CSE 203B Convex Optimization

Discussion: Convex functions
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Convex Functions:

- A function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is convex if $\text{dom } f$ is a convex set and if for all $x, y \in \text{dom } f$, and $0 \leq \theta \leq 1$, we have:
  \[
f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)\]

- Concave Functions: $-f$ is Convex
First Order Condition

- Suppose $f$ is differentiable (\(\text{dom } f\) is open and \(\nabla f\) exists at each point in \(\text{dom } f\)).
- Function $f$ is convex iff \(\text{dom } f\) is convex and for all $x, y \in \text{dom } f$

\[
f(y) \geq f(x) + \nabla f(x)^T (y - x)
\]
Second Order Conditions

- Suppose $f$ is twice differentiable ($\text{dom } f$ is open and its Hessian exists at each point in $\text{dom } f$), then $f$ is convex iff $\text{dom } f$ is convex and for all $x, y \in \text{dom } f$.

\[ \nabla^2 f(x) \succeq 0 \] (the Hessian is positive semidefinite)

\[ x^T \nabla^2 f(x) x \geq 0 \quad x \in \mathbb{R}^n \]
Operations that preserve convexity

- Verify by Using definition of convex functions (Proof of \( f(x) = \max_i x_i \))
- For twice differentiable functions, show \( \nabla^2 f(x) \succeq 0 \) (PSD Hessian)
- Show that \( f \) is obtained from simple convex functions by operations that preserve convexity (Ref. Chap. 3.2)
  - Non-negative weighted sum
  - Composition with affine function
  - Pointwise maximum or supremum
  - Composition
  - Partial Minimization
Conjugate Functions

- For a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ (that is not necessarily convex), the conjugate $f^* : \mathbb{R}^n \rightarrow \mathbb{R}$ is defined as follows:

$$f^*(y) = \sup \{ y^T x - f(x) \mid x \in \text{dom } f \}$$

- $f^*(y)$ is convex even if $f(x)$ is not convex.
- Proof: Use pointwise supremum (maximum)
- $y^T x - f(x)$ is affine function in $y$. Therefore pointwise supremum of convex function leads to a convex function.
Supporting hyperplane: if $f$ is convex in that domain

$(x, f(x))$ where $\nabla f^T(x) = y$ if $f$ is differentiable

$$\sup \left\{ y^T x - f(x) \right\}$$

$$y = \nabla f(x)$$

Figure 3.8 A function $f : \mathbb{R} \rightarrow \mathbb{R}$, and a value $y \in \mathbb{R}$. The conjugate function $f^*(y)$ is the maximum gap between the linear function $yx$ and $f(x)$, as shown by the dashed line in the figure. If $f$ is differentiable, this occurs at a point $x$ where $f'(x) = y$. 
Examples of Conjugates

- Derive the conjugates of $f : R \rightarrow R$

**Affine**

- $f(x) = ax - \beta$
  - $f^*(y) = \begin{cases} \beta & \text{if } y = \alpha \\ \infty & \text{if } y \neq \alpha \end{cases}$

**Norm**

- $f(x) = |x|$
  - $f^*(y) = \begin{cases} 0 & \text{if } |y| \leq 1 \\ \infty & \text{if } |y| > 1 \end{cases}$
Solving Conjugate Function Problems

Let \( f^*(y) = \sup_x g(x, y) = \sup_x (y^T x - f(x)) \). At a given point \( \bar{y} \), the conjugate could be one of the following three cases:

1. **Finite**: \( g(x, \bar{y}) \in \mathbb{R} \) (the good scenario).
2. **Infeasible**: \( g(x, \bar{y}) \to +\infty \) for at least one choice of \( x \). For example in conjugate of L1 norm the solution is infeasible for \( |y| > 1 \).

   Intuition: If I can keep making \( g(x, \bar{y}) \) larger and larger somehow, then all finite values that \( g(x, \bar{y}) \) can take will eventually pale in comparison.
Solving Conjugate Problems

- **Unbounded Below**: \( g(x, y) \to -\infty \) for all \( x \) values. For all sets of \( x(t) \) values; as \( t \to \infty \), \( g(x, y) \to -\infty \) (This is rare.) Intuition: If there existed any finite solutions anywhere, it would have been preferable to \(-\infty\).

Note: Different conditions may arise for different ranges of \( y \) values. To determine complete set of solutions we need to consider different values of \( y \) separately where the solution is Finite/Infeasible/Unbounded below.
Solving Conjugate Problems: (Solve in Class)

Example: \( f(x) = a^T x + b \), \( f^*(y) = ? \)

\[
\begin{align*}
f^*(y) &= \sup_{x} (y^2 - a^T x - b) \\
&= \sup_{x} (y-a)x - b)
\end{align*}
\]

Case I, \( y > a \)

\[
\sup_{x} ((y-a)x - b) \to \infty \quad \text{for} \quad x \to \infty
\]
Continued..

Case 2, \( y < a \)

\[
\sup_{x \in \mathbb{R}} (y - a) = -b \rightarrow -\infty \quad \text{for } x \rightarrow -\infty
\]

Case 3, \( y = a \)

\[
\sup_{x \in \mathbb{R}} (-b) = -b
\]
Example 2:

Example (2020 CSE203B QII.3):

\[ f(x) = \begin{cases} 
\frac{1}{2}x^2 & |x| < 1 \\
|x| - \frac{1}{2} & |x| > 1
\end{cases}, \quad f^*(y) = ? \]

[Graph or image of the function f(x) with regions and inequalities]
\[
\begin{align*}
\mathcal{P}(z) &= \sup_{x \in \mathbb{R}} \left\{ yx - f(x) \right\} \\
&= \begin{cases} 
  yx + x + \frac{1}{2} & \text{if } z \leq -1 \\
  yx - \frac{1}{2} x^2 & \text{if } -1 < z < 1 \\
  yx - x + \frac{1}{2} & \text{if } z \geq 1
\end{cases}
\end{align*}
\]

2) For \( z < -1 \) or \( z > 1 \), \( \mathcal{P}(z) \to \infty \)

2) For \(-1 < z < 1 \)

1) \( \frac{\partial}{\partial x} \left( y + 1 \right) x^{\frac{1}{2}} = y + 1 > 0 \)

max value is for \( z = -1 \) or \( z = -\frac{1}{2} \)
2. \[ \frac{d}{dx} (y\frac{e}{2} - \frac{1}{2} x^2) = y - e \Rightarrow y = e \]

3. \[ \frac{d}{dx} (y - 1) + \frac{1}{2} = y + \leq 0 \]

Max value is for \( x = 1 \)

\[ \Rightarrow y = \frac{1}{2} \]

\[ f(y) = \frac{1}{2} y^2 \]
Example 3:

\[ f(x) = \begin{cases} 
\|x\|_2^2, & \|x\|_2 \leq a, \\
\frac{a}{2}\|x\|_2 - a, & \|x\|_2 > a,
\end{cases} \]

Similar to HW 0.3
Assignment Hints

1. Use convexity preserving properties to prove/disprove convexity.
2. Apply second derivative to prove or disprove convexity.
3. Show rough sketch of approximation by trying different values of $X_i$. What happens when one of the values is 0 and rest are $x_{\text{max}}$?
4. Look for some other methods to approximate the function.
5. $\max(x^Ty) = \max(\frac{x^Ty}{\|y\|_p})$
6. Try to solve $\nabla_y \max(\frac{x^Ty}{\|y\|_p}) = 0$
Assignment Hints

- For Conjugate functions check the feasible regions for the solution to exist.
- Check what happens when ‘x’ lies in null space and range of A.
- Take the derivative of conjugate.
- $L_p$ $0<p<1$, try reading https://kconrad.math.uconn.edu/blurbs/analysis/lpspace.pdf

\[ \text{Dual norm is projection on vector } x \text{, find the maximum projection.} \]
Quasiconvex and Quasiconcave functions

- A function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is quasi-convex if its domain and all its sublevel sets:
  \[ S_\alpha = \{ x \in \text{dom } f \mid f(x) \leq \alpha \}, \alpha \in \mathbb{R} \]
  are convex.

- A function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is quasi-concave if all its super level sets:
  \[ S_\alpha = \{ x \in \text{dom } f \mid f(x) \geq \alpha \}, \alpha \in \mathbb{R} \]

- A function that is both quasiconvex and quasiconcave is said to be quasilinear.
Examples:

1) A quasiconvex function that is not convex

2) Not quasiconvex
Quasiconvex and Quasiconcave Functions:

A function $f$ is quasiconvex if and only if $\text{dom}(f)$ is convex and for any $x, y \in \text{dom}(f)$ and $0 \leq \theta \leq 1$, 

$$f(\theta x + (1-\theta) y) \leq \max\{f(x), f(y)\},$$

Figure 3.10 A quasiconvex function on $\mathbb{R}$. The value of $f$ between $x$ and $y$ is no more than $\max\{f(x), f(y)\}$. 