CSE203B Convex Optimization: Lecture 3: Convex Function

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Outlines

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1. Convex Function Definitions: Examples

$f: R^n \rightarrow R$ is convex if $dom\ f$ is a convex set and

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$$

$$\forall x, y \in dom\ f, 0 \leq \theta \leq 1$$

Example on R:

Convex Functions

Affine:	$ax + b$ on R	for any $a, b \in R$
Exponential:	e^{ax}	for any $a \in R$
Power:	x^{α} on R_{++}	for $\alpha \geq 1$ or $\alpha \leq 0$
	$\lvert x \rvert^{p}$ on R	for $p \geq 1$

Concave Functions

Affine:	$ax + b$ on R	for any $a, b \in R$
Power:	x^{α} on R_{++}	for $0 \leq \alpha \leq 1$
Logarithm:	$logx$ on R_{++}	

1. Convex Function Definitions: Examples

Concave Functions:

Log Determinant: $f(X) = \log\det X$, $dom\ f = S^n_{++}$

Proof: Let $g(t) = f(X + tV)$ $(V \in S^n)$

$$g(t) = log\ det\ (X + tV) = log\ det\ X + logdet(I + tX^{-\frac{1}{2}}VX^{-\frac{1}{2}})$$

$$= log\ det\ X + \sum_{i=1}^{n} log(1 + t\lambda_i)$$

λ_i: $eigenvalue\ of\ X^{-\frac{1}{2}}VX^{-\frac{1}{2}}$

g is concave in $t \Rightarrow f$ is concave

1. Convex Function Definitions: Examples

Example on R^n:

Affine function: $f(x) = a^T x + b$

Norms: $\|x\|_p = (\sum_{i=1}^{n} |x|^p)^{1/p}$ $for\ p \geq 1$;

$\|x\|_\infty = \max_k |x_k|$

Example on $R^{m\times n}$:

Affine function: $f(X) = tr(A^T X) = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} x_{ij}$

Spectral (max singular value):

$$f(X) = \|X\|_2 = \sigma_{max}(X) = (\lambda_{max}(X^T X))^{1/2}$$

Convex function examples: norm, max, expectation

norm: If $f: R^n \to R$ is a norm and $0 \leq \theta \leq 1$

$$f(\theta x + (1-\theta)y) \leq f(\theta x) + f\big((1-\theta)y\big) \quad \textit{triangle inequality}$$
$$= \theta f(x) + (1-\theta) f(y) \quad \textit{scalability}$$

Max function: $f(x) = \max_i x_i$, $x = [x_1, x_2, \dots, x_n]^T$

$$f(\theta x + (1-\theta)y) = \max_i (\theta x_i + (1-\theta) y_i)$$
$$\leq \theta \max_i x_i + (1-\theta) \max_i y_i$$
$$= \theta f(x) + (1-\theta) f(y) \quad \text{for } 0 \leq \theta \leq 1$$

Probability: (Expectation)

If $f(x)$ is convex with $p(x)$ a probability at x,

i.e. $p(x) \geq 0, \forall x$ and $\int p(x)\, dx = 1$

Then $f(Ex) \leq Ef(x)$,

where $Ex = \int x\, p(x) dx$

$Ef(x) = \int f(x)\, p(x) dx$

1. Definitions: Convex Function vs Convex Set

Theorem: Given $S = \{x | f(x) \leq b\}$

If function $f(x)$ is convex, then S is a convex set.

Proof: We prove by the definition of convex set.

For every $u, v \in S$, i.e. $f(u) \leq b, f(v) \leq b,$

We want to show that $\alpha u + \beta v \in S, \quad \forall \alpha + \beta = 1, \alpha, \beta \geq 0.$

We have

$$\begin{aligned} f(\alpha u + \beta v) &\leq \alpha f(u) + \beta f(v) \quad (f \text{ is convex}) \\ &\leq \alpha b + \beta b \qquad\qquad (\alpha, \beta \geq 0) \\ &= (\alpha + \beta) \cdot b = b \quad (\alpha + \beta = 1) \end{aligned}$$

Thus $\alpha u + \beta v \in S$

Remark: Convex function => Convex Set

$f(x) \leq b$ => Convex Set

$f(x) \geq b$ => ?

1.3 Views of Functions and Related Hyperplanes

Given $f(x), x \in R^n$, we plot the function in R^n and R^{n+1} spaces.

1. Draw function in R^n space

   Equipotential surface: **tangent plane** $\nabla f(\tilde{x})^T(x-\tilde{x}) = 0$ at $\tilde{x}$

2. Draw function in R^{n+1} space

   2.1 Graph of function: $\{(x,h) | x \in dom\, f, h = f(x)\}$

   hyperplane (h = $\nabla f(\tilde{x})^T(x-\tilde{x}) + f(\tilde{x})$)

   $$[\nabla f(\tilde{x})^T \quad -1]\left(\begin{bmatrix} x \\ h \end{bmatrix} - \begin{bmatrix} \tilde{x} \\ f(\tilde{x}) \end{bmatrix}\right) = 0$$

   Example: $f(x) = x^2$. We show the hyperplane with $\nabla f(x)$

   2.2. Epigraph: epi f: $\{(\mathrm{x}, t) | x \in dom\, f, f(x) \le t\}$

   A function is convex iff its epigraph is a convex set.

   Example: $f(x) = \max\{f_i(x) |\, i = 1 \dots r\}, f_i(x)\ are\ convex$.

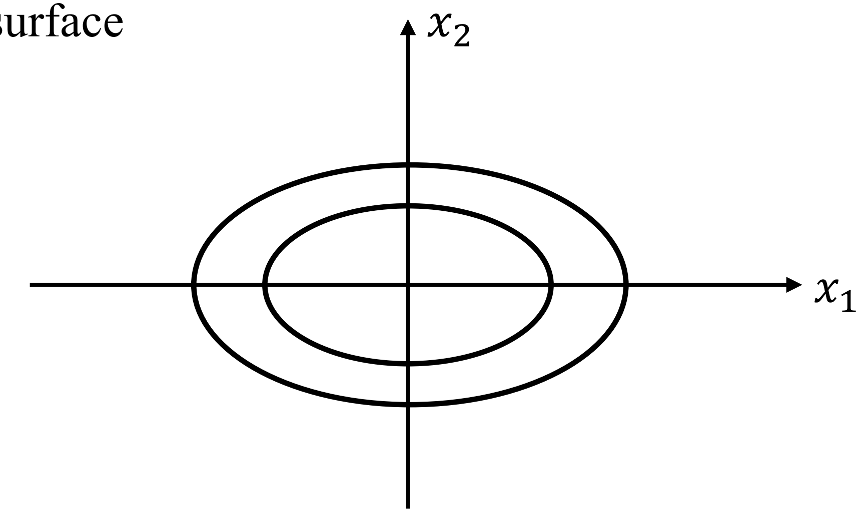
   Since epi f is the intersect of epi f_i, epi f is convex.

   Thus, function f is convex.

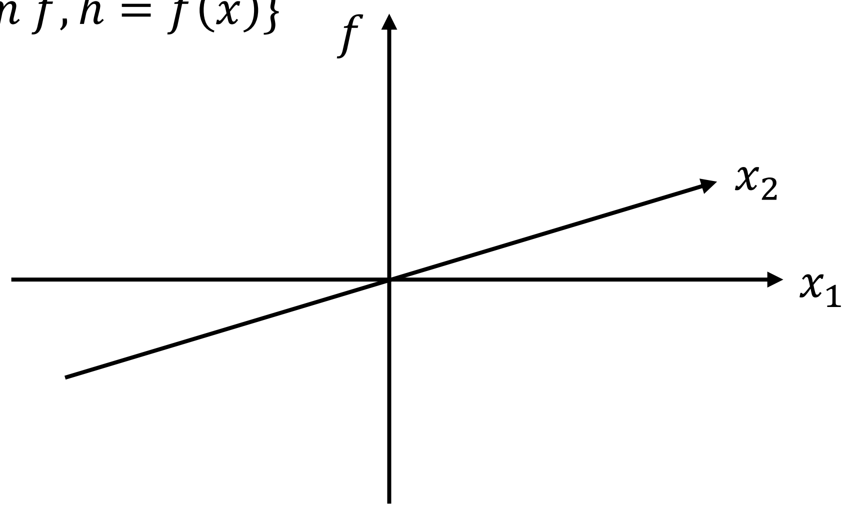
1.3 Views of Functions: Example

$Ex. f(x_1, x_2) = ax_1^2 + bx_2^2, a, b > 0.$

1. Equipotential surface



2. $\{(x, h) | x \in dom\ f, h = f(x)\}$



3. Epigraph

2. Conditions of Optimality: First Order Condition

Defintion: f is differentiable if $dom f$ is open and

$$\nabla f(x) \equiv \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \dots, \frac{\partial f(x)}{\partial x_n}\right)^T \text{ exists at each } x \in dom f$$

Theorem: Differentiable f with convex domain is convex

$$\text{iff } f(y) \geq f(x) + \nabla f(x)^{\mathrm{T}}(y - x), \forall x, y \in dom f$$

Proof => If f is convex

$$Then\ \ (1-t)f(x) + tf(y) \geq f\big((1-t)x + ty\big), \forall 0 \leq t \leq 1$$

$$t[f(y) - f(x)] \geq f\big(x + t(y - x)\big) - f(x)$$

$$f(y) - f(x) \geq \frac{1}{t}(f\big(x + t(y - x)\big) - f(x))$$

$$= \nabla f(x)(y - x) \quad when\ t \to 0$$

<= $Given\ f(y) \geq f(x) + \nabla f(x)^{\mathrm{T}}(y - x), \forall x, y \in dom f$

$$Let\ z = (1-t)x + ty$$

$$where \begin{cases} f(x) \geq f(z) + \nabla f(z)^{\mathrm{T}}(x - z) \\ f(y) \geq f(z) + \nabla f(z)^{\mathrm{T}}(y - z) \end{cases}$$

$$Thus\ (1-t)f(x) + tf(y) \geq f(z)$$

2. Conditions: Second Order Condition

Definition: f is twice differentiable if $domf$ is open and the Hessian $\nabla^2 f(x) \in S^n$

$$\nabla^2 f(x)_{ij} \equiv \frac{\partial^2 f(x)}{\partial x_i \partial x_j}, \quad i, j = 1, \dots, n \text{ exists at each } x \in domf$$

Theorem: Twice Differentiable f with convex domain is convex iff $\nabla^2 f(x) \succcurlyeq 0, \forall x \in domf$

Proof: Using Lagrange remainder, we can find a z

$$f(x + t(y - x))$$
$$= f(x) + \nabla f(x)^T t(y - x) + \frac{1}{2} t^2 (y - x)^T \nabla^2 f(z)(y - x),$$
$$\forall 0 \leq t \leq 1, z \text{ is between } x \text{ and } x + t(y - x)$$

Since the last term is always positive by assumption, the first order condition is satisfied.

2. Conditions: Second Order Condition

Example: Negative Entropy:

$f(x) = x \log x, x \in R_{++}$

$f'(x) = \frac{x}{x} + \log x = 1 + \log x, f''(x) = \frac{1}{x}$

Since $x \in R_{++}, f''(x) > 0 \Rightarrow f(x)$ is convex

Show the plot of $x \log x$

Remark:

- 1^{st} order condition can be used to design and prove the property of opt. algorithms.
- 2^{nd} order condition implies the 1^{st} order condition
- 2^{nd} order condition can be used to prove the convexity of the functions.

2. Conditions: Examples

- Quadratic Function: $f(x) = \frac{1}{2}x^T Px + q^T x + r, P \in S^n$
  $$\nabla f(x) = Px + q,\ \nabla^2 f(x) = P$$
- Least Square: $f(x) = \|Ax - b\|_2^2$
  $$\nabla f(x) = 2A^T(Ax - b),\ \nabla^2 f(x) = A^T A$$
- Quadratic over linear: $f(x, y) = \frac{x^2}{y},\ y > 0$

$$\nabla f(x, y) = \left(\frac{2x}{y}, -\frac{x^2}{y^2}\right)^T,$$

$$,$$

$$\nabla^2 f(x) = \begin{bmatrix} \frac{2}{y} & -\frac{2x}{y^2} \\ -\frac{2x}{y^2} & \frac{2x^2}{y^3} \end{bmatrix} = \frac{2}{y^3}\begin{bmatrix} y \\ -x \end{bmatrix}\begin{bmatrix} y & -x \end{bmatrix}$$

2. Conditions: Examples

- Log-sum-exp: $f(x) = \log \sum_{K=1}^{n} e^{x_k}$ (Smooth max of softmax function)

  $\nabla^2 f(x) = \frac{1}{1^T z} diag(z) - \frac{1}{(1^T z)^2} zz^T, z_k = e^{x_k}$

  $v^T \nabla^2 f(x) v = \frac{1}{(1^T z)^2} [(\sum_{i=1}^{n} z_i)(\sum_{i=1}^{n} v_i^2 z_i) - (\sum_{i=1}^{n} v_i z_i)^2] \geq 0,$

  for all $v \in R^n$ (Cauchy-Schwarz inequality)

  Thus, $f(x)$ is a convex function

Cauchy-Schwarz inequality: $[(a^T a)(b^T b) \geq (a^T b)^2,\ a_i = \sqrt{z_i}, b_i = v_i \sqrt{z_i}]$

Proof 1: Let $z = a - \frac{a^T b}{b^T b} b$, or $a = z + \frac{a^T b}{b^T b} b$

We have

$$\mathrm{a}^T \mathrm{a} = \mathrm{z}^\mathrm{T} \mathrm{z} + \frac{(a^T b)^2}{(b^T b)^2} b^T b \geq \frac{(a^T b)^2}{(b^T b)^2} b^T b = \frac{(a^T b)^2}{b^T b}$$

Proof 2: By induction

3. Operations that preserve convexity

- Nonnegative multiple: αf, where $\alpha \geq 0$, f is convex
- Sum: $f_1 + f_2$, where $f_1, and\ f_2$ are convex
- Composition with affine function: $f(Ax + b)$, where f is convex

Proof: $\nabla_x^2 f(Ax + b) = A^T \nabla_y^2 f(y|y = Ax + b)A$

E.g. $f(x) = -\sum_{i=1}^{m} \log(b_i - a_i^T x_i)$,

$$dom\ f = \{x | a_i^T x < b_i, i = 1, \ldots, m\}$$

$f(x) = \|Ax + b\|$ (if f is twice differentiable)

3. Operations that preserve convexity

- Pointwise maximum: $f(x) = \max\{f_1(x), \dots, f_r(x)\}$, f_i are convex
- Pointwise supremum:

  $g(x) = \sup_{y \in C} f(x, y)$, where $f(x, y)$ is convex in x and C is an arbitrary set

  Examples
  - $S_c(x) = \sup_{y \in C} y^T x$, for an arbitrary set C
  - $f(x) = \sup_{y \in C} \|x - y\|$, for an arbitrary set C
  - $\lambda_{max}(X) = \sup_{\|y\|_2 = 1} y^T X y, \quad X \in S^n$

3. Operations that preserve convexity: Dual norm

Example:

$$f(x) = \max_{\|y\|_2 \leq 1} y^T x$$

$$f(x) = \max_{\|y\|_1 \leq 1} y^T x$$

$$f(x) = \max_{\|y\|_p \leq 1} y^T x$$

3. Operations that preserve convexity: max function

Theorem: Pointwise maximum of convex functions is convex
Given $f(x) = \max\{f_1(x), f_2(x)\}$, where f_1 and f_2 are convex and $dom\ f = dom\{f_1\} \cap dom\{f_2\}$ is convex, then $f(x)$ is convex.

Proof: For $0 \leq \theta \leq 1,\ x, y \in dom\ f$

$$
\begin{aligned}
&f(\theta x + (1-\theta)y) \\
&\quad = \max\{f_1(\theta x + (1-\theta)y), f_2(\theta x + (1-\theta)y)\} \\
&\quad \leq \max\{\theta f_1(x) + (1-\theta)f_1(y), \theta f_2(x) + (1-\theta)f_2(y)\} \\
&\quad \leq \theta \max\{f_1(x),\ f_2(x)\} + (1-\theta)\max\{f_1(y), f_2(y)\} \\
&\quad = \theta f(x) + (1-\theta)f(y)
\end{aligned}
$$

i.e. $f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$

Thus, function $f(x)$ is convex.

3. Operations that preserve convexity: minimization

Theorem: Partial minimization

If $g(x, y)$ is convex in x and y, and a set C is convex

Then $f(x) = \min_{y \in C} g(x, y)$ is convex.

Proof: Let $y_1 \in \{y | \min_{y \in C} g(x_1, y)\}$ and $y_2 \in \{y | \min_{y \in C}(g(x_2, y)\}$,

we can write

$$
\begin{aligned}
&\theta f(x_1) + (1-\theta) f(x_2) \\
&\quad = \theta g(x_1, y_1) + (1-\theta) g(x_2, y_2) \\
&\quad \geq g(\theta x_1 + (1-\theta)x_2, \theta y_1 + (1-\theta) y_2)\ (g\ is\ convex) \\
&\quad \geq \min_{y \in C} g(\theta x_1 + (1-\theta)x_2, y)\ (C\ is\ convex) \\
&\quad = f(\theta x_1 + (1-\theta)x_2)
\end{aligned}
$$

i.e. we have $\theta f(x_1) + (1-\theta) f(x_2) \geq f(\theta x_1 + (1-\theta)x_2)$

Therefore, $f(x) = \min_{y \in C} g(x, y)$ is convex.

3. Operations that preserve convexity

Examples for Partial Minimization

Given $f(x, y) = \begin{bmatrix} x^T & y^T \end{bmatrix} \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$$x \in R^n, y \in R^m, A \in S_+^n, C \in S_+^m, \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \in S_+^{n+m}$$

Let $g(x) = \min_y f(x, y) = x^T(A - BC^+B^T)x$,

C^+: **pseudo inverse** of matrix C. (Drazin inverse, or generalized inverse)

We can claim that function $g(x)$ is convex.

Proof:

(1) $f(x, y)$ is convex

(2) $y \in R^m$ where R^m is a convex non-empty set

(3) Therefore, $g(x)$ is convex, i.e. $A - BC^+B^T \succcurlyeq 0$

3. Operations that preserve convexity

Composition:

Given $g: R^n \to R$ and $h: R \to R,$ we set $f(x) = h(g(x))$

f is convex if g convex, h convex, $\tilde{h}$ nondecreasing

g concave, h convex, $\tilde{h}$ nonincreasing

f is concave if g convex, h concave, $\tilde{h}$ nonincreasing

g concave, h concave, $\tilde{h}$ nondecreasing

Proof : for n=1

$$f''(x) = h''\big(g(x)\big)g'(x)^2 + h'\big(g(x)\big)g''(x)$$

Ex1: exp $g(x)$ is convex if g is convex

Ex2: $1/g(x)$ is convex if g is concave and positive

Note that we set $\tilde{h}(x) = \infty$ if $x \notin dom\ h$, h is convex

$\tilde{h}(x) = -\infty$ if $x \notin dom\ h$, h is concave

3. Operations that preserve convexity

Show that $h\big(g(\theta x + (1-\theta)y)\big) \le \theta h\big(g(x)\big) + (1-\theta)h(g(y))$

for the case that g, h are convex, and $\tilde{h}$ is nondecreasing

(1) g is convex

$$g(\theta x + (1-\theta)y) \le \theta g(x) + (1-\theta)g(y)$$

(2) h is nondecreasing: From (1), we have

$$h\big(g(\theta x + (1-\theta)y)\big) \le h\big(\theta g(x) + (1-\theta)g(y)\big)$$

(3) h is convex

$$h\big(\theta g(x) + (1-\theta)g(y)\big) \le \theta h\big(g(x)\big) + (1-\theta)h\big(g(y)\big)$$

(4) From (2) & (3)

$$h\big(g(\theta x + (1-\theta)y)\big) \le \theta h\big(g(x)\big) + (1-\theta)h\big(g(y)\big)$$

4. Conjugate Functions

The setting of conjugate functions starts from the following problem (which may not be convex)

$$min\, f(x)$$

subject to

$$x \leq 0$$

We convert to a function of y

$$\inf_x f(x) - y^T x$$

The conjugate function is

$$f^*(y) = \sup_x y^T x - f(x)$$

In the class, we interchange min and inf; max and sup to simplify the notation.

4. Conjugate Functions

Given $f: R^n \to R$, we have $f^*: R^n \to R$

$$f^*(y) = \sup_{x \in dom\, f} y^T x - f(x); \quad \left(-f^*(y) = \min_{x \in dom\, f} -y^T x + f(x)\right)$$

Constraint: $y \in R^n$ for which the supremum is finite (bounded)

$f^*(y)$ is called the conjugate of function f

Theorem : $f^*(y)$ is convex (pointwise maximum)

Proof : $f^*(\theta y_1 + (1-\theta) y_2) = \sup_x (\theta y_1 + (1-\theta) y_2)^T x - f(x)$

$$\leq \sup_x \left(\theta y_1^T x - \theta f(x)\right) + \sup_x ((1-\theta) y_2^T x - (1-\theta) f(x))$$

$$= \theta f^*(y_1) + (1-\theta) f^*(y_2)$$

Remark: $f^*(y)$ is convex even if $f(x)$ is not convex

4. Conjugate Functions

Suppose we have a pair $\bar{x}, \bar{y}$, such that $f^*(\bar{y}) = \bar{y}^T\bar{x} - f(\bar{x})$,

we can show that $\bar{y} = \nabla_x f(\bar{x})$ (exercise 3.40)

And the supporting hyperplane : $\bar{y}^T x - h = f^*(\bar{y})$

$$[\bar{y}^T \quad -1]\begin{bmatrix} x \\ h \end{bmatrix} = f^*(\bar{y})$$

Ex. $f(x) = x^2 - 2x, \quad x \in R$

$$f^*(y) = \sup_x yx - x^2 + 2x, y \in R$$

4. Conjugate Functions

One way to view conjugate function

$$f^*(y) = \sup_{x \in dom\, f} y^T x - f(x)$$

x : negative slack

y : shadow price (loss) to accommodate the slack

$f^*(y)$: balance between price slack product ($y^T x$) and objective function $f(x)$.

Remark: When $f^*(y)$ is unbounded, the shadow price y is not reasonable.

4. Conjugate Functions: Examples (single variable)

Ex: $f(x) = ax + b,\ x \in R$

$$f^*(y) = \sup_x (yx - ax - b)$$

(1) If $y \neq a, f^*(y) = \infty$

(2) If $y = a, f^*(y) = -b$ → $dom\ f^* = a, f^*(y) = -b$

4. Conjugate Functions: Examples (single variable)

Ex: $f(x) = -logx, \;\; x \in \mathrm{R}_{++}$

$$f^*(y) = \sup_{x \in R_{++}} yx + logx$$

(1) If $y \geq 0, \; f^*(y) = \infty$

(2) If $y < 0, \; f^*(y) = \max_{x \in R_{++}} xy + logx$

Let $g(x) = xy + logx, \; g'(x) = y + \frac{1}{x}$

If $g'(x) = 0, x = -\frac{1}{y}$

Thus, $f^*(y) = -1 + \log\left(-\frac{1}{y}\right) = -1 - \log(-y)$

→ $dom\, f^* = -R_{++}, \; f^*(y) = -1 - \log(-y)$

4. Conjugate Functions

Ex: $f(x) = e^x,\ x \in R$

$$f^*(y) = \sup_x xy - e^x$$

(1) $y < 0$: $f^*(y) = \infty$

(2) $y > 0$: Let $g(x) = xy - e^x$ → $g'(x) = y - e^x$

If $g'(x) = 0$, then $x = logy$

Thus $f^*(y) = ylogy - y$

(3) $y = 0$: $f^*(y) = 0$ → $dom\ f^* = R_+,\ f^*(y) = ylogy - y$

Therefore, we have

$$f^*(y) = ylogy - y, \text{ where } y \geq 0.$$

4. Conjugate Functions

Ex: $f(x) = xlogx,\ x \in R_+,\ f(0) = 0$

$$f^*(y) = \sup_x xy - xlogx$$

Let $g(x) = xy - xlogx$ → $g'(x) = y - logx - 1$

Suppose $g'(x) = 0$, we have $y = 1 + logx$ or $x = e^{y-1}$

Thus $f^*(y) = ye^{y-1} - e^{y-1}(y-1) = e^{y-1}$ where $y \in R$

4. Conjugate Functions

Ex: $f(x) = \frac{1}{2}x^T Qx,\ x \in R^n, Q \in S_{++}^n$

$f^*(y) = \sup_x x^T y - \frac{1}{2}x^T Qx$

Let $g(x) = x^T y - \frac{1}{2}x^T Qx \rightarrow \nabla g(x) = y - Qx$

If $\nabla g(x) = 0$, we have $x = Q^{-1}y$

Thus, $f^*(y) = \frac{1}{2}y^T Q^{-1}y$

Remark: Suppose that $f^*(\bar{y}) = \bar{y}^T\bar{x} - f(\bar{x})$ and $\nabla^2 f(\bar{x}) \succ 0$

We have $\nabla f^*(\bar{y}) = \bar{x}$ and $\nabla^2 f^*(\bar{y}) = \left(\nabla^2 f(\bar{x})\right)^{-1}$ (exercise 3.40)

4. Conjugate Functions

Basic Properties

(1) $f(x) + f^*(y) \geq x^T y$

Fenchel's inequality. Thus, in the above example

$x^T y \leq \frac{1}{2} x^T Q x + \frac{1}{2} y^T Q^{-1} y, \ \forall x, y \in R^n, Q \in S_{++}^n$

(2) $f^{**} = f$, if *f* is convex & *f* is closed (i.e. *epi f* is a closed set)

(3) If *f* is convex & differentiable, $dom\ f = R^n$

For max $y^T x - f(x)$, we have $y = \nabla f(x^*)$

Thus, $f^*(y) = x^{*T} \nabla f(x^*) - f(x^*), \ y = \nabla f(x^*)$

4. Conjugate Functions

Ex : $f(x) = log \sum_{i=1}^{n} e^{x_i} \leftrightarrow f^*(y) = \sum_{i=1}^{n} y_i log y_i$

$$f^*(y) = \sup_x y^T x - f(x) = \sup_x y^T x - log \sum_{i=1}^{n} e^{x_i}$$

Let $g(x) = y^T x - log \sum_{i=1}^{n} e^{x_i}$

$$\frac{\partial g(x)}{\partial x_i} = y_i - \frac{e^{x_i}}{\sum_{i=1}^{n} e^{x_i}} = 0$$

Thus, $y_i = \frac{e^{x_i}}{\sum_{i=1}^{n} e^{x_i}}$, i.e. $1^T y = 1$

(1) $1^T y \neq 1$ → unbounded

(2) $y_i < 0$ → unbounded

(3) $f^*(y) = \sum_{i=1}^{n} y_i log y_i$, $y \geq 0, 1^T y = 1$

5. Log-Concave, Log-Convex Functions

Log function : $log f(x), \quad f: R^n \to R, f(x) > 0, \forall x \in dom\ f$

Suppose f is twice differentiable, $dom\ f$ is convex.

$$\nabla^2 log f(x) = \frac{1}{f(x)} \nabla^2 f(x) - \frac{1}{f(x)^2} \nabla f(x) \nabla f(x)^T$$

Then

f is log-convex iff $\forall x \in dom\ f$

$$f(x) \nabla^2 f(x) \geq \nabla f(x) \nabla f(x)^T$$

f is log-concave iff $\forall x \in dom\ f$

$$f(x) \nabla^2 f(x) \leq \nabla f(x) \nabla f(x)^T$$

5. Log-Concave, Log-Convex Functions

$f : R^n \to R, \qquad f(x) > 0, \forall x \in dom\ f$

Definition: If $\log f$ is concave, f is log-concave.

Definition: If $\log f$ is convex, f is log-convex.

Ex : $f(x) = a^T x + b, dom\ f = \{x | a^T x + b\}$: log-concave

$f(x) = x^a,\ x \in R_{++}, \quad a \leq 0$: log-convex

$a > 0$: log-concave

$f(x) = e^{\alpha x}$: log convex & log-concave

$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du$: cumulative distribution function of Gaussian density log-concave

$f(x) = \frac{1}{\sqrt{(2\pi)^n det \Sigma}} e^{-\frac{1}{2}(x-\bar{x})^T \Sigma^{-1} (x-\bar{x})}$: log-concave

5. Log-Concave, Log-Convex Functions

Properties

$$\nabla^2 log f(x) = \frac{1}{f(x)}\nabla^2 f(x) - \frac{1}{f(x)^2}\nabla f(x)\nabla \mathrm{f(x)}^{\mathrm{T}}$$

$f(x)\nabla^2 f(x) \geq \nabla f(x)\nabla f(x)^T,\ \forall x \in dom\ f$: log-convex

$f(x)\nabla^2 f(x) \leq \nabla f(x)\nabla f(x)^T,\ \forall x \in dom\ f$: log-concave

Outlines

1. Definitions: Convexity, Examples & Views
2. Conditions of Optimality
   1. First Order Condition
   2. Second Order Condition
3. Operations that Preserve the Convexity
   1. Pointwise Maximum
   2. Partial Minimization
4. Conjugate Function
5. Log-Concave, Log-Convex Functions