CSE 203B W21 Homework 4

Due Time : 11:50pm, Wednesday Feb. 22, 2023 Submit to Gradescope

In this homework, we work on exercises from the textbook. Problems 4.1, 4.8, 4.11, and 4.15 are related to LP. Problems 4.21, 4.39, and 4.47 are related to QCQP, and SDP. Problem 5.3 is about the basic definition of duality. Problems 5.4, 5.5, 5.6, 5.7, 5.8, 5.9, and 5.10 are examples and applications of duality. Also, we practice using the convex optimization tools on a linear programming problem, and a quadratic programming problems.

Total points: 50. Exercises are graded by completion, and assignments are graded by correctness.

I. Exercises from textbook chapters 4 & 5 (15 pts, 1pt for each problem)

4.1, 4.8, 4.11, 4.15, 4.21, 4.39, 4.47, 5.3, 5.4, 5.5, 5.6, 5.7, 5.8, 5.9, 5.10.

II. Assignments (35 pts)

II.1 Linear Programming: You are free to use any software packages. (15 pts)

Given

$$A = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 2 & 3 & -1 & 2 \\ 0 & 2 & 2 & 0 \end{bmatrix},$$

$$b^T = \begin{bmatrix} 2 & 1 & 3 & 5 & 5 \end{bmatrix},$$

$$c^T = \begin{bmatrix} -1 & -2 & -3 & -1 \end{bmatrix},$$

and $n = 4$, solve the following linear programming problems. If a solution is found, validate that the solution satisfies the optimality criteria (which was talked about in class or textbook). Otherwise, explain why a solution is not feasible and suggest how to mitigate the issue if you are the project leader:

II.1.1. minimize $f_0(x) = c^T x$ subject to $Ax \leq b$, $x \in R^n$.

II.1.2. minimize $f_0(x) = c^T x$ subject to $Ax = b$, $x \in R^n$.

II.1.3. minimize $f_0(x) = c^T x$ subject to $Ax \leq b$, $x \in R^n_+$.

II.1.4. minimize $f_0(x) = c^T x$ subject to $Ax = b$, $x \in R^n_+$.

II.2 Graph embedding (20 pts) Graph embedding is an important problem in machine learning and graph theory. Given an undirected graph $G = (V, E)$ with n vertices, the problem is to assign coordinates in $\mathbb{R}^m$ to each vertex $v \in V$. Typically there are desired qualities or constraints imposed on the embedding—e.g. the coordinates assigned to connected nodes should be close with respect to some notion of distance. For example, the choice of Euclidean distance yields a quadratically constrained quadratic program (QCQP). Let $A \in \{0, 1\}^{n \times n}$ be the symmetric adjacency of G, and let D be the corresponding diagonal degree matrix such that $D_{ii} = \sum_j A_{i,j}$. The *graph Laplacian* is defined to be $L = D - A$.

One well known way to do graph embedding (Laplacian Eigenmaps) is to solve the following problem:

$$\begin{aligned} \min_{X \in \mathbb{R}^{n \times m}} & \langle X, LX \rangle \\ \text{s.t. } & X^\top X = I, \ \mathbf{1}^\top X = 0 \end{aligned} \tag{1}$$

Where the inner product $\langle A, B \rangle$ is defined to be $\text{tr}(A^\top B)$.

(i) Show that when $m = 1$, $\langle x, Lx \rangle = x^\top Lx = \sum_{i,j \in E} (x_i - x_j)^2$. Is (1) convex? Why or why not?

(ii) Show how the linear constraint may be eliminated and derive the dual and KKT conditions of the relaxation, where r is a scalar constant. Are there any issues with this formulation?

$$\begin{aligned} \min_{X \in \mathbb{R}^{n \times m}} & \langle X, \widetilde{L}X \rangle \\ \text{s.t. } & \langle X, X \rangle \leq r^2 \end{aligned} \tag{2}$$

(iii) One way to condition the relaxation is to introduce additional constraints. Consider a "semi-supervised" modification of problem 1: where first k vertices are "labeled" or "anchored"—i.e. we have the constraints $X_i = y_i$ for $1, \ldots, k,$, where $y_i \in \mathbb{R}^m$.

$$\begin{aligned} \min_{X \in \mathbb{R}^{n \times m}} & \langle X, \widetilde{L}X \rangle \\ \text{s.t. } & \langle X, X \rangle \leq r^2, \quad X_i = y_i, \ldots i = 1, \ldots, k \end{aligned} \tag{3}$$

Derive the associated optimization problem and its dual in *standard form*. (Hint: there are multiple derivations. Two potential directions: (a.) consider the partitioning $X = \begin{bmatrix} X_l \\ X_u \end{bmatrix}$, where X_l denote the "labeled" first-k rows of X (b.) consider the geometry of the label constraints.)

(iv) Let $X \in \mathbb{R}^{n \times 3}$ represent the coordinates of n vertices in 3-d. The vertex coordinate assignment can be visualized—a "drawing" of G. Implement your solution to Problem (iii) and show your result for the given graph. We have written a partial framework in Python+CVXPY to get you started[1]:
`https://colab.research.google.com/drive/1nvwWug_eowcAf4HrznlTm4YhcBJuokgc?usp=sharing`.

[1] If you prefer a different language or framework, you can also download a .txt file containing L, x, y, and the indices of the fixed nodes: `https://piazza.com/class_profile/get_resource/kx85xrdgigl5m5/kzfw6ud6fd964c` (idx, x, y are the first 3 columns)