CSE 203B Week 5 Discussion: Convex Optimization problems

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Outline

- Standard form
- Linear program
- Graph embedding
- Code demo

Optimization problem in standard form

$$
\begin{aligned}
&\min_x f_0(x) \\
&f_i(x) \leq 0, \quad i = 1, 2, \dots, m \\
&h_i(x) = 0, \quad i = 1, 2, \dots, p
\end{aligned}
$$

- Domain of the problem $\mathcal{D} = \bigcap_{i=0}^{m} \textbf{dom}\ f_i \ \cap \ \bigcap_{i=1}^{p} \textbf{dom}\ h_i$
- The optimal value $p^* = \inf\{f_0(x) | f_i(x) \leq 0, h_i(x) = 0\}$
- If the problem is infeasible, $p^* = \inf \emptyset = +\infty$
- If there are feasible points x_k s.t. $f_0(x_k) \to -\infty$ as $k \to \infty$ then the problem is unbounded below

Globally optimal and locally optimal

A feasible point x is locally optimal if there is an $R > 0$ s.t. x solves the following optimization problem

$$
\begin{aligned}
\min_z \; & f_0(z) \\
f_i(z) &\leq 0, \quad i = 1, 2, \ldots, m \\
h_i(z) &= 0, \quad i = 1, 2, \ldots, p \\
||x - z|| &\leq R
\end{aligned}
$$

Convex optimization problem in standard form

$$\min_x f_0(x)$$
$$f_i(x) \le 0, \quad i = 1, 2, \ldots, m$$
$$a_i^\top x - b_i = 0, \quad i = 1, 2, \ldots, p$$

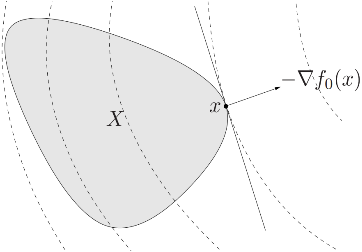
- All f_i's have to be convex and note that equality constraints are affine
- The above implies that the intersection of domains of objective functions and constraints is convex which in turn implies the domain of the problem is convex
- Thus, in a convex optimization problem we minimize a convex objective function over a convex set

Optimiality criterion for differentiable f_0

x is optimal if for all feasible point y

$$\nabla f_0(x)^\top (y - x) \geq 0$$

Supporting hyperplane to feasible set at x



Linear Program (LP)

$$
\begin{aligned}
min_x c^\top x & \\
Gx - h \leq 0 & \quad G \in \mathbb{R}^{m\times n} \\
Ax - b = 0 & \quad A \in \mathbb{R}^{p\times n}
\end{aligned}
$$

- Objective function as well as constraints are affine
- The feasible set of the problem is a polyhedron

Linear Programming cases

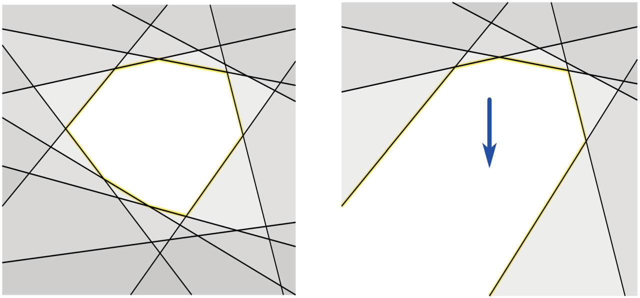
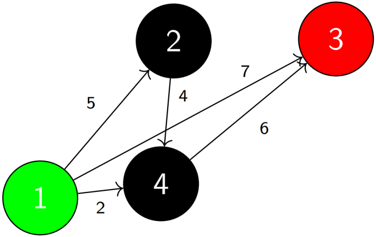


Figure: Bounded v/s unbounded

LP shortest path



$$\min_x \sum_{(i,j)\in E} w_{ij} x_{ij}$$

$$\sum_j x_{ij} - \sum_j x_{ji} = \begin{cases} 1 & i = s \\ -1 & i = t \\ 0 & \text{otherwise} \end{cases}$$

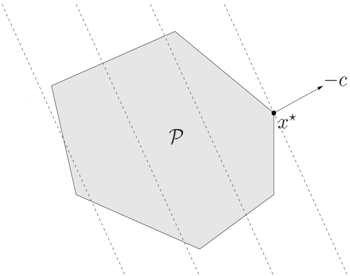
$$x \geq 0$$

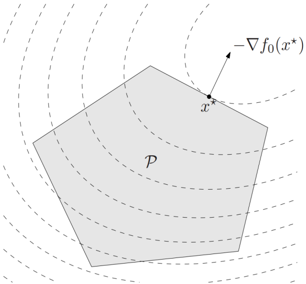
Quadratic program (QP)

$$
\begin{aligned}
& min_x \frac{1}{2} x^\top P x + q^\top x + r \\
& \qquad\qquad Gx - h \leq 0 \quad G \in \mathbb{R}^{m \times n} \\
& \qquad\qquad Ax - b = 0 \quad A \in \mathbb{R}^{p \times n}
\end{aligned}
$$

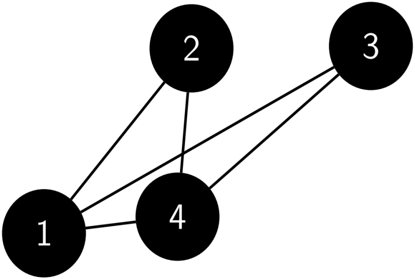
- $P \in \mathbb{S}_+^n$. Objective function is convex quadratic and constraints are affine
- Like in LP, the feasible set of the problem is a polyhedron

LP vs QP geometry





Graph embedding



Laplacian = D - A

$$L = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

Embedding in 1-D

Find points in eucledian space such that

1. Connected nodes are close
2. The points are around the origin
3. Avoid trivial solution

$$\min_x x^\top L x = \min_x \sum_{(i,j)\in E} (x_i - x_j)^2$$

$$1^\top x = 0$$

$$x^\top x = c$$

- Convexity?

Embedding in 1-D with relaxed constraints

1. Convex relaxation

$$\min_x x^\top \tilde{L} x$$
$$x^\top x \leq c$$

   - Can we transform x into another space where the constraint $1^\top x = 0$ always holds?
   - Can we come up with $\tilde{L}$ such that it implicitly has the constraint $1^\top x = 0$?

2. Addition of fixed nodes
   - $x^\top = [x_{fixed}, x_{free}]$

Code demo

Questions?