Outline

- Standard form
- Linear program
- Graph embedding
- Code demo
Optimization problem in standard form

\[
\min_x f_0(x) \\
 f_i(x) \leq 0, \quad i = 1, 2, \ldots, m \\
 h_i(x) = 0, \quad i = 1, 2, \ldots, p
\]

- Domain of the problem \( \mathcal{D} = \bigcap_{i=0}^m \text{dom } f_i \cap \bigcap_{i=1}^p \text{dom } h_i \)
- The optimal value \( p^* = \inf \{ f_0(x) | f_i(x) \leq 0, h_i(x) = 0 \} \)
- If the problem is infeasible, \( p^* = \inf \emptyset = +\infty \)
- If there are feasible points \( x_k \) s.t. \( f_0(x_k) \to -\infty \) as \( k \to \infty \) then the problem is unbounded below
A feasible point \( x \) is locally optimal if there is an \( R > 0 \) s.t. \( x \) solves the following optimization problem

\[
\begin{align*}
\min_z & \quad f_0(z) \\
\text{s.t.} & \quad f_i(z) \leq 0, \quad i = 1, 2, \ldots, m \\
& \quad h_i(z) = 0, \quad i = 1, 2, \ldots, p \\
& \quad ||x - z|| \leq R
\end{align*}
\]
Convex optimization problem in standard form

\[
\begin{align*}
\min_{x} f_0(x) \\
f_i(x) &\leq 0, \quad i = 1, 2, \ldots, m \\
a_i^\top x - b_i &\equiv 0, \quad i = 1, 2, \ldots, p
\end{align*}
\]

- All $f_i$’s have to be convex and note that equality constraints are affine.
- The above implies that the intersection of domains of objective functions and constraints is convex which in turn implies the domain of the problem is convex.
- Thus, in a convex optimization problem we minimize a convex objective function over a convex set.
Optimality criterion for differentiable $f_0$

$x$ is optimal if for all feasible point $y$

$$\nabla f_0(x)^\top (y - x) \geq 0$$

Supporting hyperplane to feasible set at $x$
Objective function as well as constraints are affine
The feasible set of the problem is a polyhedron
Figure: Bounded v/s unbounded
\[ \begin{align*} \min_x & \quad \sum_{(i,j) \in E} w_{ij} x_{ij} \\ \sum_j x_{ij} - \sum_j x_{ji} & = \begin{cases} 1 & i = s \\ -1 & i = t \\ 0 & \text{otherwise} \end{cases} \\ x & \geq 0 \end{align*} \]
Quadratic program (QP)

\[
\min_x \frac{1}{2} x^\top P x + q^\top x + r \\
G x - h \leq 0 \quad G \in \mathbb{R}^{m \times n} \\
A x - b = 0 \quad A \in \mathbb{R}^{p \times n}
\]

- \( P \in \mathbb{S}_+^n \). Objective function is convex quadratic and constraints are affine
- Like in LP, the feasible set of the problem is a polyhedron
LP vs QP geometry
Laplacian = D - A

\[
L = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}
\]
Embedding in 1-D

Find points in Euclidean space such that

1. Connected nodes are close
2. The points are around the origin
3. Avoid trivial solution

\[
\min_x x^\top L x = \min_x \sum_{(i,j) \in E} (x_i - x_j)^2
\]

\[
1^\top x = 0
\]

\[
x^\top x = c
\]

Convexity?
Convex relaxation

\[
\min_{x} \quad x^{\top} \tilde{L} x \\
\quad \quad x^{\top} x \leq c
\]

- Can we transform \( x \) into another space where the constraint \( 1^{\top} x = 0 \) always holds?
- Can we come up with \( \tilde{L} \) such that it implicitly has the constraint \( 1^{\top} x = 0 \)?

Addition of fixed nodes

\[
x^{\top} = [x_{fixed}, x_{free}]
\]
Code demo
Questions?