CSE 203B Convex Optimiztion

Discussion: Convex functions

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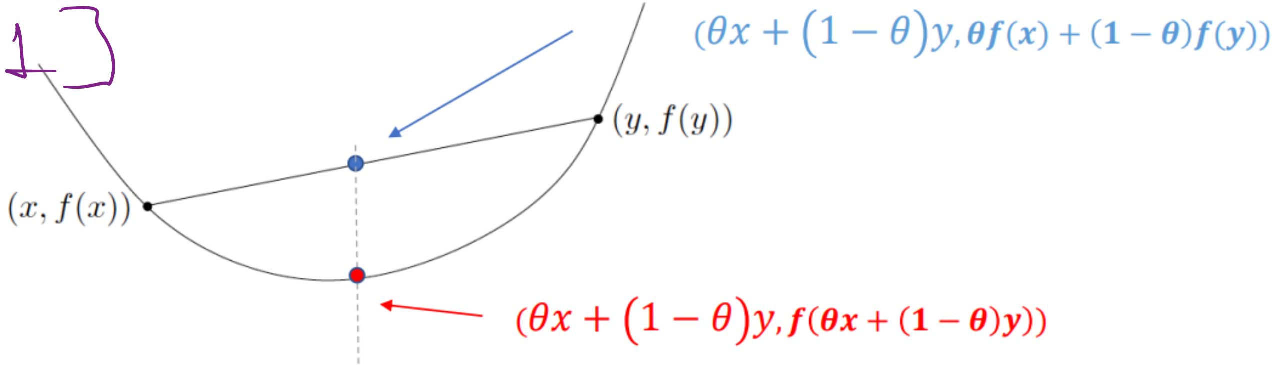
Convex Functions:

- A function $f: R^n \rightarrow R$ is convex if dom f is a convex set and if for all x, $y \in$ dom f, and $0 \le \theta \le 1$, we have:

$$f(\theta x + (1 - \theta)y) \le \theta f(x) + (1 - \theta) f(y)$$

- Concave Functions: -f is Convex

$\theta \in [0, 1]$



First Order Condition

Equation of hyperplane: $f(x) \geq f(\bar{x}) + \nabla f(x)^T(x - \bar{x})$

$\Rightarrow \nabla f(x)^T \bar{x} - f(\bar{x}) \geq \nabla f(x)^T (x) - f(x)$

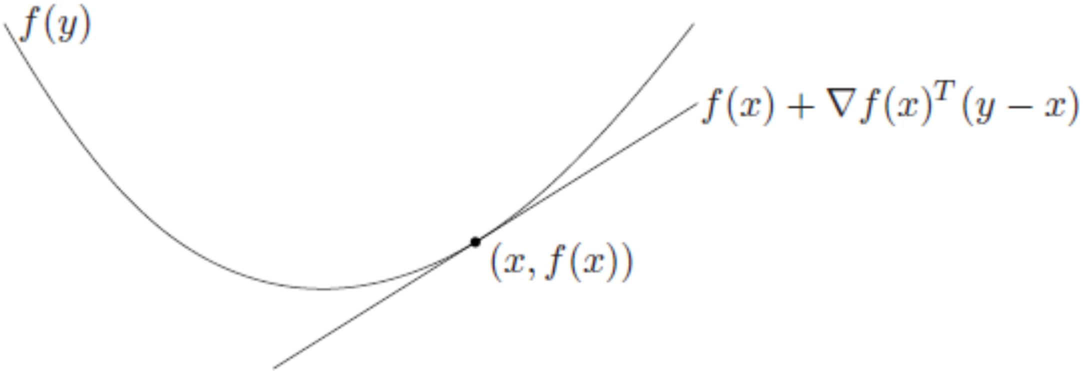
$\Rightarrow \begin{bmatrix} \nabla f(\bar{x}) & -1 \end{bmatrix} \begin{bmatrix} \bar{x} \\ f(\bar{x}) \end{bmatrix} \geq \begin{bmatrix} \nabla f(x) & -1 \end{bmatrix} \begin{bmatrix} x \\ f(x) \end{bmatrix}$

- Suppose f is differentiable ($dom\ f$ is open and ∇f exists at each point in $dom\ f$).

$\Rightarrow b \geq \begin{bmatrix} \nabla f(x)^T & -1 \end{bmatrix} \begin{bmatrix} x \\ f(x) \end{bmatrix}$

- Function f is convex iff dom f is convex and for all $x, y \in dom\ f$

$$f(y) \geq f(x) + \nabla f(x)^T(y - x)$$



Second Order Conditions

- Suppose f is twice differentiable ($dom\ f$ is open and its Hessian exists at each point in $dom\ f$), then f is convex iff dom f is convex and for all x, y $dom\, f$.

  $\nabla^2 f(x) \succcurlyeq 0$ (the Hessian is positive semidefinite)

$x^T \nabla^2 f(x) x \geqslant 0 \qquad x \in \mathbb{R}^n$

Operations that preserve convexity

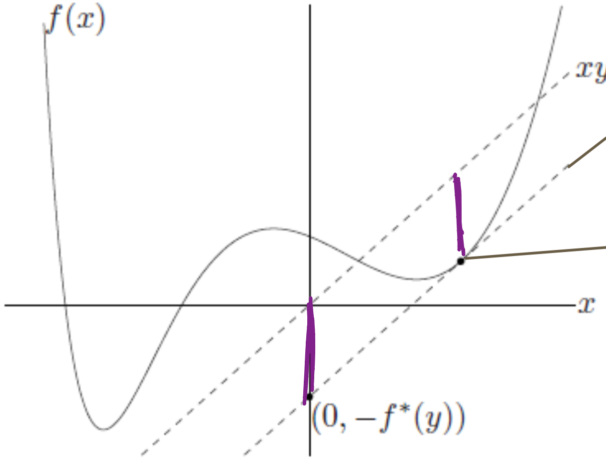
- Verify by Using definition of convex functions (Proof of f(x) = $\max_i$ x_i)
- For twice differentiable functions, show $\nabla^2 f(x) \succcurlyeq 0$. (PSD Hessian)
- Show that f is obtained from simple convex functions by operations that preserve convexity (Ref. Chap. 3.2)
  - Non-negative weighted sum $h(x) = f(x) + g(x)$
  - Composition with affine function
  - Pointwise maximum or supremum $\sup_y f(x, y)$ {convex w.r.t x}
  - Composition $f(g(x))$
  - Partial Minimization

Conjugate Functions

- For a function $f : R^n \rightarrow R$ (that is not necessarily convex), the conjugate $f^* : R^n \rightarrow R$ is defined as follows:

$$f^*(y) = \sup \left(y^T x - f(x)\right) \quad \{x \in \textbf{dom } f\}$$

- $f^*(y)$ is convex even $f(x)$ is not convex.
- Proof: Use pointwise supremum (maximum)
- $\boldsymbol{y^T x - f(x)}$ is affine function in y. Therefore pointwise supremum of convex function leads to a convex function.



Supporting hyperplane: if f is convex in that domain

$(x$, f(x)) where ∇f^T(x) = y if f is differentiable

sup { $y^T x - f(x)$ }

$y = \nabla f(x)$

Figure 3.8 A function $f : \mathbf{R} \to \mathbf{R}$, and a value $y \in \mathbf{R}$. The conjugate function $f^*(y)$ is the maximum gap between the linear function yx and $f(x)$, as shown by the dashed line in the figure. If f is differentiable, this occurs at a point x where $f'(x) = y$.

$f^*(y) = \{yx - \alpha x + \beta\}$

$\sup\{(y-\alpha)x + \beta\}$

Examples of Conjugates

$\begin{cases} \beta & \text{if } y = \alpha \\ \infty & \text{otherwise} \end{cases}$

$f^*(y) = \sup_x \left(yx - \frac{c}{2}x^2\right)$

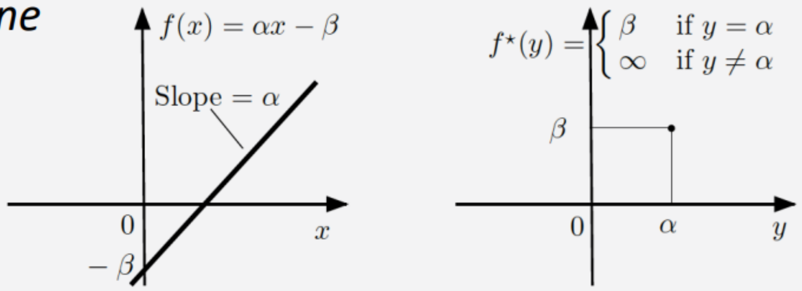
$\nabla f = y - cx \Rightarrow 0$

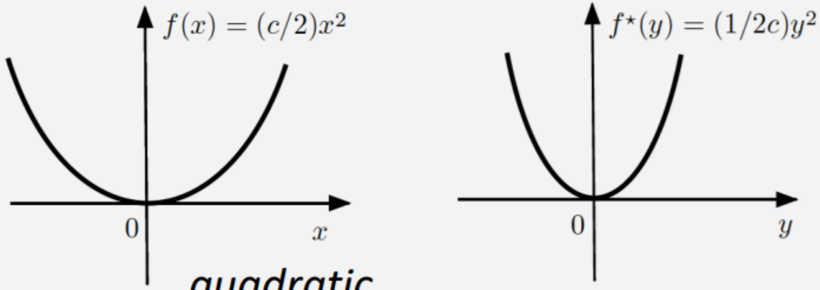
$\frac{y}{c} = x$

$= \frac{1}{2c}y^2$

- Derive the conjugates of $f: R \to R$

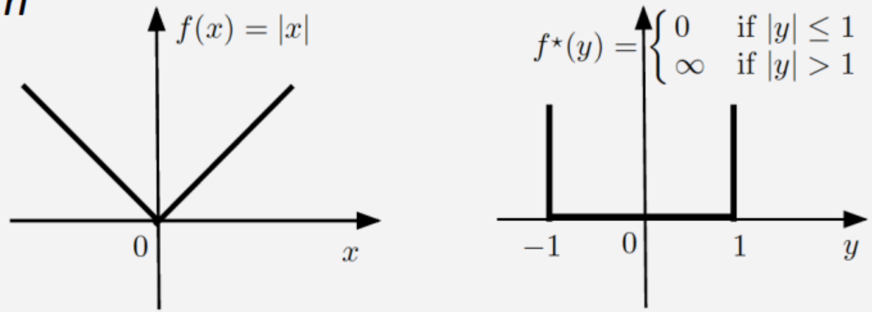
Affine





quadratic

Norm



$\sup_x y^T x - f(x)$

$yx - |x|$

$|y||x| - |x| \geq yx - |x|$

$|x|(|y| - 1)$

Solving Conjugate Function Problems

Let $f*(y) = \sup_x g(x, y) = \sup_x (y^T x - f(x))$. At a given point $\bar{y}$,the conjugate could be one of the following three cases:

1. **Finite:** $g(x, \bar{y}) \in R$ (the good scenario).
2. **Infeasible:** $g(x, \bar{y}) \to +\infty$ for at least one choice of x. For example in conjugate of L_1 norm the solution is infeasible for $|y|>1$.

Intuition: If I can keep making $g(x, \bar{y})$ larger and larger somehow, then all finite values that $g(x, \bar{y})$ can take will eventually pale in comparison.

Solving Conjugate Problems

- **Unbounded Below:** g(x, $\bar{y}$) $\to -\infty$ for all x values. For all sets of x(t) values; as t $\to \infty$, g(x, y) $\to -\infty$ (This is rare.) Intuition: If there existed any finite solutions anywhere, it would have been preferable to $-\infty$.

Note: Different conditions may arise for different ranges of y values. To determine complete set of solutions we need to consider different values of y separately where the solution is Finite/Infeasible/Unbounded below.

Solving Conjugate Problems: (Solve in Class)

Example: $f(x) = a^T x + b$, $f^*(y) = ?$

$$f^*(y) = \sup_x (yx - a^T x - b)$$

$$= \sup_x ((y-a)x - b)$$

Case I, $y > a$

$\sup_x ((y-a)x - b) \to \infty$ for $x \to \infty$

Continued..

Case 2, $y < a$

$\sup_x (y-a)x - b \to \infty$ for $x \to -\infty$

Case 3 $y = a$

$\sup_x (-b) = -b$

Example 2:

Example (2020 CSE203B QII.3):

$$f(x) = \begin{cases} \frac{1}{2}x^2 & |x| < 1 \\ |x| - \frac{1}{2} & |x| > 1 \end{cases}, \quad f^*(y) = ?$$

$$f(x) = \begin{cases} -x - \frac{1}{2} & , \; x < -1 \\ \frac{1}{2}x^2 & , \; 1 < x < -1 \\ x - \frac{1}{2} & \; x > 1 \end{cases}$$

$$f^*(y) = \sup_x \{ yx - f(x) \}$$

$$= \sup_x \begin{cases} yx + x + \frac{1}{2} & x < -1 \\ yx - \frac{1}{2}x^2 & -1 < x < 1 \\ yx - x + \frac{1}{2} & 1 < x \end{cases}$$

1) For $y < -1$ or $y > 1$, $f^*(y) \to \infty$

2) For $-1 < y < 1$

1) $\frac{d}{dx}\left((y+1)x + \frac{1}{2}\right) = y+1 \geq 0$

max value is for $x = -1$

$= -y - \frac{1}{2}$

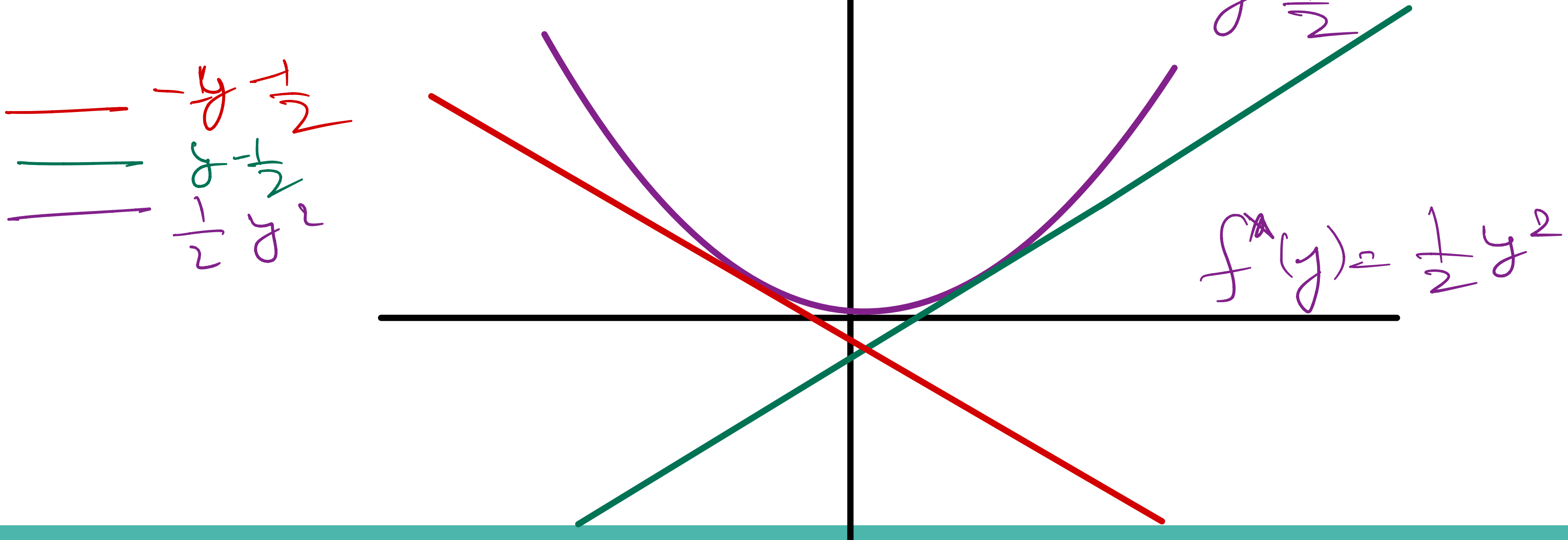
② $\frac{d}{dx}\left(yx - \frac{1}{2}x^2\right) = y - x \Rightarrow y = x$

$= \frac{1}{2}y^2$

③ $\frac{d}{dx}\left((y-1)x + \frac{1}{2}\right) = y - 1 \le 0$

max value is for $x = 1$

$\to y - \frac{1}{2}$



Example 3:

$$f(x) = \begin{cases} ||x||_2^2, & ||x||_2 \leq a, \\ a(2||x||_2 - a), & ||x||_2 > a, \end{cases}$$

Similar to HW Q.3

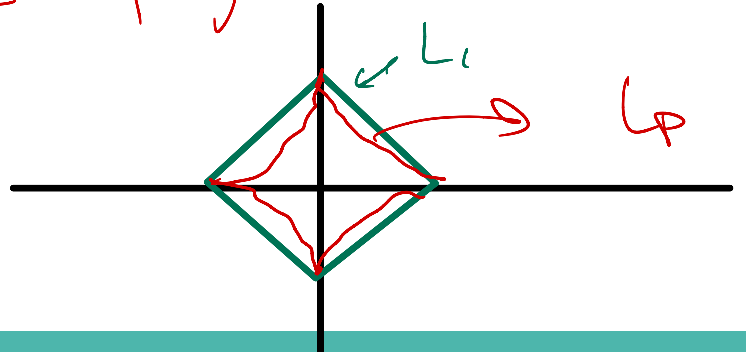
Assignment Hints

1. Use convexity preserving properties to prove/disprove convexity.
2. Apply second derivative to prove or disprove convexity
3. Show rough sketch of approximation by trying different values of X_i. What happens when one of the values is 0 and rest are x_max?
4. Look for some other methods to approximate the function.
5. $\max(x^T y) = \max(\frac{x^T y}{||y||_p})$
6. Try to solve $\nabla_y \max(\frac{x^T y}{||y||_p})=0$

Assigment Hints

- For Conjugate functions check the feasible regions for the solution to exist.
- Check what happens when ‘x’ lies in null space and range of A.
- Take the derivative of conjugate.
- Lp 0<p<1, try reading https://kconrad.math.uconn.edu/blurbs/analysis/lpspace.pdf (Not required)

→ Dual Norm is projection on vector x. Find the maximum projection)



Quasiconvex and Quasiconcave functions

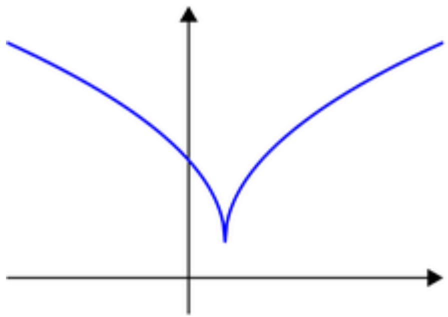
- A function $f\colon R^n \to R$ is quasi-convex if its domain and all its sublevel sets:

  $$S_\alpha = \{x \in dom\ f \mid f(x) \leq \alpha\},\ \alpha \in \mathcal{R}$$

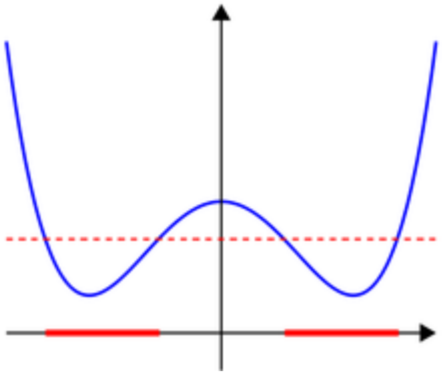
  are convex.
- A function $f\colon R^n \to R$ is quasi-concave if all its super level sets:

  $$S_\alpha = \{x \in dom\ f \mid f(x) \geq \alpha\},\ \alpha \in \mathcal{R}$$
- A function that is both quasiconvex and quasiconcave is said to be quasilinear.

Examples:



1) A quasiconvex function that is not convex



2) Not quasiconvex

Quasiconvex and Quasiconcave Functions:

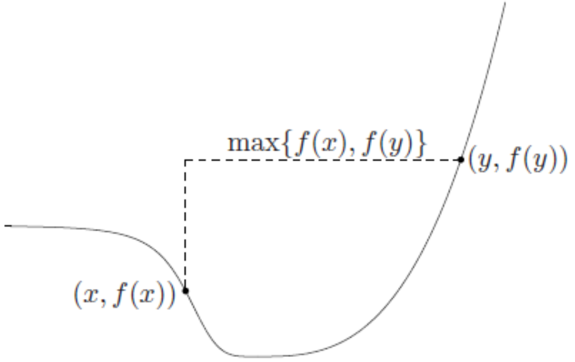


Figure 3.10 A quasiconvex function on **R**. The value of f between x and y is no more than $\max\{f(x), f(y)\}$.

A function f is quasiconvex if and only if dom(f) is convex and for any x, y ∈ dom(f) and 0 ≤ θ ≤ 1,

$$f(\theta x + (1 - \theta)y) \le \max\{f(x), f(y)\},$$