Discussion Week 2

Meng Song

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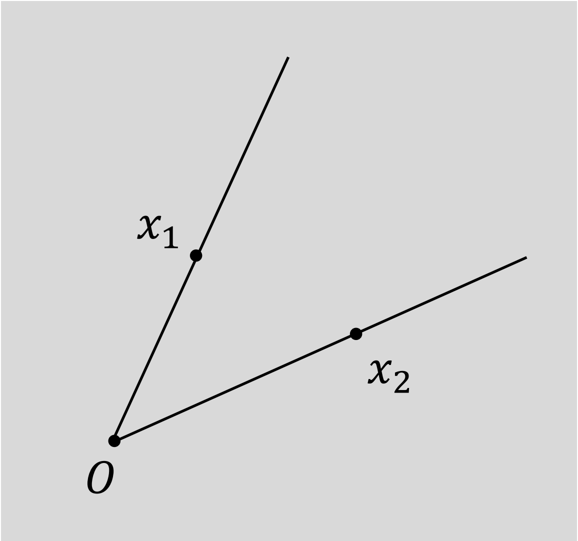
Outline

- Represent a set as a combination of points
- Solution set of $Ax = b$
- Solution set of $Ax \leq b$
- Dual cone

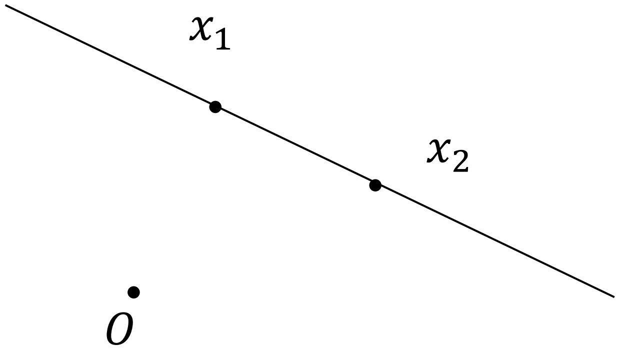
Vector space: linear combination

- $C = \{\theta_1 x_1 + \theta_2 x_2 \mid \theta_1, \theta_2 \epsilon \mathbb{R}\}$
- C is a 2-dimensional vector space
  - A vector space must contain the origin.



Affine set: affine combination

- $C = \{\theta_1 x_1 + \theta_2 x_2 \mid \theta_1 + \theta_2 = 1, \theta_1, \theta_2 \in \mathbb{R}\}$
- C is a 1-dimensional line
  - C may not contain the origin.



Hyperplane

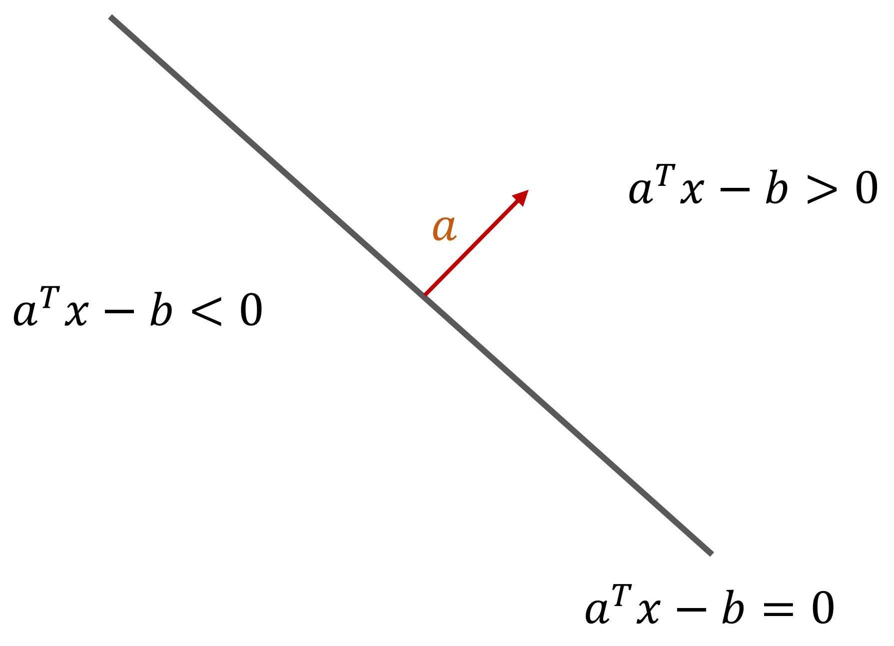
- A hyperplane H is an affine set whose dimension is one less than that of its ambient space
- The degree freedom of H is $n - 1$ when H is a hyperplane in $\mathbb{R}^n$
- Examples
  - A line in $\mathbb{R}^2$
  - A plane in $\mathbb{R}^3$

Hyperplane

- Let H be a hyperplane in $\mathbb{R}^n$, it corresponds to a linear equation
- $H = \{x|a^T x = b\} = \{x|a^T(x - x_0) = 0\}$
  - x is a n-dimensional vector representing n variables
  - a is a n-dimensional vector aligning with the normal direction of H
  - x_0 is a point on H, $x - x_0$ is always orthogonal to a
  - b is a number

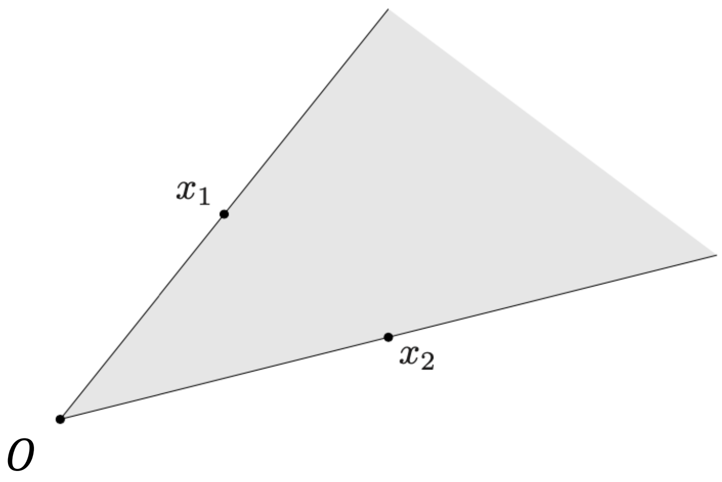
Hyperplane

- Let H be a hyperplane in $\mathbb{R}^n$
  - Let $f(x) = a^T(x - x_0) = a^T x - \mathrm{b}$
    - $\nabla f(x) = a$
  - H separates $\mathbb{R}^n$ into three parts:
    - H
    - $S_1 = \{x \mid f(x) > 0\}$
    - $S_2 = \{x \mid f(x) < 0\}$
  - Two half spaces: $S_1 \cup H$ and $S_2 \cup H$



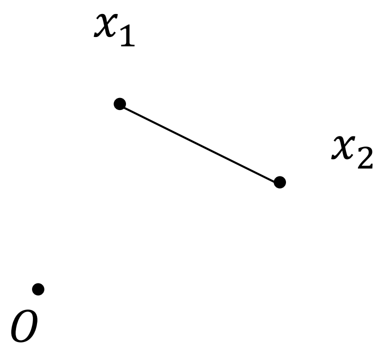
Cone: conic combination

- $C = \{\theta_1 x_1 + \theta_2 x_2 \mid \theta_1, \theta_2 \epsilon \mathbb{R}_+\}$
- C is a cone
  - C must contain the origin.



Convex combination

- $C = \{\theta_1 x_1 + \theta_2 x_2 \mid \theta_1 + \theta_2 = 1, \theta_1, \theta_2 \in \mathbb{R}_+\}$
- C is a line segment
  - C may not contain the origin.

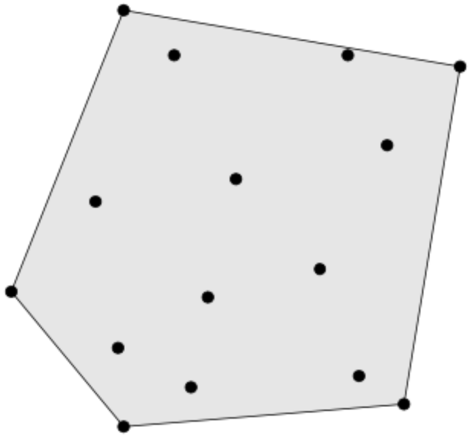


Convex set

- A set C is convex if and only if C contains every convex combination of its points.
  - Affine set
  - Vector space
  - Convex cone
  - …

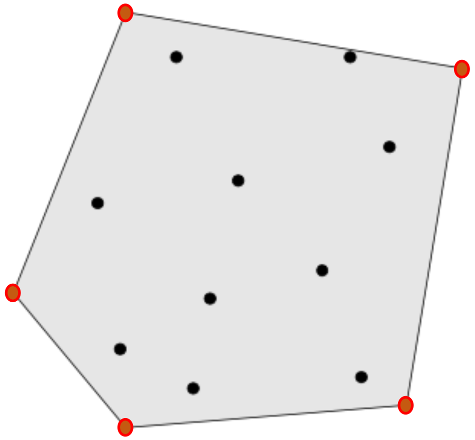
Convex hull

- The convex hull of a set C is the set of all convex combinations of points in C
  - A minimal convex set enclosing C



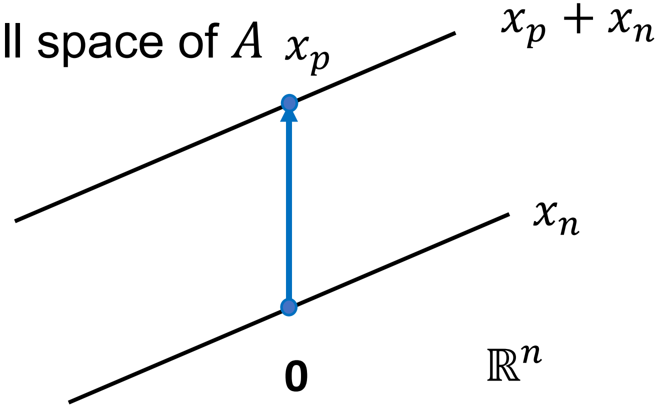
Vertices of a convex set

- Let C be a convex set. For a $z \in C$, it is a vertex of C if there is no distinct $x, y \in C$ and $\theta \in (0,1)$ such that $z = (1-\theta)x + \theta y$.



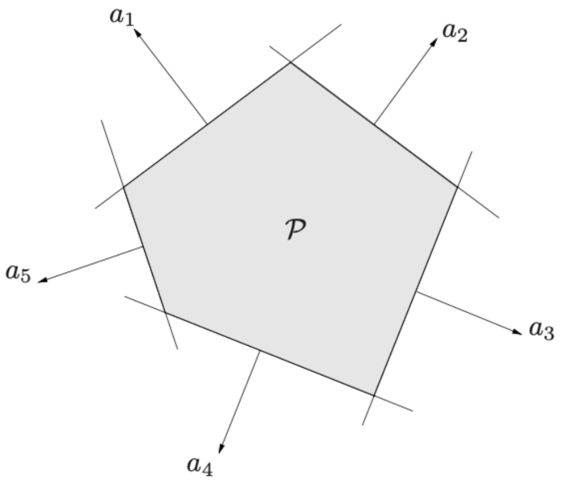
Solve $Ax = b$

- Let A be a $m \times n$ matrix, $Ax = b$ containing m equations $a_i^T x = b_i$ with n variables $x = [x_1, x_2, \dots, x_n]^T$
- The solution set is the intersection of m hyperplanes
  - No solution: No point lie on all m hyperplanes
  - One unique solution: m hyperplanes intersect at exactly one point
  - Infinitely many solutions: m hyperplanes intersect at k-dimensional affine set $(1 \le k \le n-1)$
    - The affine set (complete solution) is a translation of the null space of A
      - $Ax_n = 0$
      - $Ax_p = b$



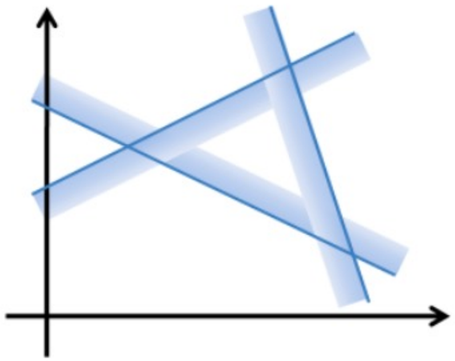
Solve $Ax \leq b$

- Let A be a $m \times n$ matrix, $Ax \leq b$ containing m inequalities $a_i^T x \leq b_i$ with n variables $x = [x_1, x_2, \dots, x_n]^T$
- The solution set is the intersection of m halfspaces, i.e. a polyhedron
  - Enclosed by k hyperplanes ($0 \leq k \leq m$)

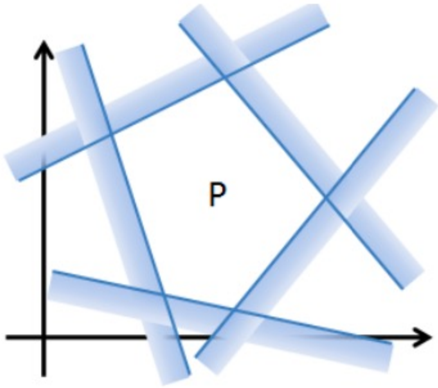


Polyhedron

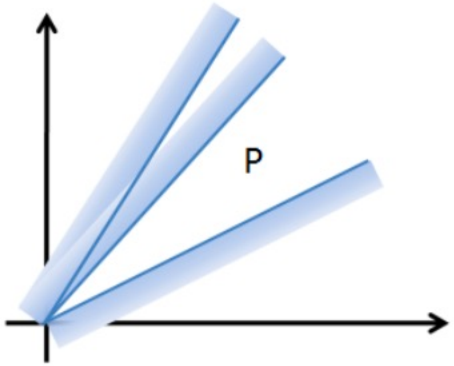
- Empty set
- Polytope: A bounded polyhedron
- Cone
- Combination of cone and polytope



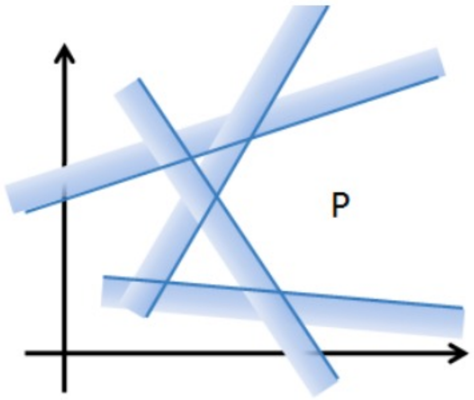
empty set



polytope



cone



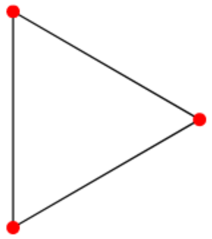
combination of cone and polytope

Simplex

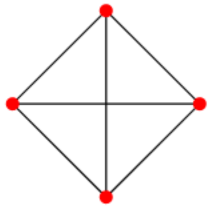
- A n-simplex is a n-dimensional **polytope** which is the convex hull of its $n + 1$ vertices.

0-simplex

1-simplex



2-simplex



3-simplex

Two representations of the solution set

- Explicit / Enumeration-oriented
  - Represent the solution set by its vertices

- Implicit / Qualification-oriented
  - Represent the solution set by its n-1 dimensional faces

Two representations of the solution set

Qualification-oriented

- $\{\, x|Ax = b, x \in\ R^n\}$
- $\{\, x|Ax \leq b, x \in\ R^n\}$
- $\{\, x|Ax \leq 0, x \in\ R^n\}$

Enumeration-oriented

- Affine set: $\{\, U\theta|1^T\theta = 1, \theta \in R_+^k\}$
- Polyhedron: $\{\, U\theta|1^T\theta = 1, \theta \in R_+^k\}$
- Cone: $\{\, U\theta|\ \theta \in R_+^k\}$

$Ax \leq b$: From qualification to enumeration

- Let A be a $m \times n$ matrix, given $Ax \leq b$, find the vertices of its solution set
  - Enumerate all C_m^n possibilities
    - Set n of the m inequalities to equalities (by turns) and get a $n \times n$ system $\tilde{A}x = \tilde{b}$,
    - When $rank(\tilde{A}) = n$, solve $\tilde{A}x = \tilde{b}$ and get a candidate vertex x
    - x is a real vertex when x satisfies the remaining $m - n$ inequalities, discard if doesn't
- The solution set can be represented as a convex combination of the k vertices
  - $0 \leq k \leq C_m^n$

$Ax \leq b$: From qualification to enumeration

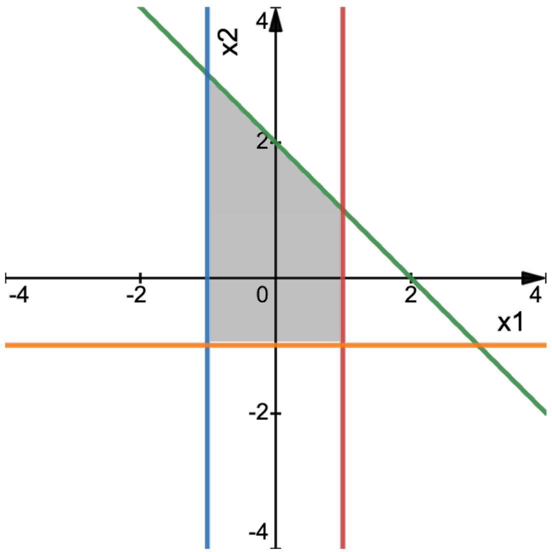
- Note that in homework 2, $x \in R_+^6$, so $x_i \geq 0$ are also inequality constraints

Example: From qualification to enumeration

Qualification-oriented:

$$\{\, x | Ax \leq b, x \in \ R^2\}$$

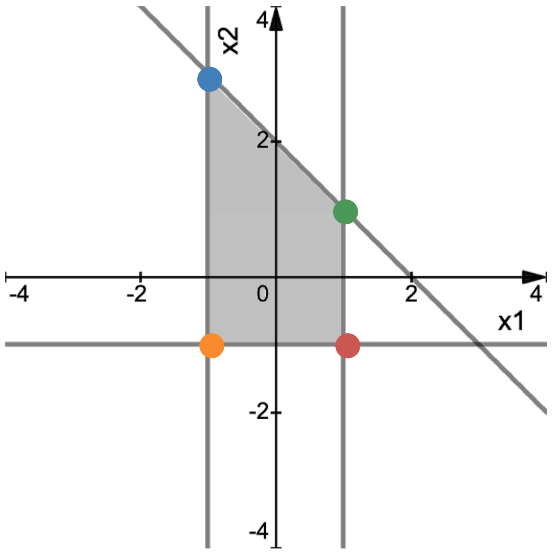
$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$



Enumeration-oriented:

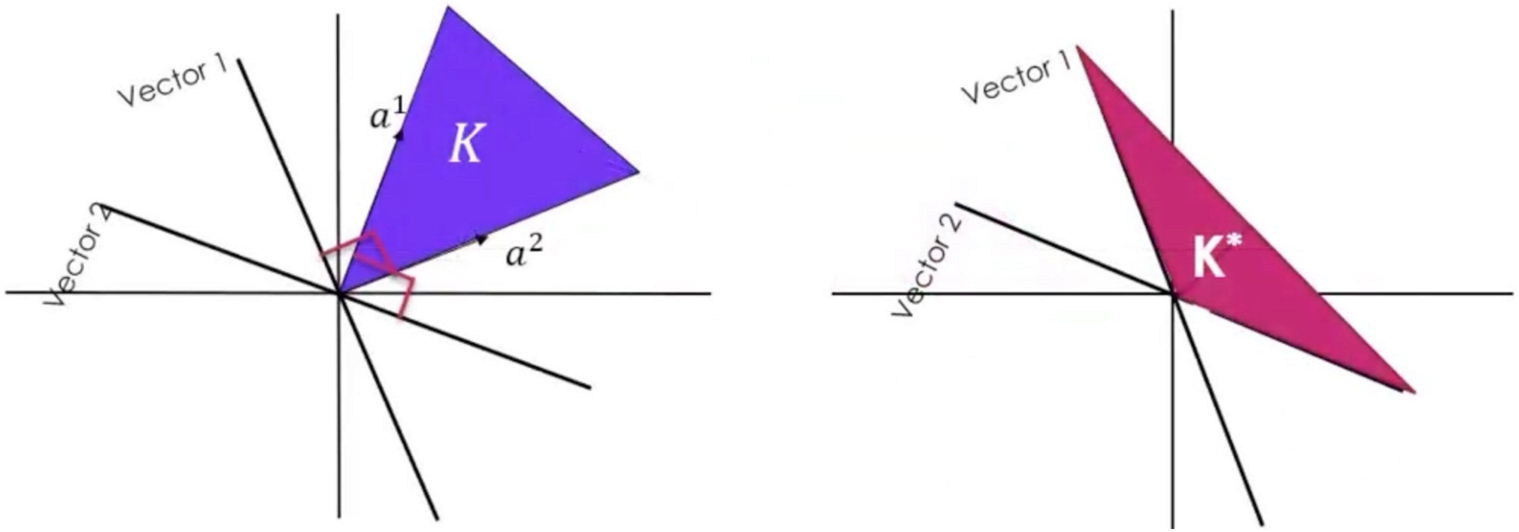
$$\{\, U\theta | 1^T\theta = 1, \theta \in R_+^4\}$$

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}$$



Dual cone

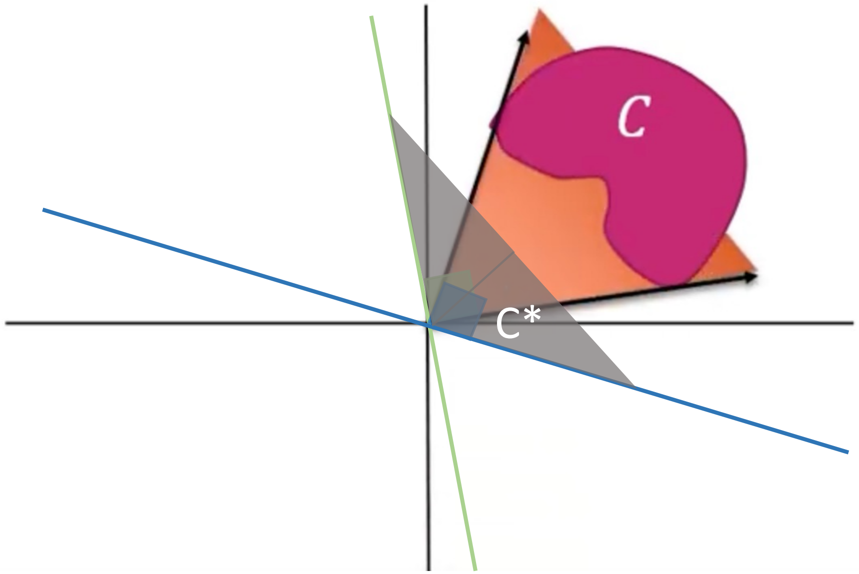
- Let C be a set, C^* is called the dual cone of C
  - $C^* = \{\, y \mid x^T y \geq 0, \forall x \in C\}$



Example: C is a cone

Dual cone

- C^* is always convex even if the original set C is non-convex



Example: C is not a cone

Q & A

Thank you !