3.2 Quadratic Opt. Problems (SOCP)

SOCP : (Second-Order Cone Program)

$$\min f^T x$$
$$s.t.\ \ ||A_i x + b_i||_2 \le c_i^T x + d_i, i = 1, \dots, m$$
$$Fx = g$$

$SOCP$: $(Ax + b, c^T x + d)$ lies in the second order cone

$$\{(y,t) \mid ||y||_2 \le t, y \in R^k\}$$

SOCP viewed as a Semidefinite Program Problem

SOCP constraint: $||Ax + b||_2 \le c^T x + d$

can be expressed as a Semidefinite Program constraint:

$$\begin{bmatrix} (c^T x + d)I & Ax + b \\ (Ax + b)^T & c^T x + d \end{bmatrix} \succcurlyeq 0 \rightarrow \begin{bmatrix} (c^T x + d)I & 0 \\ 0 & c^T x + d - (Ax+b)^T \cdot \end{bmatrix}$$

$$\underline{(c^T x + d)^{-1} I (Ax+b)}$$

$$\begin{bmatrix} I & 0 \\ -E^{-1}F & I \end{bmatrix} \begin{bmatrix} E & F \\ F^T & G \end{bmatrix} \begin{bmatrix} I & -E^{-1}F \\ 0 & I \end{bmatrix} = \begin{bmatrix} E & 0 \\ 0 & G - F^T E^{-1} F \end{bmatrix}$$

$$\Downarrow$$

$$\underline{(c^T x + d)^2 - (Ax+b)^T (Ax+b)}$$

R^T $\qquad\qquad$ R

3.3 Geometric Programming

$$f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \dots x_n^{a_{nk}}, \qquad c_k > 0, a_{ik} \in R, x \in R_{++}^n$$

Each term is called monomial

$f(x)$ is called posynomial

Obj: $f_0(x) = x_1 x_2^{-2} + 2 x_1^2 x_3^{-1/3} + 1.5 x_2^{-1} x_3^{-2}$

Geometric Program:

$\min f_o(x)$

s.t.

$f_i(x) \le 1, i = 1, \dots, m$

$h_i(x) = 1, i = 1, \dots, p$

$x > 0$

s.t. $f_1(x) = x_2^{-1} x_4 + x_1^{-1/2} x_3 + \cdots \le b \; 1$

$\vdots$

$h_1(x) = x_1^{-1/3} x_2^{4} x_3^{4.5} = \cancel{b} \; 1$

f_is are posynomials

h_is are monomials

$x_i \Rightarrow e^{\log x_i} \xRightarrow{y = \log x_i} e^{y}$

3.3 Geometric programing in convex form

$x_i^{a_i} \to e^{a_i y_i}$

monomial $f(x) = cx_1^{a_1} \dots x_n^{a_n},\ x \in R_{++}^n$

$\log x_i \to y_i$ $\log f(e^{y_1}, \dots, e^{y_n}) = a^T y + b,\ \ b = \log c$ $\quad c x_1^{a_1} \cdots x_n^{a_n}$

$x_i \to e^{y_i}$ $\quad \to e^{\log c + a_1 y_1 + a_2 y_2 + \cdots a_n y_n}$

polynomial $f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} \dots x_n^{a_{nk}}$

$$\log f(e^{y_1} \dots e^{y_n}) = \log \sum_{k=1}^{K} e^{a_k^T y + b_k},\ \ b_k = \log c_k$$

$\xrightarrow{\log} \log c + a_1 y_1 + a_2 y_2 \cdots + a_n y_n$

Geometric program transform

$$\min \log\left(\sum_{k=1}^{K_0} e^{a_{ok}^T y + b_{ok}}\right)$$

$$\textit{subject to}\ \ \log \sum_{k=1}^{K_i} e^{a_{ik}^T y + b_{ik}} \le 0,\ i = 1, \dots, m$$

$$Gy + d = 0$$

3.4 Generalized Inequality Constraints

$\min f_o(x)$

$s.t.\ \ f_i(x) \preccurlyeq_{K_i} 0$

$Ax = b$

$(x \preccurlyeq_K y \to y - x \in K)$

$f_i(\bar{x}) \succeq_{K_i} f_i(\tilde{x})$

iff $f_i(\bar{x}) - f_i(\tilde{x}) \in K_i$

Semidefinite Programming (SDP)

$\min c^T x$

$s.t.\ \ x_1 F_1 + \dots + x_n F_n + G \preccurlyeq 0$ ← negative semidefinite

$Ax = b$

$G, F_1, \dots, F_n \in S^k, A \in R^{p \times n}$

Standard Form SDP

$\min tr(CX)$

$s.t.\ \ tr(A_i X) = b_i,\ i = 1, \dots, p$

$X \succcurlyeq 0$

$C, A_1, \dots, A_p \in S^n, X \in S^n$

CSE203B Convex Optimization: Chapter 5 Duality

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Chapter 5 Duality

- Primal and Dual Problem (Mechanism)
  - Primal Problem
  - Lagrangian Function
  - Lagrange Dual Problem
- Examples (Primal Dual Conversion Procedure)
  - Linear Programming
  - Quadratic Programming
  - Conjugate Functions (Duality)
  - Entropy Maximization
- Interpretation (Duality) (Theory)
  - Saddle-Point Interpretation
  - Geometric Interpretation
  - Slater's Condition
  - Shadow-Price Interpretation
- KKT Conditions (Optimality Conditions)
- Sensitivity (Shadow-Price) (Perturbation)
- Generalized Inequalities

Duality

<u>Primal Problem (Solution x is feasible)</u>

$\min\ f_0(x) \qquad x \in R^n$

$s.t.\ f_i(x) \le 0 \quad i = 1, \dots, m \qquad domain\ D$

$\qquad h_i(x) = 0 \quad i = 1, \dots, p \qquad = dom\ f_0 \cap_i\ dom\ f_i\ \cap_i dom\ h_i$

Notation: Opt: $x^*, p^* = f_0(x^*)$

<u>Lagrangian: $L: R^n \times R^m \times R^p \to R$</u>

$$L(x, \lambda, v) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} v_i h_i(x)$$

λ_i, v_i: *Lagrange multiplier*, $\lambda_i, \in R_+, v_i \in R$.

<u>Lagrange dual function (Solution x may not be feasible)</u>

$$g(\lambda, v) = \inf_{x \in D} L(x, \lambda, v) = \min_x f_0(x) + \sum_i \lambda_i f_i(x) + \sum_i v_i h_i(x)$$

<u>Dual Problem</u>

$$max_{\lambda, v}\ g(\lambda, v) \quad s.t.\ \lambda \in R_+^m, v \in R^p$$

Shadow Price.

$$-g(\lambda, v) = \max_x -\sum \lambda_i f_i(x) - \sum v_i h_i(x) - f_0(x)$$

$$= \max_x \underbrace{\underbrace{-\sum \lambda_i f_i(x) - \sum v_i h_i(x)}_{\text{convex } (\lambda_i, v_i)} - f_0(x)}_{\text{convex}(\lambda_i, v_i)}$$

Duality

<u>Dual Problem (Solution x may not be feasible)</u>

$max_{\lambda, v}\ g(\lambda, v) \quad s.t.\ \lambda \succcurlyeq 0$

1. $g(\lambda, v)$ *is concave*
2. $g(\lambda, v) \le p^*$ an optimal value where $\lambda \succcurlyeq 0$

Proof 1: By definition of $g(\lambda, v)$ and the convexity of pointwise max operation on convex functions.

Proof 2: For any feasible $\tilde{x}$ and $\lambda \succcurlyeq 0$

$f_0(\tilde{x}) \ge L(\tilde{x}, \lambda, v)$ (Because $\sum \lambda_i f_i(\tilde{x}) + \sum v_i h_i(\tilde{x}) \le 0$)

$\qquad L(\tilde{x}, \lambda, v) \ge g(\lambda, v)$ by definition of $g(\lambda, v)$

Thus $p^* = f_0(x^*) \ge g(\lambda, v)$