

CSE 203B W23 Midterm 10AM 2/26/2023-10AM 2/28/2023

Submit your solution to gradescope before the due time.

Policy of the Exam: Here is the policy of the exam:

1. This is an open-book take-home exam. Internet search is permitted. However, you are required to work by yourself. Consultation or discussion with any other parties is not allowed.
2. You are not required to typeset your solutions. We do expect your writing to be legible and your final answers clearly indicated. Also, please allow sufficient time to upload your solutions.
3. You are allowed to check your answers with programs in Matlab, CVX, Mathematica, Maple, NumPy, etc. Be aware that these programs may not produce the intermediate steps needed to receive credit.
4. If something is unclear, state the assumptions that seem most natural to you and proceed under those assumptions. Out of fairness, we will not be answering questions about the technical content of the exam on Piazza or by email. The solution will then be graded based on the reasonable assumptions made.

Part I: True or False: Explain your answer with one sentence (30 pts)

I.1 (convex set): Set $\{(x, y) | x^2 + y^2 \geq 1, x, y \in R\}$ is convex.

T/F:

I.2 (dual cone): Given cone $K = \{x | a_1^T x \geq 0, a_2^T x \leq 0, \text{ where } a_1, a_2, x \in R^n\}$, its dual cone is $K^* = \{a_1 \theta_1 + a_2 \theta_2 | \theta_1 \geq 0, \theta_2 \leq 0, \text{ and } \theta_1, \theta_2 \in R\}$.

T/F:

I.3 (dual cone): The **dual of the dual** cone $K = \{x | \|Ax + b\|_2 < c^T x + d\}$ is itself, where $A \in R^{mn}, x, b, c \in R^n$ and $d \in R$.

T/F:

I.4 (Convex Function): Given a function $f(x) = 1.4x^{1.3} + 2.2x^{2.5}$, where $x \in R_+$. The function is convex.

T/F:

I.5 (Convex Function): Function $g(x) = \max_y f(x, y)$ is a convex function for any $f(x, y) \in R$, where $x, y \in R^n$.

T/F:

I.6 (Conjugate Function): Given function $f(x) = x^T Ax + b^T x + c$, where $x, b \in R^n, A \in R^{nn}$, and $c \in R$. Suppose that matrix A is not positive semidefinite. We can claim that the conjugate function, $f^*(y) = \infty$, for all $y \in R^n$.

T/F:

I.7 (Supporting Hyperplane): Given a differentiable and convex function $f(x)$, where $x \in R^n$, and a fixed point $\bar{x} \in R^n$. Suppose that in a $n + 1$ dimension

space $[x^T, t]^T$, we draw the supporting hyperplane

$$[\nabla f(\bar{x})^T, -1] \left(\begin{bmatrix} x \\ t \end{bmatrix} - \begin{bmatrix} \bar{x} \\ f(\bar{x}) \end{bmatrix} \right) = 0.$$

We can claim that the supporting hyperplane intersects the t axis at its conjugate function i.e. $[x^T = \mathbf{0}^T, t = -f^*(y)]^T$ where $y = \nabla f(\bar{x})$, and $\mathbf{0}$ is a vector of 0.

T/F:

I.8 (Problem Formulation/Duality): Given a convex programming problem:

$$\text{minimize } f_0(x), \text{ subject to } Ax \leq b, x \in R^n, A \in R^{m \times n}, b \in R^m,$$

where $f_0(x)$ is a differentiable convex function, we can claim that

$$\nabla f_0(\bar{x}) \in \{-A^T \theta \mid \theta \in R_+^m\}$$

is a necessary and sufficient condition for \bar{x} to be an optimal solution.

T/F:

I.9 (Duality): In the textbook subsection (5.1.1), we have the problem formulation (5.1) and its Lagrangian $L(x, \lambda, \nu)$. The variable x of the Lagrangian has to be feasible, i.e. x satisfies the constraint in formulation (5.1).

T/F:

I.10 (Min Max Problem): Given a function $f(x, y) = x^T A y$, we have $x, y \in R^n$ and matrix $A \in R^{n \times n}$. We can claim that the equality,

$\min_x \max_y f(x, y) = \max_y \min_x f(x, y)$ is true when there is a bounded (not infinite) solution.

T/F:

Part II: Problem Solving: Show your process

Problem 1. Dual Cone: Find the dual cone of the following cones. (20 pts)

1.1 $K = \left\{ \begin{bmatrix} x \\ t \end{bmatrix} \mid \|Ax\|_2 \leq t \right\}$, where $A \in R^{n \times n}$, $x \in R^n$, and $t \in R_+$. Matrix A is nonsingular. (hint: If you have no clue, start with a small but nontrivial case, e.g. $n=1$ and/or 2.

1.2 $K = \left\{ \begin{bmatrix} x \\ t \end{bmatrix} \mid \|Ax\|_p \leq t \right\}$, where $A \in R^{n \times n}$, $x \in R^n$, $p \geq 1$, and $t \in R_+$. Matrix A is nonsingular.

Problem 2. Conjugate Function: Find the conjugate function of the following functions. (20 pts)

2.1 $f(x) = -x^3 + 3x + 2$, where $x \in R$.

2.2 $f(x) = \frac{x_1^2}{x_2}$, where $x \in R_{++}^2$.

Problem 3. Graph embedding and graph learning. (30 pts)

Recall that given a connected, undirected graph $G = (V, E)$ with n_0 vertices, the graph embedding problem is to assign coordinates in \mathbb{R}^m to each vertex $v \in V$. Let $A \in \{0, 1\}^{n_0 \times n_0}$ be the symmetric adjacency of G , and let D be the corresponding diagonal degree matrix such that $D_{ii} = \sum_j A_{i,j}$. The *graph Laplacian* is defined to be $L_0 = D - A$.

One way to define the graph embedding problem (i.e. Laplacian Eigenmaps) is to solve the following problem:

$$\begin{aligned} \min_{X \in \mathbb{R}^{n_0 \times m}} \quad & \langle X, L_0 X \rangle \\ \text{s.t.} \quad & X^\top X = I, \quad \mathbf{1}^\top X = 0 \end{aligned} \tag{1}$$

Where the inner product $\langle U, V \rangle$ is defined to be $\text{tr}(U^\top V)$.

(i) Problem (1) is nonconvex. Prove that the solution is given by m eigenvectors corresponding to the smallest nonzero m eigenvalues of L_0 .

(ii) Derive the dual and KKT conditions of the semi-supervised problem (you may assume L is nonsingular).

$$\begin{aligned} \min_{X \in \mathbb{R}^{n \times m}} \quad & \langle X, LX \rangle + \langle B, X \rangle \\ \text{s.t.} \quad & \mathbf{1}^\top X = 0 \end{aligned} \tag{2}$$

(iii) Derive the dual and KKT conditions of the problem

$$\begin{aligned} \min_{X \in \mathbb{R}^{n \times m}} \quad & \langle X, LX \rangle + \langle B, X \rangle \\ \text{s.t.} \quad & \mathbf{1}^\top X = 0, \quad \|X\|_1 \leq c, \end{aligned} \tag{3}$$

where $\|X\|_1 = \sum_{ij} |X_{ij}|$ and c is some constant, say 1000. (Hint: if you get stuck, try to write the l1 ball in standard form as an lp). Use the framework¹ from homework 4 to implement this problem (where L and B are now given). Note that the random seed has been updated from homework 4 (different graph and fixed vertices).

¹<https://colab.research.google.com/drive/1suB03RgKaQzJBh-tzpGLI1Xn625hmzH2?usp=sharing>