CSE203B Convex Optimization: Chapter 4: Problem Statement

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Convex Optimization Formulation

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1.1 Introduction: Problem Statement

Formulation: One of the most critical processes to conduct a project.

Format:
$$
\begin{aligned}
&\min\ f_0(x)\\
&s.t.\ f_i(x) \le 0\ \ i = 1, \dots, m\\
&\qquad h_i(x) = 0\ \ i = 1, \dots, p\ (Ax = b \text{ Affine set})
\end{aligned}
$$

$$
\begin{aligned}
&x \in R^n\\
&D_{f_0}\ f_0 : R^n \to R\\
&D_{f_i}\ f_i : R^n \to R\\
&D_{h_i}\ h_i : R^n \to R\\
&f_0, f_i, \dots, f_m \textit{ are convex}
\end{aligned}
$$

Domain of functions, $D = \cap_{i=0,m} D_f \cap_{i=0,p} D_{h_i}$.

Feasible Set: The subset of D that satisfies the constraints

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1.2 Constraints: Eliminating Equality Constraints

$$
\begin{aligned}
&\min\ f_0(x)\\
&s.t.\ f_i(x) \le 0\ \ i = 1, \dots, m\\
&\qquad Ax = b
\end{aligned}
$$

a. Convert $\{x | Ax = b\}$ *to* $\{Fz + x_0 | z \in R^k\}$
b. We have an equivalent problem

$$
\begin{aligned}
&\min\ f_0(Fz + x_0)\\
&s.t.\ f_i(Fz + x_0) \le 0,
\end{aligned}
$$

where $Ax_0 = b$, *and* matrix F contains columns of null space basis

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1.2 Constraints: Slack Variables

$$\begin{aligned} &\min\ f_0(x) \\ &s.t.\ f_i(x) \le 0, i = 1, \dots, m \\ &\quad\ \ Ax = b \end{aligned}$$

Add slack variables to convert to an equivalent problem

a. Convert the objective function with variable t

$$\begin{aligned} &\min\ t \\ &s.t. f_0(x) - t \le 0 \\ &\quad f_i(x) \le 0, i = 1, \dots, m \\ &\quad A^T x = b \end{aligned}$$

b. Convert the inequality with variables s_i

$$\begin{aligned} &\min\ f_0(x) \\ &s.t. f_i(x) + s_i = 0 \\ &\quad A^T x = b \\ &\quad s_i \in R_+, i = 1, \dots, m \end{aligned}$$

1.3 Objective Functions: Absolute values and Softmax

a. Absolute values

$$\begin{aligned} &|f_i(x)| \le b \\ &\Rightarrow f_i(x) \le b \text{ and} \\ &\ \ -f_i(x) \le b \end{aligned}$$

b. Maximum values

$$\max\{f_1, f_2, \dots, f_m\}$$

$\begin{cases} \min t \\ f_i \le t \ \forall i \end{cases}$

$$\text{Softmax: } \frac{1}{\alpha}\log\left(e^{\alpha f_1} + e^{\alpha f_1} + \dots + e^{\alpha f_m}\right)$$

$Example$: $\max\{1, 5, 10, 2, 3\} \Rightarrow$ Softmax

$$\frac{1}{\alpha}\log(e^{\alpha} + e^{5\alpha} + e^{10\alpha} + e^{2\alpha} + e^{3\alpha}) \approx 10$$

2.1 Optimality Conditions: Local vs. Global Optima

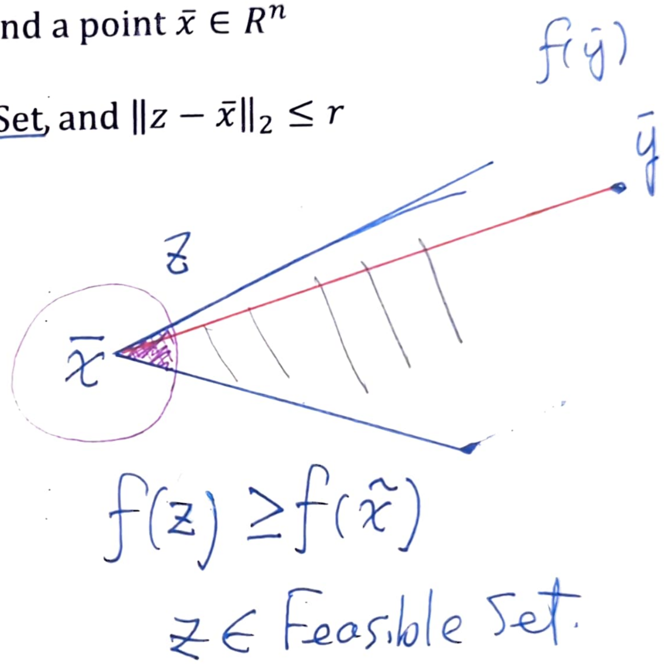
Definition: Local Optima

Given a convex optimization problem and a point $\bar{x} \in R^n$

If there exists a $r > 0$

$s.t.$ $f_0(z) \geq f_0(\bar{x})$ for all $z \in$ Feasible Set, and $\|z - \bar{x}\|_2 \leq r$

Then $\bar{x}$ is a local optimum.



2.1 Optimality Conditions: Local vs. Global Optima

Theorem: Given a convex opt. problem

If $\bar{x}$ is a local optimum, then $\bar{x}$ is a global optimum

Proof: By contradiction

Suppose that $\exists \bar{y} \in \textit{Feasible Set}$

$s.t.$ $f_0(\bar{x}) > f_0(\bar{y})$ $\qquad 0 < \theta \leq 1$

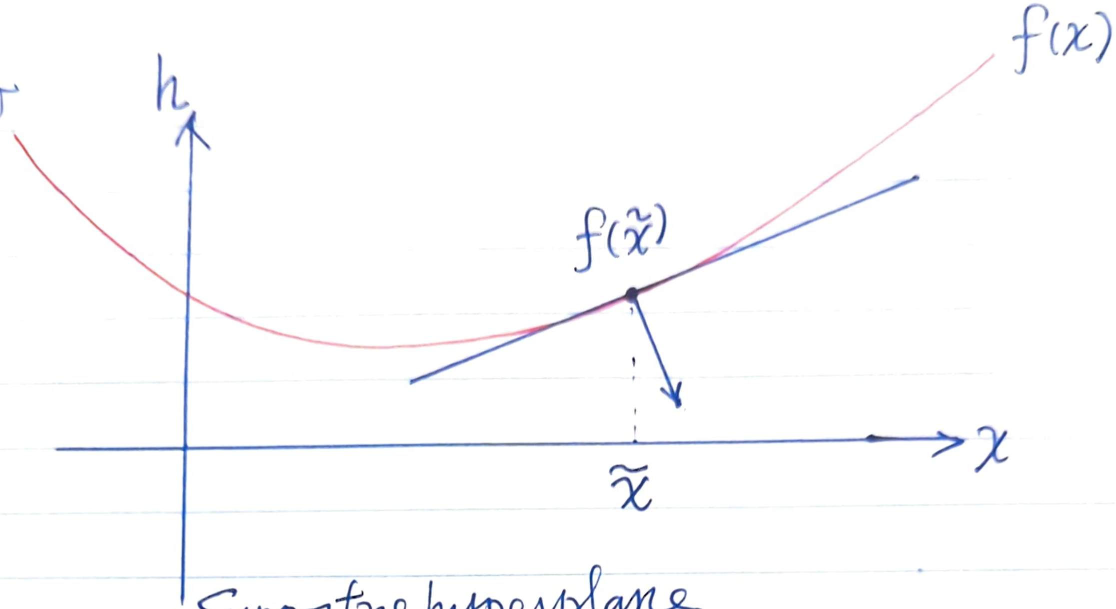
We have $f_0(\bar{x}) > (1-\theta) f_0(\bar{x}) + \theta f_0(\bar{y})$ (*by assumption*)

$> f_0((1-\theta)\bar{x} + \theta\bar{y})$ (f_0 *is convex*)

And $(1-\theta)\bar{x} + \theta\bar{y}$ is feasible (Feasible set is convex)

The inequality contradicts to the assumption of local optima.

R^{n+1} view



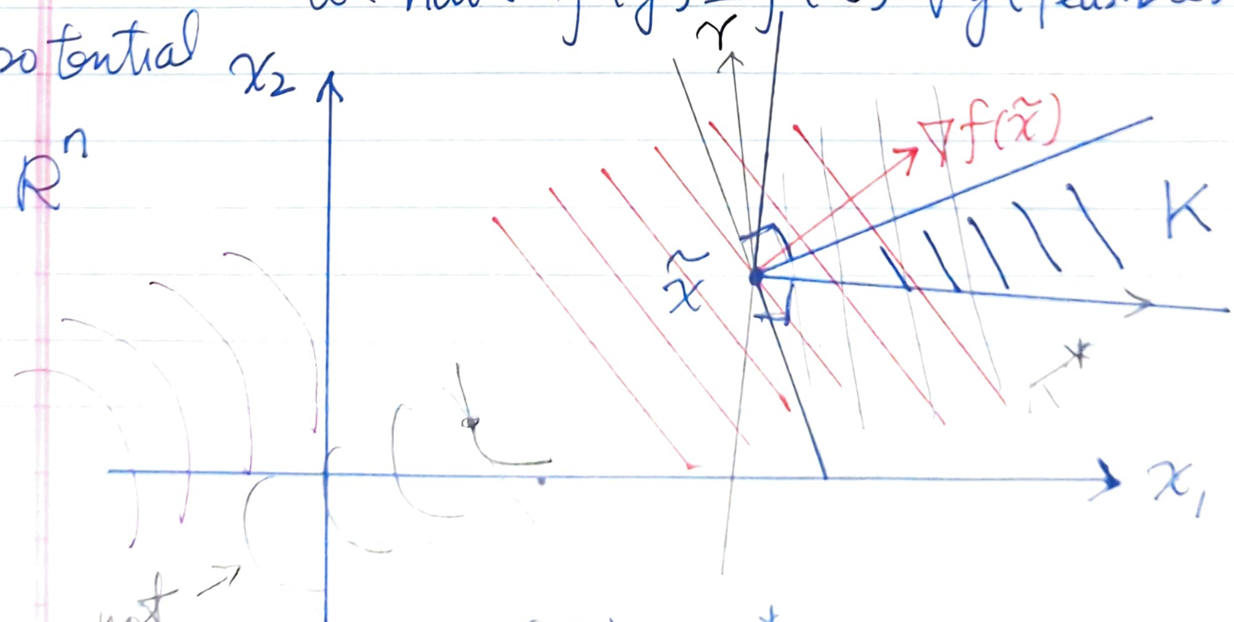
Supporting hyperplane

$$f(y) \geq h(y) = f(\tilde{x}) + \nabla f(\tilde{x})^T (y - \tilde{x})$$

If $\nabla f(\tilde{x})^T (y - \tilde{x}) \geq 0 \quad \forall y \in$ Feasible Set

we have $f(y) \geq f(\tilde{x}) \quad \forall y \in$ Feasible Set.

Equipotential view R^n



not a convex function

If $\nabla f(\tilde{x}) \in K^*$ i.e.

$\nabla f(\tilde{x})^T (y - \tilde{x}) \geq 0 \quad \forall (y - \tilde{x}) \in K$.

Then $\tilde{x}$ is a local opt. sol.

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2.2 Optimality Criterion for Differentiable $f_0(x)$

Theorem: If $\nabla f_0(\tilde{x})^T(x-\tilde{x}) \geq 0$, for a given $\tilde{x}$ $\in$Feasible Set and for all $x \in$ Feasible Set, then x is optimal.
(i. e. $K = \{x - \tilde{x} | x \in feasible\ set\}, \nabla f_0(\tilde{x}) \in K^*$)

Proof: From the first order condition of convex function, we have $f_0(x) \geq f_0(\tilde{x}) + \nabla f_0(\tilde{x})^T(x-\tilde{x})$.
Given the condition that $\nabla f_0^T(\tilde{x})(x-\tilde{x}) \geq 0$, $\forall x$ in feasible set.
We have $f_0(x) \geq f_0(\tilde{x})$, $\forall x$ in feasible set, which implies that $\tilde{x}$ is optimal.

Remark: $\nabla f_0^T(\tilde{x})(x-\tilde{x}) = 0$ is a supporting hyperplane to feasible set at $\tilde{x}$, because $\nabla f_0(\tilde{x})^T(x-\tilde{x}) \geq 0$, for all $x \in$ Feasible Set.

$5 \geq \underline{2} + 3$

$5 \geq 7 \underbrace{- 4}$

2.2.1 Optimality Criterion without Constraints

Theorem: For problem min $f_0(x), x \in R^n$, where f_0 is convex, the optimal condition is $\tilde{x}$, when $\nabla f_0(\tilde{x}) = 0$.

Proof: ($\nabla f_0(\tilde{x}) = 0 \Rightarrow$ Optimality)
Since $f_0(y) \geq f_0(x) + \nabla f_0(x)^T(y-x), \forall x, y \in R^n$ (first order condition of convex function)
We have $f_0(y) \geq f_0(\tilde{x})$, for all $y \in R^n$(assumption $\nabla f_0(\tilde{x}) = 0$)
Therefore, $\tilde{x}$ is an optimal solution.
($\nabla f_0(\tilde{x}) = 0 \Leftarrow$ Optimality)
By contradiction (Taylor's exp.)

2.2.2 Opt. with Inequality Constraints

Problem: Min $f_0(x)$

$$s.t.\ Ax \le b,\ A \in R^{m \times n}$$

$a_1^T x \le b_1$

$a_2^T x \le b_2$

$a_m^T x \le b_k$

Suppose that $A\bar{x} = b$ (one particular case).
Replace $x = \bar{x} + u$.

We can write $\begin{cases} \min f_0(\bar{x} + u) \\ Au \le 0 \end{cases}$

Opt. condition: $\nabla f_0(x)^T u \ge 0,\ \forall \{u | Au \le 0\} \equiv K$

In other words,

$$\nabla f_0(\bar{x}) \in K^* \ of\ K = \{u | Au \le 0\}\ and\ K^* = \{-A^T v | v \ge 0\}$$

i.e. $\nabla f_0(\bar{x}) = -A^T v, \exists v \in R_+^m$

Or $\nabla f_0(\bar{x}) + A^T v = 0,\ v \ge 0.$

$n \times m$

2.2.3 Opt. with Equality Constraints

Problem: $\begin{cases} \min f_0(x) \\ s.t.\ Ax = b \end{cases}$

Replace $x = \bar{x} + u$, where $A\bar{x} = b$,

we have $\begin{cases} \min f_0(\bar{x} + u) \\ Au = 0 \end{cases}, K = \{u | Au = 0\}$

Opt. Cond. $\nabla f_0(\bar{x}) \in K^*,\ K^* = \{A^T v | v \in R^p\}$

Or $\nabla f_0(\bar{x}) + A^T v = 0$

Let $K_1 = \{u | Au \ge 0\}$

$K_2 = \{u | -Au \ge 0\}$

$K = K_1 \cap K_2 = \{u | Au \ge 0, -Au \ge 0\}$

We have

$$K^* = (K_1 \cap K_2)^* = \{A^T v_1 + (-A)^T v_2 | v_1, v_2 \ge 0\}$$
$$= \{A^T v | v \in R^p\}$$

2.2.3 Opt. with Equality Constraints: Example

$$\min_x f(x) = x_1^2 + x_2^2$$
$$s.t.\ [2\ 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3$$

We can derive $x^* = (x_1^*, x_2^*) = (\frac{6}{5}, \frac{3}{5})$

$$\nabla f(x^*) = \begin{bmatrix} 2x_1^* \\ 2x_2^* \end{bmatrix} = \begin{bmatrix} \frac{12}{5} \\ \frac{6}{5} \end{bmatrix},\ \nabla f(x^*) + A^T v = \begin{bmatrix} \frac{12}{5} \\ \frac{6}{5} \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \times \left(-\frac{6}{5}\right) = 0$$

New Problem:

$$\begin{matrix} \nabla f(x) + A^T v = 0 \\ Ax = b \end{matrix} \Rightarrow \begin{matrix} \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} v = 0 \\ [2\ 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3 \end{matrix}$$

2.3 Quasiconvex Functions

$f: R^n \rightarrow R$ is called quasiconvex (unimodal)
sublevel set $S_t = \{x | x \in dom\ f,\ f(x) \leq t\}$
if its domain and all sublevel sets $S_t, \forall t \in R$ are convex,
$f: R^n \rightarrow R$ is called quasiconcave if $-f$ is quasiconvex.

$f(x)$ quasiconvex and quasiconcave → quasilinear

Ex: $\log x, x \in R_{++}$