{||z||_p} \in \mathbb{R}^n_+ \) and \( ||\frac{z}{||z||_p}||_p \leq 1, \) by assumption, we have \( y^T \frac{z}{||z||_p} ||z||_p \leq 1 \)
Thus, \( \frac{z}{||z||_p} - 1 \leq 0, \) and since \( ||z||_p \geq 0, \) we get \( ||z||_p (\frac{z}{||z||_p} - 1) \leq 0 \)
Thus, \( y^T z - f_1(z) \leq 0 \)
Since \( z \in \mathbb{R}^n_+ \) is arbitrary, we conclude \( f^*(y) = \sup_{x \in \mathbb{R}^n} y^T x - f_1(x) \leq 0, \) and the equality holds when \( y = 0. \)
Thus, \( f^*(y) = 0. \)

Case 3: \( ||y||_q \geq 1 \) and \( \exists z \in \mathbb{R}^n_+, ||z||_p \leq 1, \) such that \( y^T z > 1. \)
Then \( y^T z - ||z||_p > 0. \)
Thus, \( \lim_{t \to \infty} y^T (tz) - f(tz) = \lim_{t \to \infty} y^T (tz) - ||tz||_p = \lim_{t \to \infty} t(y^T z - ||z||_p) = \infty \)
Then by (1), \( f^*(y) \geq \lim_{t \to \infty} y^T (tz) - f(tz) = \infty. \)
Thus, \( f^*(y) = \infty. \)

\( \square \)

II.2.2 \( f_2(x) = \log \sum_i \exp(x_i/\gamma), \) where \( x \in \mathbb{R}^n. \) (6 pts)

[Rubrics]

- (1) +1 pt: Mentions \( f^*(y) = \gamma (\sum_i y_i (\log y_i + \log \gamma) \) when \( 1^T y = \frac{1}{\gamma}, y \geq 0. \)
- (2) +2.5 pts: Proof of (1).
- (3) +1 pt: Mentions \( f^*(y) = \infty \) otherwise.
- (4) +1 pt: Proof for \( f^*(y) = \infty \) when \( 1^T y \neq \frac{1}{\gamma}, y \geq 0. \)
- (5) +0.5 pt: Proof for \( f^*(y) = \infty \) when \( 3t \leq k \leq n, y_k < 0. \)
- (6) +1 pt: Fails to meet all other requirements, but attempts to solve.
- (7) Positive point adjustment if there is significant effort but the result is wrong.
[Solution]
Result:
\[ f_2^*(y) = \begin{cases} \gamma \sum_i y_i (\log(y_i) + \log \gamma) & \text{if } 1^T y = 1, y \geq 0 \\ \infty & \text{otherwise} \end{cases} \]

Proof:
The conjugate function of \( f_2(x) \) is \( f_2^*(y) = \sup_{x \in \text{dom} f} (y^\top x - f_2(x)) \)

Note that \( \sum_i \exp(x_i/\gamma) = 1^\top \exp(x/\gamma) \). Then \( f_2(x) = \log(1^\top \exp(x/\gamma)) \).

**Case 1:** \( 1^\top y = \frac{1}{\gamma}, y \geq 0 \).

Now set \( \nabla_x (y^\top x - f_2(x)) = 0 \) to find \( x \in \text{dom } f \) that achieves the max value of \( y^\top x - f_2(x) \),

which gives \( y = \nabla_x f_2(x) = \frac{1}{1 - \exp(x/\gamma)} \exp(x/\gamma) = \frac{\exp(x/\gamma)}{\sum_i \exp(x_i/\gamma)} = \frac{\exp(x/\gamma)}{\gamma \exp(f_2(x))} \).

Thus, we get (1): \( y > 0, 1^\top y = 1 \).

∀1 ≤ i ≤ n, \( y_i = \frac{\exp(x_i/\gamma)}{\exp(f_2(x))} \exp(x_i/\gamma) = y_i \gamma \exp(f_2(x)) \)

By (1), ∀1 ≤ i ≤ n, LHS = RHS > 0, so \( \log \text{LHS} = \log \text{RHS} \), \( x_i = \gamma (\log(y_i) + \log(\gamma) + f_2(x)) \) (2)

Such \( x \in \text{dom } f \) achieves the max value of \( y^\top x - f_2(x) \), so plugging (2) back gives

\[ f_2^*(y) = \left[ \sum_i y_i (\log(y_i) + \log(\gamma) + f_2(x)) \right] - f_2(x) \]

\[ = \left[ \gamma \sum_i y_i (\log(y_i) + \log(\gamma)) \right] + (\gamma \sum_i y_i - 1) f_2(x) \]

By (1), \( (\gamma \sum_i y_i - 1) = 0 \).

Thus, in case 1, \( f_2^*(y) = \gamma \sum_i y_i (\log(y_i) + \log(\gamma)) \).

**Case 2:** \( 1^\top y \neq \frac{1}{\gamma}, y \geq 0 \).

**Way 1:** Let \( x = t1 \), then \( y^\top x - f_2(x) = t(y^\top 1) - \log(n \exp(t/\gamma)) = t(y^\top 1) - \log n - t/\gamma = t(1 - \gamma) - \log n \)

If \( y^\top > 1/\gamma \), then \( f_2^*(y) \geq \lim_{t \to -\infty} y^\top x - f_2(x) = \infty \)

Otherwise \( y^\top < 1/\gamma \), then \( f_2^*(y) \geq \lim_{t \to -\infty} y^\top x - f_2(x) = \infty \)

Thus, \( f_2^*(y) = \infty \)

**Way 2** (more redundant but also more general):

**Case 2-1:** \( 1^\top y \geq \frac{1}{\gamma} \). Let \( x = t1 \). Then \( f^*(y) = \sup_{x \in \text{dom } f} y^\top x - \log \sum_i \exp(x_i/\gamma) \)

\[ f^*(y) \geq \lim_{t \to -\infty} y^\top (t1) - \log \sum_i \exp(t/\gamma) = \lim_{t \to -\infty} t1^\top y - \log \sum_i \exp(x_i/\gamma) \]

Since log is concave, by Jensen’s inequality, \( \log \sum_i \frac{\exp(x_i/\gamma)}{n} \geq \frac{\sum_i \log \exp(x_i/\gamma)}{n} = \frac{1}{n} \sum_i x_i / (n \gamma) \)

Thus, \( \log \sum_i \exp(x_i/\gamma) - \log n \geq \sum_i t / (n \gamma) \), which is that log \( \sum_i \exp(x_i/\gamma) \geq \frac{1}{n} \)

Thus, \( f^*(y) \geq \lim_{t \to -\infty} t1^\top y - \frac{1}{n} = \infty \).

**Case 2-2:** \( 1^\top y < \frac{1}{\gamma} \)

By the same token, \( f^*(y) \geq \lim_{t \to -\infty} t1^\top y - \frac{1}{n} = \infty \).

**Case 3:** \( \exists 1 \leq k \leq n, y_k < 0 \)

**Case 3-1:** \( n = 1 \)

\[ y^\top x - f_2(x) = yx - x/\gamma = x(y - 1/\gamma) \]

Since \( y - 1/\gamma < 0 \), we have \( \lim_{x \to -\infty} x(y - 1/\gamma) = \infty \)

Thus, \( f^*(y) = \sup_{x \in \text{dom } f} y^\top x - f_2(x) \geq \lim_{x \to -\infty} x(y - 1/\gamma) = \infty \)

**Case 3-2:** \( n \geq 2 \)

Let \( x = te_k \), then \( y^\top x - f_2(x) = -ty_k - \log(n - 1 + \exp(t/\gamma)) \)

\[ f^*(y) = \sup_{x \in \text{dom } f} y^\top x - f_2(x) \geq \lim_{t \to -\infty} ty_k - \log(n - 1 + \exp(t/\gamma)) = \lim_{t \to -\infty} ty_k - \lim_{t \to -\infty} \log(n - 1 + \exp(t/\gamma)) = \infty - \lim_{t \to -\infty} \log(n - 1 + \exp(t/\gamma)) \]
Since the function log function uniformly continuous on \([1, \infty)\), we get \(\lim_{t \to -\infty} \log(n - 1 + \exp(t/\gamma)) = \log(n - 1 + \lim_{t \to -\infty} \exp(t/\gamma)) = \log(n - 1 + 0) = \log(n - 1)\).

Thus, \(f^*(y) \geq -\infty - \log(n - 1) = \infty\).

\[\text{II.2.3 } f_3(x) = \sum_i x_i \exp(x_i/\gamma) / \sum_i \exp(x_i/\gamma), \text{ where } x \in \mathbb{R}^n. \ (6 \text{ pts})\]

**[Rubrics]**

- **(1)** +2 pts: Authentic attempt to solve the problem, including but not limited to setting \(\nabla_x f^*(y) = 0\). The gradient must be concerning \(x\) rather than \(y\).
- **(2)** +2 pts: Correct expression of \(y_i\) in terms of \(x\), tolerating minor mistakes.
- **(3)** +2 pts: Meet (2) with enough steps.
- **(4)** +3 pts: Does not meet (2), but enough work done.
- **(5)** +1 pt: Completion: fail to meet any of the other requirements, but attempt to solve the problem.
- **(6)** +1 pt: Regular Extra Credit: attempt to derive the conjugate function with enough work shown. For instance, \(f_3^*(y) = \infty\) when \(\sum_i y_i \neq 1\).
- **(7)** +1 pt: Special Extra Credit 1: providing a tight bound for the conjugate function’s value when \(\sum_i y_i \neq 1\).
- **(8)** +2 pts: Special Extra Credit 2: Proving the conjugate function’s value when \(\sum_i y_i \neq 1\). Either proving it’s positive/negative infinite or providing the value in analytical form (without variable \(y\)).

**[Solution]**

Since this question is still unresolved, there will be an overleaf document published later to collect ideas to solve it.

**[Part 1: a way of getting full points]**

The conjugate function of \(f_3(x)\) is \(f_3^*(y) = \sup_{x \in \text{dom} f} (y^\top x - f_3(x))\)

Note that \(\sum_i x_i \exp(x_i/\gamma) = x^\top \exp(x/\gamma)\) and \(\sum_i \exp(x_i/\gamma) = 1^\top \exp(x/\gamma)\)

Then let \(f_3(x) = g(x)/h(x)\), where \(h(x) = x^\top \exp(x/\gamma)\), \(g(x) = 1^\top \exp(x/\gamma)\)

Now set \(\nabla_x (y^\top x - f_3(x)) = 0\) to find \(x \in \text{dom} f\) that achieves the max value of \(y^\top x - f_3(x)\), which gives \(y = \nabla_x f_3(x)\)

\[
y = \frac{\partial (g(x)/h(x))}{\partial x} = \frac{(\nabla_x g(x))(x) - g(x)(\nabla_x h(x))}{h(x)}
\]

\[
\nabla_x g(x) = \exp(x/\gamma) + (1/\gamma) \text{diag}(x) \exp(x/\gamma), \text{ where } \text{diag}(x) \text{ is the square matrix whose diagonal entries are components of } x
\]

\[
y = \frac{1}{h(x)} [\exp(x/\gamma) h(x) + (1/\gamma) \text{diag}(x) \exp(x/\gamma) h(x) - g(x)(1/\gamma) \exp(x/\gamma)]
\]

\[
= \frac{1}{h(x)} [\exp(x/\gamma) h(x) + (1/\gamma) \text{diag}(x) \exp(x/\gamma) h(x) - g(x)(1/\gamma) \exp(x/\gamma)]
\]

\[
= \frac{1}{h(x)} [\exp(x/\gamma) + (1/\gamma) \text{diag}(x) \exp(x/\gamma) - f_3(x)(1/\gamma) \exp(x/\gamma)]
\]

(currently do not know how to express \(x\) in terms of \(y\))
II.2.4 Suppose that we use the above functions as a softmax function. Derive the worst error bounds, i.e. $|\max_i x_i - f(x)|$, of the above functions. (7 pts)

[Part 2: a possible solution following Albert’s approach]

Claim 1: \(\forall x \in \mathbb{R}^n, f_3(x) \leq \max_i x_i\), and the equality holds if and only if all entries of \(x\) are equal to each other, i.e. \(\exists s \in \mathbb{R}, \forall 1 \leq i \leq n, x_i = s\)

**Proof.** Let \(x \in \mathbb{R}^n\).

Since \(\forall 1 \leq j \leq n, \frac{\exp(x_j/\gamma)}{\sum_i \exp(x_i/\gamma)} > 0\), then \(\frac{\sum_i \exp(x_i/\gamma)}{\sum_j \exp(x_j/\gamma)} < \frac{\sum_i \exp(x_i/\gamma)}{\sum_j \exp(x_j/\gamma)}\).

Thus, \(\sum_j x_j \frac{\exp(x_j/\gamma)}{\sum_i \exp(x_i/\gamma)} \leq \sum_j (\max_i x_i) \frac{\exp(x_j/\gamma)}{\sum_i \exp(x_i/\gamma)} = (\max_i x_i) \frac{\sum_j \exp(x_j/\gamma)}{\sum_i \exp(x_i/\gamma)}\), which is \(\sum_j x_j \frac{\exp(x_j/\gamma)}{\sum_i \exp(x_i/\gamma)} \leq (\max_i x_i) \frac{\sum_j \exp(x_j/\gamma)}{\sum_i \exp(x_i/\gamma)} = \max_i x_i\).

By simply changing the symbol, we get \(f_3(x) = \sum_j x_j \frac{\exp(x_j/\gamma)}{\sum_i \exp(x_i/\gamma)} \leq (\max_i x_i) \sum_j \exp(x_j/\gamma) = \max_i x_i\).

\(\square\)

Claim 2: \(\forall (y \in \mathbb{R}^n, \sum_i y_i \neq 1), f^*(y) = \infty\).

**Proof.** Let \(y\) satisfy \((y \in \mathbb{R}^n, \sum_i y_i - 1 \neq 0)\).

Let \(x\) satisfy \((x = (s, \ldots, s) \in \mathbb{R}^n, \text{i.e. } \forall 1 \leq i \leq n, x_i = s)\).

Then \(y^\top x - f_3(x) = (\sum_i y_i s) - s = s(\sum_i y_i - 1)\).

By assumption, either \((\sum_i y_i - 1) > 0\) or \((\sum_i y_i - 1) < 0\).

If \((\sum_i y_i - 1) > 0\), we get \(\lim_{y \to \infty} y^\top x - f_3(x) = \infty\).

Then \(f^*(y) = \sup_{x \in \text{dom} f} y^\top x - f_3(x) \geq \lim_{y \to \infty} y^\top x - f_3(x) = \infty\).

Otherwise, \((\sum_i y_i - 1) < 0\), we get \(\lim_{y \to -\infty} y^\top x - f_3(x) = \infty\).

By the same token, we also get \(f^*(y) \geq \infty\).

Thus, \(f^*(y) = \infty\).

\(\square\)

Claim 3: \(\forall (y \in \mathbb{R}^n, \sum_i y_i = 1, y \geq 0), f^*(y) = 0\).

**Proof.** Let \(y \in \mathbb{R}^n, \sum_i y_i = 1, y \geq 0\).

Obviously, \(\forall x \in \mathbb{R}^n, y^\top x = \sum_i y_i x_i \leq \max_i x_i\).

\(\square\)

[Rubrics]

- (1) +0 pts: The previous piazza post suggesting \(|\max_i f^*(e_i)|\) was wrong, but students will not be penalized for that. Check piazza and the solution for the latest update :)+0pts
- (2) +1.5 pts: Some expression of \(f_1\)’s max error bound, which could be conjugate function \(|\max_i f^*_1(e_i)|, (\max_i x_i)(n^{1/\gamma} - 1), \text{etc.}\)
- (3) +2 pts: Enough work for \(f_1\)’s max error bound, such as splitting cases, the transformation of inequalities, etc., regardless of correctness.
- (4) +1.5 pts: Some expression of \(f_2\)’s max error bound, which could be conjugate function \(|\max_i f^*_2(e_i)| \text{ when } \gamma = 1, (\frac{1}{\gamma} - 1)(\max_i x_i) + \log n \text{ when } \ldots, \text{etc.}\)
• (5) +2 pts: Enough work for \( f_2 \)'s max error bound, such as splitting cases, the transformation of inequalities, etc., regardless of correctness.

• (6) +1 pt: Fails to meet any of the other requirements, but attempt to solve the problem.

• (7) +1 pt: Regular Extra Credit: extraordinary work beyond the above requirements, such as analyzing exhaustive cases for \( f_2 \)'s max error bound rigorously.

• (8) +1 pt: Special Extra Credit: the max error bound provided is one of the tightest among all submissions.

[Solution]
1. \( f_1 \)'s max error bound
   Let \( c = \max_i x_i \), then \( \max_i (x_i - f_1(x)) = (\max_i x_i) - f_1(x) = c - (\sum_i (x_i)^p)^{\frac{1}{p}} \).
   Since each \( 0 \leq x_i \leq c \), it must be \( \max_i (x_i - f_1(x)) \geq c - (ne^p)^{\frac{1}{p}} = c - n^\frac{1}{p} c = c(1 - n^\frac{1}{p}) \).
   Since \( n \geq 1 \), we have \( 1 - n^\frac{1}{p} \leq 0 \), and thus \( c(1 - n^\frac{1}{p}) \leq 0 \)
   In general, \( \max_i (x_i - f_1(x)) \leq 0 \)
   We have \( c(1 - n^\frac{1}{p}) \leq \max_i (x_i - f_1(x)) \leq 0 \), so the max error bound is \( c(1 - n^\frac{1}{p}) \), which might not be the tightest.
2. \( f_2 \)'s max error bound
   Pending. Will combine all ideas from students.

Appendix: Classic example of conjugate function: \( p \)-norm's conjugate function.

(There still needs a further investigation into whether \( p \) must be rational).
Claim 1: \( \forall y, x \in \mathbb{R}^n, p \in [1, \infty), y^\top x \leq ||x||_p ||y||_1^{\frac{1}{p}} \).

**Proof.** Let \( y, x \in \mathbb{R}^n, p \in [1, \infty) \).
\[
y^\top x = ||x||_p (\frac{x}{||x||_p})^\top y
\]

From \( ||\frac{x}{||x||_p}||_p \leq 1 \) we get \( (\frac{x}{||x||_p})^\top y \leq \max_{||z||_1 \leq 1} z^\top y = ||y||_1 \)
Then \( y^\top x = x^\top y = ||x||_p (\frac{x}{||x||_p})^\top y \leq ||x||_p ||y||_1 \).
\(\square\)

Claim 2: For \( p \)-norm \( f(x) = ||x||_p \), where \( x \in \mathbb{R}^n, p \in [1, \infty) \), the conjugate function of is \( f^\star(y) = \begin{cases} 0, & \text{if } ||y||_p \leq 1 \\ \infty, & \text{otherwise} \end{cases} \) (Refers to textbook pp. 93).

**Proof.** By definition, the conjugate function is \( \forall y \in \mathbb{R}^n, f^\star(y) = \sup_{z \in \mathbb{R}^n} y^\top z - f(z) \)

Then \( \forall z \in \mathbb{R}^n, y^\top x - f(x) \leq \sup_{z \in \mathbb{R}^n} y^\top z - f(z) = f^\star(y) \) (1)
Case 1: \( ||y||_1 \leq 1 \).
II.2.1's case 1's proof and conclusion still holds if the domain is changed from \( \mathbb{R}^n_+ \) to \( \mathbb{R}^n \).
This is trivial to show. Thereby we proved that in this case, \( f^\star(y) = \infty \).

Case 2: \( ||y||_1 > 1 \)
\[
\max_{||z||_1 \leq 1} z^\top y = ||y||_1 > 1 \), then \( \exists z \in \mathbb{R}^n, ||z||_1 \leq 1 \), such that \( y^\top z > 1 \). Then \( y^\top z - ||z||_p > 0 \).
Thus, \( \lim_{t \to \infty} y^\top (t z) - f(t z) = \lim_{t \to \infty} y^\top (t z) - ||t z||_p = \lim_{t \to \infty} t(y^\top z - t ||z||_p) = \lim_{t \to \infty} t(y^\top z - ||z||_p) = \infty \)

Then by (1), \( f^*(y) \geq \lim_{t \to \infty} y^\top (t z) - f(t z) = \infty. \)
Thus, \( f^*(y) = \infty. \)