CSE 203B W21 Homework 4

Due Time : 11:50pm, Wednesday Feb. 15, 2023 Submit to Gradescope

In this homework, we work on exercises from the textbook. Problems 4.1, 4.8, 4.11, and 4.15 are related to LP. Problems 4.21, 4.39, and 4.47 are related to QCQP, and SDP. Problem 5.3 is about the basic definition of duality. Problems 5.4, 5.5, 5.6, 5.7, 5.8, 5.9, and 5.10 are examples and applications of duality. Also, we practice using the convex optimization tools on a linear programming problem, and a quadratic programming problems.

Total points: 50. Exercises are graded by completion, and assignments are graded by correctness.

I. Exercises from textbook chapters 4 & 5 (15 pts, 1pt for each problem)

4.1, 4.8, 4.11, 4.15, 4.21, 4.39, 4.47, 5.3, 5.4, 5.5, 5.6, 5.7, 5.8, 5.9, 5.10.

II. Assignments (35 pts)

II.1 Linear Programming: You are free to use any software packages. (15 pts)

Given

$$A = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 2 & 3 & -1 & 2 \\ 0 & 2 & 2 & 0 \end{bmatrix},$$

$$b^T = \begin{bmatrix} 2 & 1 & 3 & 5 & 5 \end{bmatrix},$$

$$c^T = \begin{bmatrix} -1 & -2 & -3 & -1 \end{bmatrix},$$

and $n = 4$, solve the following linear programming problems. If a solution is found, validate that the solution satisfies the optimality criteria (which was talked about in class or textbook). Otherwise, explain why a solution is not feasible and suggest how to mitigate the issue if you are the project leader:

II.1.1. minimize $f_0(x) = c^T x$ subject to $Ax \le b$, $x \in R^n$.

$$\bar{x}^\top := [-k/2, -k, k, 0] \tag{1}$$

$$A\bar{x} \le b \implies -k \le 3, -5k \le 5 \tag{2}$$

$$c^\top \bar{x} = -k/2 \tag{3}$$

$$\lim_{k\to\infty} c^\top \bar{x} = \lim_{k\to\infty} -k/2 = -\infty \tag{4}$$

For $k \to \infty$, the constraints in (2) are satisfied. Therefore, the problem is feasible but unbounded below (4). We can make the problem bounded by imposing two more positivity constraints like $x_2 \ge 0, x_3 \ge 0$ where $x^\top = [x_1, x_2, x_3, x_4]$

II.1.2. minimize $f_0(x) = c^T x$ subject to $Ax = b$, $x \in R^n$.

$$[A|b] \xrightarrow{RREF} I_{5\times 5}$$

So, $Ax = b$ does not have a solution. Hence, the problem is infeasible. We can make the problem feasible by turning the last equality constraint to inequality i.e., $2x_2 + 2x_3 \leq 5$ where $x^\top = [x_1, x_2, x_3, x_4]$

II.1.3. minimize $f_0(x) = c^T x$ subject to $Ax \leq b$, $x \in R^n_+$.

All dimensions of c are negative. So, for the problem to be unbounded at least one of the dimensions of x has to go to infinity. We can see that under the constraint $x \in R^n_+$, none of the dimensions of x can go to infinity without violating the constraint $Ax \leq b$. Hence, the problem is bounded.

$$\underbrace{\begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 2 & 3 & -1 & 2 \\ 0 & 2 & 2 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{\bar{A}} x \leq \underbrace{\begin{bmatrix} 2 \\ 1 \\ 3 \\ 5 \\ 5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\bar{b}} \tag{5}$$

Note that $\text{rank}(\bar{A}) = 4$. Recall from previous HWs that we can get the vertices of polyhedron by solving all combinations of 4 inequalities from $\bar{A}x \leq \bar{b}$ and then verifying if the solution satisfies rest of the inequalities. We get the following vertices $\bar{X}$

$$\bar{X} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.5 & 1 & 0.25 & 0.5 & 11/14 & 0 & 1/7 \\ 0 & 0 & 0 & 0 & 1.5 & 1 & 1.5 & 0.5 & 0.5 & 0.5 & 1 & 0.5 & 1.2 & 0 & 0 & 1 & 1.5 & 2/3 & 18/14 & 0.5 & 8/7 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 2 & 2 & 2 & 1 & 2 & 0.6 & 0 & 0 & 0 & 0 & 0 & 6/14 & 2 & 5/7 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0.25 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$c^\top \bar{X} = [0.0, -1, -6, -4, -3, -3, -3.25, -7, -7, -7, -6, -7, -5.2, -1, -1.5, -3, -3.25, -2.83, -4.64, -7, -5.57]$

We can achieve a minimum of -7 for $\bar{x}^\top = [0, 1/2, 2, 0]$

To check optimality criterion, notice that all feasible solution can be written as $\{\bar{X}\theta | 1^\top \theta = 1, \theta \geq 0\}$ and $\nabla f_0(x) = c$. Hence,

$$\begin{aligned} \min_\theta \nabla f_0(\bar{x})^\top (\bar{X}\theta - \bar{x}) &= \min_\theta c^\top (\bar{X}\theta - \bar{x}) \\ &= \min_\theta c^\top \bar{X}\theta - c^\top \bar{x} \\ &= \min_\theta c^\top \bar{X}\theta + 7 \\ &= 0 \\ \implies \nabla f_0(\bar{x})^\top (\bar{X}\theta - \bar{x}) \geq 0 & \qquad \forall \theta \end{aligned}$$

II.1.4. minimize $f_0(x) = c^T x$ subject to $Ax = b$, $x \in R^n_+$.

$$[A|b] \xrightarrow{RREF} I_{5\times 5}$$

So, $Ax = b$ does not have a solution. Hence, the problem is infeasible. We can make the problem feasible by turning the last equality constraint to inequality i.e., $2x_2 + 2x_3 \leq 5$ where $x^\top = [x_1, x_2, x_3, x_4]$

II.2 Graph embedding (20 pts) Graph embedding is an important problem in machine learning and graph theory. Given an undirected graph $G = (V, E)$ with n vertices, the problem is to assign coordinates in $\mathbb{R}^m$ to each vertex $v \in V$. Typically there are desired qualities or constraints imposed on the embedding—e.g. the coordinates assigned to connected nodes should be close with respect to some notion of distance. For example, the choice of Euclidean distance yields a quadratically constrained quadratic program (QCQP). Let $A \in \{0,1\}^{n\times n}$ be the symmetric adjacency of G, and let D be the corresponding diagonal degree matrix such that $D_{ii} = \sum_j A_{i,j}$. The *graph Laplacian* is defined to be $L = D - A$.

One well known way to do graph embedding (Laplacian Eigenmaps) is to solve the following problem:

$$\begin{aligned} &\min_{X\in\mathbb{R}^{n\times m}} \langle X, LX\rangle \\ &\text{s.t. } X^\top X = I,\ \mathbf{1}^\top X = 0 \end{aligned} \tag{6}$$

Where the inner product $\langle A, B\rangle$ is defined to be $\mathrm{tr}(A^\top B)$.

(i) Show that when $m = 1$, $\langle x, Lx\rangle = x^\top Lx = \sum_{i,j\in E}(x_i - x_j)^2$. Is (1) convex? Why or why not?

$$\begin{aligned} x^\top Lx = x^\top(D-A)x &= x^\top Dx - x^\top Ax \\ &= \sum_i D_{ii}x_i^2 - \sum_{i,j\in E} 2x_ix_j \\ &= \sum_i \sum_{i,j\in E} x_i^2 - \sum_{i,j\in E} 2x_ix_j \\ &= \sum_{i,j\in E}(x_j^2 + x_i^2 - 2x_ix_j) = \sum_{i,j\in E}(x_i - x_j)^2 \end{aligned}$$

It's worth noting that this is a rare non-convex problem ($x^\top x = 1$ characterizes a sphere) that has a closed form solution (given by the eigenvectors/eigenvalues of L) and zero duality gap.

(ii) Show how the linear constraint may be eliminated and derive the dual and KKT conditions of the relaxation. Are there any issues with this formulation?

$$\begin{aligned} &\min_{X\in\mathbb{R}^{n\times m}} \langle X, \widetilde{L}X\rangle \\ &\text{s.t. } \langle X, X\rangle \leq r^2 \end{aligned} \tag{7}$$

Let P be the projection onto the orthogonal complement of the subspace spanned by $\mathbf{1}$, i.e. $P = I - \frac{1}{n}\mathbf{1}\mathbf{1}^\top$. Apply the substitution $X \leftarrow PX$. The objective can be re-written: $\langle PX, LPX\rangle = \langle X, PLPX\rangle = \langle X, \widetilde{L}X\rangle$. $\widetilde{L}$ is PSD. The relaxation admits a degenerate solution of $X^* = 0$.

(iii) One way to condition the relaxation is to introduce additional constraints. Consider a "semi-supervised" modification of problem 1: where first k vertices are "labeled" or "anchored"—i.e. we have the constraints $X_i = y_i$ for $1, \ldots, k$, , where $y_i \in \mathbb{R}^m$.

$$\begin{aligned} &\min_{X\in\mathbb{R}^{n\times m}} \langle X, \widetilde{L}X\rangle \\ &\text{s.t. } \langle X, X\rangle \leq r^2, \quad X_i = y_i, \ldots i = 1, \ldots, k \end{aligned} \tag{8}$$

Derive the associated optimization problem and its dual in *standard form*. (Hint: there are multiple derivations. Two potential directions: (a.) consider the partitioning $X = \left[\begin{smallmatrix} X_l \\ X_u \end{smallmatrix}\right]$, where X_l denote the "labeled" first-k rows of X (b.) consider the geometry of the label constraints.)

For simplicity, consider the $m = 1$ case: a partitioning of L that is implied by the partitioning of x mentioned in the hint:

$$L = \begin{bmatrix} L_{uu} & L_{ul} \\ L_{lu} & L_{ll} \end{bmatrix}$$

Now, expanding the quadratic:

$$\begin{aligned} x^\top Lx &= [x_u : x_l]L[x_u : x_l]^\top \\ &= [x_u : x_u] \begin{bmatrix} L_{uu} & L_{ul} \\ L_{lu} & L_{ll} \end{bmatrix} [x_u : x_l]^\top \\ &= x_1^\top L_{uu} x_1 + 2x_l^\top L_{lu} x_1 + x_l^\top L_{ll} x_l \end{aligned}$$

So, $b = 2x_l^\top L_{lu}$. Additionally, note that the minimizer of $x_u^\top Lx_u + x_u^\top b$ is equivalent to the minimizer of $x_u^\top Lx_u + x_u^\top b + x_l^\top L_{ll} x_l$. Alternatively, the equality constriants on rows of X correspond to a set of affine constraints. With an appropriate choice of D and Y, these constraints imply the equality constraint $DX = Y$.
Dual of a QCQP:

$$\begin{aligned} &\min_{X \in \mathbb{R}^{n-k \times m}} \langle X, AX \rangle + 2\langle B, X \rangle \\ &\text{s.t. } \langle X, X \rangle \leq r^2 \end{aligned} \tag{9}$$

define the Lagrangian where λ are the Lagrange multipliers and $X^\top X = \langle X, X \rangle$:

$$\mathcal{L}(X, \lambda) = \langle X, \widetilde{L}X \rangle + 2\langle X, B \rangle + \lambda(X^\top X - r^2) = \langle X, (\widetilde{L} + \lambda I)X \rangle + 2\langle X, B \rangle - \lambda r^2. \tag{10}$$

The first-order condition is then

$$\widetilde{L}X + B = -X\lambda \tag{11}$$

The associated dual problem is

$$\max_{\lambda \geq 0} -\langle B, (\widetilde{L} + \lambda I)^{-1} B \rangle - \lambda r^2 \tag{12}$$

Note that Slater's condition, and thus strong duality, holds for the primal problem. However, there is no closed form solution (unlike in (i)).

(iv) Let $X \in \mathbb{R}^{n \times 3}$ represent the coordinates of n vertices in 3-d. The vertex coordinate assignment can be visualized—a "drawing" of G. Implement your solution to Problem (iii) and show your result for the given graph. We have written a partial framework in Python+CVXPY to get you started[1]:
`https://colab.research.google.com/drive/1nvwWug_eowcAf4HrznlTm4YhcBJuokgc?usp=sharing`.
Solution: `https://colab.research.google.com/drive/1i51iYvJe5MguooGo6cdCOUK6KDi9fvf2?usp=sharing`

[1]If you prefer a different language or framework, you can also download a .txt file containing L, x, y, and the indices of the fixed nodes: `https://piazza.com/class_profile/get_resource/kx85xrdgigl5m5/kzfw6ud6fd964c` (idx, x, y are the first 3 columns)