

CSE 127: Introduction to Security

Public-Key Cryptography

Today

- ▶ Key Exchange
- ▶ Public Key Encryption
- ▶ Digital Signatures

Asymmetric cryptography/public-key cryptography

Main insight: Separate keys for different operations.

Keys come in pairs, and are related to each other by the specific algorithm

- ▶ Public key: used to encrypt or verify signatures
- ▶ Private key: used to decrypt and sign

Public-key encryption

- ▶ Encryption: (public key, plaintext) $\rightarrow$ ciphertext

$$\text{Enc}_{pk}(m) = c$$

- ▶ Decryption: (secret key, ciphertext) $\rightarrow$ plaintext

$$\text{Dec}_{sk}(c) = m$$

Properties:

- ▶ Encryption and decryption are inverse operations:

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Properties:

- ▶ Encryption and decryption are inverse operations:

$$\text{Dec}_{sk}(\text{Enc}_{pk}(m)) = m$$

- ▶ Secrecy: ciphertext reveals nothing about plaintext
 - ▶ Computationally hard to decrypt without secret key
- ▶ The point:
 - ▶ Anybody with your public key can send you a secret message!
Solves key distribution problem.

Modular Arithmetic Review

Division: Let n, d, q, r be integers.

$$\lfloor n/d \rfloor = q$$

$$n = qd + r \quad 0 \leq r < d$$

$$n \equiv r \pmod{d}$$

Facts about remainders/modular arithmetic:

Add: $(a \bmod d) + (b \bmod d) \equiv (a + b) \bmod d$

Subtract: $(a \bmod d) - (b \bmod d) \equiv (a - b) \bmod d$

Multiply: $(a \bmod d) \cdot (b \bmod d) \equiv (a \cdot b) \bmod d$

Modular Inverse: “Division” for modular arithmetic

If $a \cdot b \bmod d = c \bmod d$ we would like $c/b \bmod d = a \bmod d$.

Let's try this: let $a = 3$, $b = 2$, and $d = 4$

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This doesn't quite work, it says $3 = 1 \bmod 4$!

Fix: For rationals, $\frac{a}{b} = a \cdot \frac{1}{b}$ $b \cdot \frac{1}{b} = 1$.

Define modular inverse: $\frac{1}{b}$ means $b^{-1} \bmod d$.

- ▶ $b^{-1} \bmod d$ is a value such that $b \cdot b^{-1} \equiv 1 \bmod d$.
- ▶ Example: $3 \cdot (3^{-1} \bmod 5) \equiv$

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- ▶ $b^{-1} \bmod d$ is a value such that $b \cdot b^{-1} \equiv 1 \bmod d$.
- ▶ Example: $3 \cdot (3^{-1} \bmod 5) \equiv 3 \cdot 2 \equiv 1 \bmod 5$.
- ▶ If $\gcd(a, d) = 1$ then a^{-1} is well defined.
- ▶ Efficient to compute.

Modular exponentiation and discrete log

Modular exponentiation

- ▶ Over the integers, $g^a = g \cdot g \cdot g \dots g$
- ▶ $g^a \bmod d$ it's the same:
$$g^a \bmod d = (((g \bmod d) \cdot g \bmod d) \dots g \bmod d) \bmod d$$
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Input b, d, y , discrete log is a such that $b^a \equiv y \bmod d$.
- ▶ No known polynomial-time algorithm to compute this.

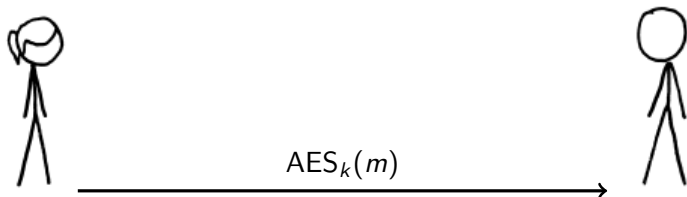


New Directions in Cryptography

Invited Paper

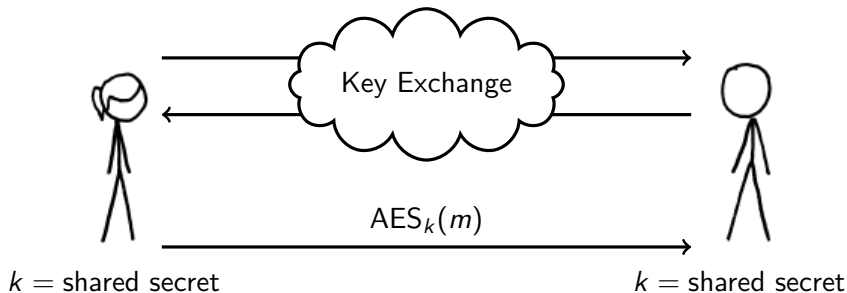
WHITFIELD DIFFIE AND MARTIN E. HELLMAN, MEMBER, IEEE

Symmetric cryptography



Public key crypto idea # 1: Key exchange

Solving key distribution without trusted third parties



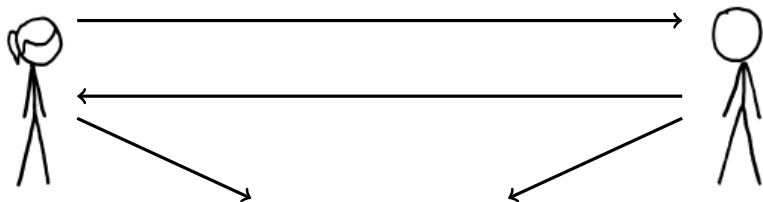
Textbook Diffie-Hellman Key Exchange

Public Parameters

p a prime

g an integer mod p

Key Exchange



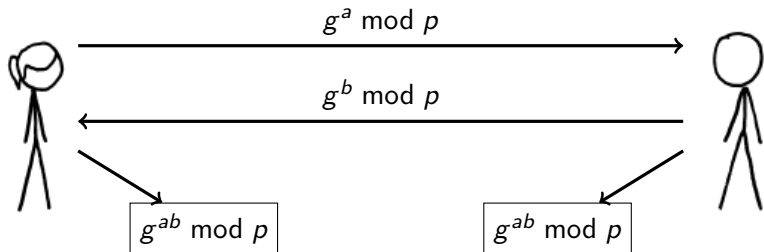
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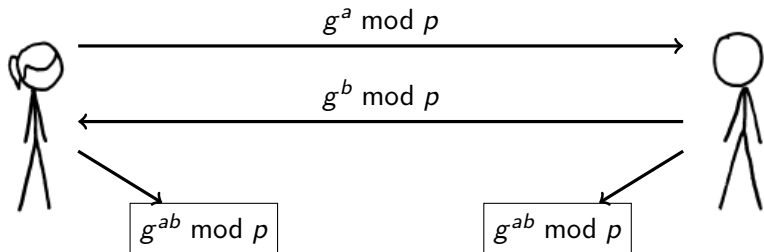
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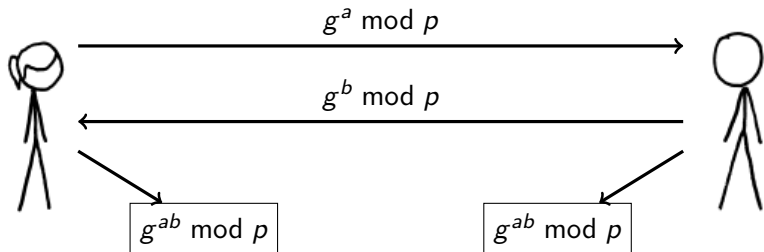
Note: $(g^a)^b \bmod p = g^{ab} \bmod p = g^{ba} \bmod p (g^b)^a \bmod p$.

Diffie-Hellman Security



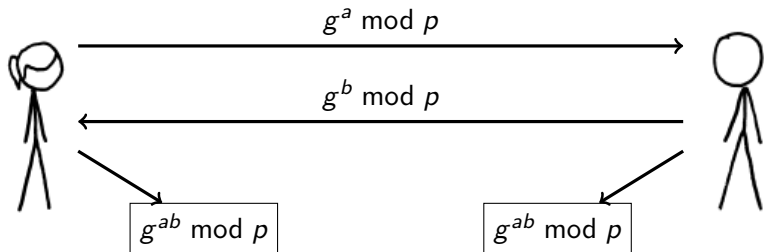
- ▶ Most efficient algorithm for passive eavesdropper to break:
Compute discrete log of public values $g^a \bmod p$ or $g^b \bmod p$.

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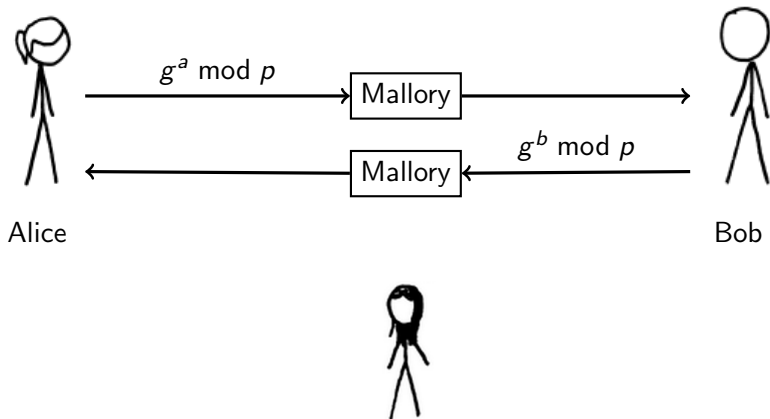
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- ▶ Parameter selection: p should be ≥ 2048 bits.

Diffie-Hellman Security



- ▶ Most efficient algorithm for passive eavesdropper to break: Compute discrete log of public values $g^a \bmod p$ or $g^b \bmod p$.
- ▶ Parameter selection: p should be ≥ 2048 bits.
- ▶ **Do not implement this yourself ever: discrete log is only hard for certain choices of p and g .**
- ▶ Best current choice: Use elliptic curve Diffie-Hellman. (Similar idea, more complicated math.)

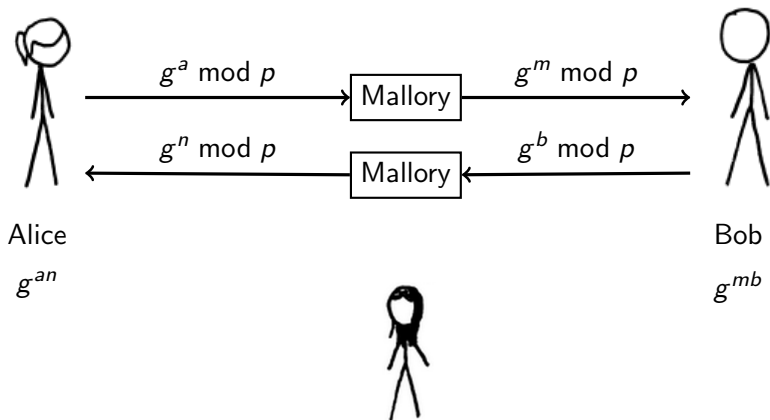
Diffie-Hellman insecure against man-in-the-middle



Active adversary can modify Diffie-Hellman messages in transit and learn both shared secrets.

Allows transparent MITM attack against later encryption.

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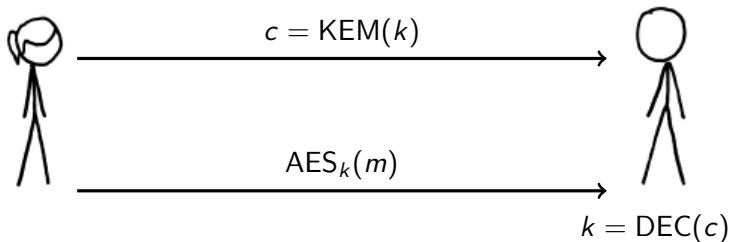
Fix: Need to authenticate messages.

Computational complexity for integer problems

- ▶ Integer multiplication is efficient to compute.
- ▶ There is no known polynomial-time algorithm for general-purpose factoring.
- ▶ Efficient factoring algorithms for many types of integers. Easy to find small factors of random integers.
- ▶ Modular exponentiation is efficient to compute.
- ▶ Modular inverses are efficient to compute.

Idea # 2: Key encapsulation/public-key encryption

Solving key distribution without trusted third parties





A Method for Obtaining Digital Signatures and Public-Key Cryptosystems

R.L. Rivest, A. Shamir, and L. Adleman*

Textbook RSA Encryption

[Rivest Shamir Adleman 1977]

Public Key pk

$N = pq$ modulus

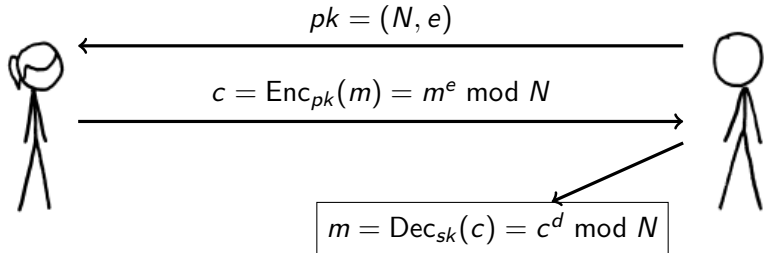
e encryption exponent

Secret Key sk

p, q primes

d decryption exponent

$$(d = e^{-1} \bmod (p-1)(q-1) = e^{-1} \bmod \phi(N))$$



$\text{Dec}(\text{Enc}(m)) = m^{ed} \bmod N \equiv m^{1+k\phi(N)} \equiv m \bmod N$ by Euler's theorem ($m^{\phi(N)} \equiv 1 \bmod N$).

RSA Security

- ▶ Best algorithm to break RSA: Factor N and compute d .
- ▶ Factoring is not efficient in general.
- ▶ Current key size recommendations: N should be ≥ 2048 bits.
- ▶ Do not ever implement this yourself. Factoring is only hard for some integers, and textbook RSA is insecure.

Textbook RSA is super insecure

Unpadded RSA encryption is homomorphic under multiplication.

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Attack: Malleability

Given a ciphertext $c = \text{Enc}(m) = m^e \bmod N$, attacker can forge ciphertext $\text{Enc}(ma) = ca^e \bmod N$ for any a .

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Attack: Chosen ciphertext attack

Given a ciphertext $c = \text{Enc}(m)$ for unknown m , attacker asks for $\text{Dec}(ca^e \bmod N) = d$ and computes $m = da^{-1} \bmod N$.

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Fix: always use padding on messages.

RSA PKCS #1 v1.5 padding

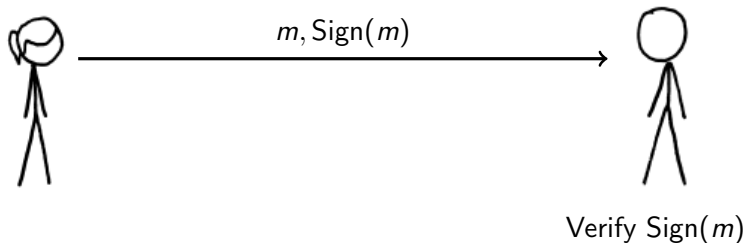
Most common implementation choice even though it is insecure

$\text{pad}(m) = 00\ 02\ [\text{random padding string}]\ 00\ [m]$

- ▶ Encrypter pads message, then encrypts padded message using RSA public key: $\text{Enc}_{pk}(m) = \text{pad}(m)^e \bmod N$
- ▶ Decrypter decrypts using RSA private key, strips off padding to recover original data: $\text{Dec}_{sk}(c) = c^d \bmod N = \text{pad}(m)$

PKCS#1v1.5 padding is vulnerable to a number of padding attacks. It is still commonly used in practice.

Idea #3: Digital Signatures



Bob wants to verify Alice's signature using only a public key.

- ▶ Signature verifies that Alice was the only one who could have sent this message.
- ▶ Signature also verifies that the message hasn't been modified in transit.

Digital Signatures

- ▶ Signing: (secret key, message) $\rightarrow$ signature

$$\text{Sign}_{sk}(m) = s$$

- ▶ Verification: (public key, message, signature) $\rightarrow$ bool

$$\text{Verify}_{pk}(m, s) = \text{true} \mid \text{false}$$

Signature properties:

- ▶ Verification of signed message succeeds:

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Signature properties:

- ▶ Verification of signed message succeeds:
 - ▶ $\text{Verify}_{pk}(m, \text{Sign}_{sk}(m)) = \text{true}$
- ▶ Unforgeability: Can't compute signature for message m that verifies with public key without corresponding secret key.
- ▶ The point:
 - ▶ Anybody with your public key can verify that you signed something!

Textbook RSA Signatures

[Rivest Shamir Adleman 1977]

Public Key pk

$N = pq$ modulus

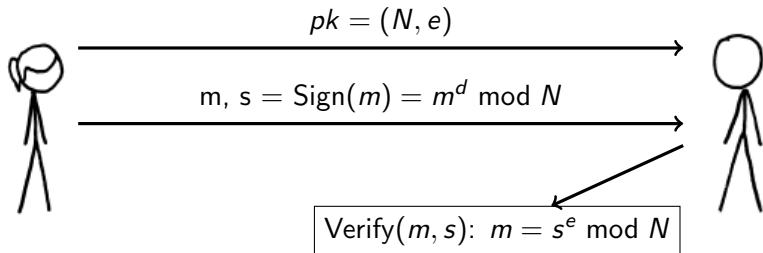
e public exponent

Secret Key sk

p, q primes

d private exponent

$(d = e^{-1} \bmod (p-1)(q-1))$



Works for the same reason RSA encryption does.

Textbook RSA signatures are super insecure

Attack: Signature forgery

1. Attacker wants $\text{Sign}(x)$.
2. Attacker computes $z = xy^e \bmod N$ for some y .
3. Attacker asks signer for $s = \text{Sign}(z) = z^d \bmod N$.
4. Attacker computes $\text{Sign}(x) = sy^{-1} \bmod N$.

Countermeasures:

- ▶ **Always use padding with RSA.**
- ▶ **Sign hash of m and not raw message m .**

Positive viewpoint:

- ▶ Blind signatures: Lots of neat crypto applications.

RSA PKCS #1 v1.5 signature padding

Most widely used padding scheme in practice

$\text{pad}(m) = 00\ 01\ [\text{FF}\ \text{FF}\ \text{FF}\ \dots\ \text{FF}\ \text{FF}]\ 00\ [\text{data}\ H(m)]$

- ▶ Signer hashes and pads message, then signs padded message using RSA private key.
- ▶ Verifier verifies using RSA public key, strips off padding to recover hash of message.

Q: What happens if a decrypter doesn't correctly check padding length?

A: **Bleichenbacher low exponent signature forgery attack.**

Bleichenbacher RSA Signature Forgery

$\text{pad}(m) = 00\ 01\ [\text{FF}\ \text{FF}\ \text{FF}\ \dots\ \text{FF}\ \text{FF}]\ 00\ [\text{data}\ H(m)]$

If victim shortcuts padding check: just looks for padding format but doesn't check length, and signature uses $e = 3$:

1. Construct a perfect cube over the integers, ignoring N , such that

$$x = 0001FF \dots FF00[\text{hash of forged message}][\text{garbage}]$$

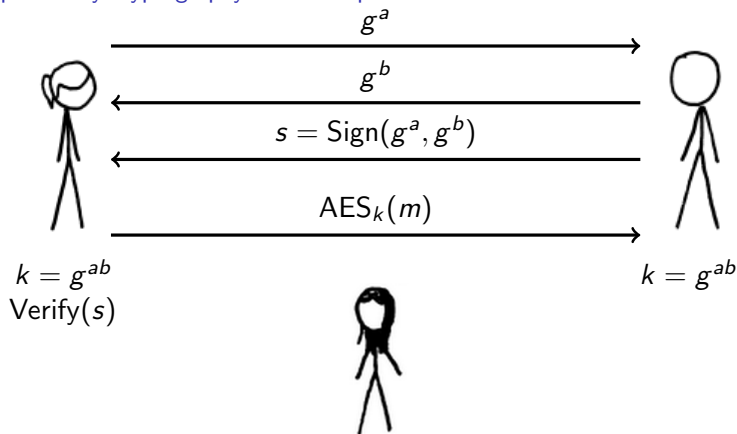
2. Compute s such that $s^3 = x$.
(Easy way: set garbage to zero and take cube root, i.e., $s = \lceil x \rceil^{1/3}$.)
3. Lazy implementation validates bad signature!

Security for RSA signatures

- ▶ Same as RSA encryption.
- ▶ Recommendation: Use ECDSA or ed25519 instead.

Putting it all together

How public-key cryptography is used in practice



- ▶ Diffie-Hellman used to negotiate shared session key.
- ▶ Alice verifies Bob's signature to ensure that key exchange was not man-in-the-middle.
- ▶ Shared secret used to symmetrically encrypt data.

Public-key cryptography and quantum computers

Right now, all public-key cryptography used in the real world involves three “hard” problems:

- ▶ Factoring
- ▶ Discrete log mod primes
- ▶ Elliptic curve discrete log

All of these problems can be solved efficiently by a general-purpose quantum computer.

Big standardization effort now to develop replacements:

- ▶ Lattice-based cryptography
- ▶ Multivariate cryptography
- ▶ Hash-based signatures
- ▶ Supersingular isogeny Diffie-Hellman

These will likely be used more in the real world in the next few years.