

## CSE 203B W23 Homework 2

Due Time : 11:50pm, Wednesday, Jan. 25, 2023. Submit to Gradescope  
Gradescope: <https://gradescope.com/>

In this homework, we work on exercises from the textbook including midpoint convexity (2.3), Voronoi diagram (2.7, 2.9), quadratic function (2.10), general sets (2.12), cones and dual cones (2.28, 2.31, 2.32), and separation of cones (2.39). Extra assignments are given on convex sets.

Total points: 30. Exercises are graded by completion, assignments are graded by content.

### I. Exercises from textbook chapter 2 (9 pts, 1pt for each problem)

2.3, 2.7, 2.9, 2.10, 2.12, 2.28, 2.31, 2.32, 2.39.

### II. Assignments (30 pts)

II. 1. Qualification vs. enumeration of convex sets:

Given

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$b^T = [2 \quad 1 \quad 2 \quad 2],$$

we describe the convex sets as follows.

II.1.1. Convert set  $\{x | Ax \leq b, x \in \mathbb{R}_+^6\}$  from a qualification oriented expression to an enumeration oriented expression in the format of  $\{U\theta | 1^T\theta = 1, \theta \in \mathbb{R}_+^m\}$ . (8 pts)

**[Solution]** Let  $x = [x_1, \dots, x_6]^T$ , the convex set  $\{x | Ax \leq b, x \in \mathbb{R}_+^6\}$  is the solution set of the following 10 inequalities:

$$x_i \geq 0 \quad i = 1, \dots, 6$$

$$2x_1 \leq 2$$

$$x_2 + 2x_4 \leq 1$$

$$x_3 + x_5 \leq 2$$

$$x_6 \leq 2$$

whose matrix form is

$$\underbrace{\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}}_{\bar{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}}_x \leq \underbrace{\begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\bar{b}}$$

Geometrically, this convex set is a polyhedron in  $\mathbb{R}^6$  which can be represented as a convex combination of its vertices in the numeration-oriented expression. This means that each column of  $U$  is a vertex in  $\mathbb{R}^6$ . A vertex is a point satisfying two conditions: (1) It is the intersection of 6 hyperplanes from  $\bar{A}x = \bar{b}$ . That is, each vertex corresponds to the solution of  $\tilde{A}x = \tilde{b}$  which are 6 equations selected from  $\bar{A}x = \bar{b}$  and  $rank(\tilde{A}) = 6$ . (2) A vertex should satisfy the rest inequalities in  $Ax \leq b$ . To find all of the vertices, we first construct  $C_{10}^6 = 210$  possible submatrices from  $\bar{A}$ , keep those having rank 6. Then solve for  $\tilde{A}x = \tilde{b}$  and keep the solution satisfying the rest inequalities in  $Ax \leq b$ . This results in **36** vertices which form the columns of  $U$  as

$$col(U)_{1-18} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 \end{bmatrix}$$

$$col(U)_{19-36} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 \end{bmatrix}$$

II.1.2. Convert set  $\{x|Ax = 0, x \in \mathbb{R}^6\}$  from a qualification oriented expression to an enumeration oriented expression in the format of  $\{Px|x \in \mathbb{R}^6\}$ . (4 pts)

**[Solution]** The set  $\{x|Ax = 0, x \in \mathbb{R}^6\}$  is the nullspace of  $A$ , which can be generated by  $Px$  ( $x \in \mathbb{R}^6$ ).  $P$  is a projection matrix projecting  $\forall x \in \mathbb{R}^6$  to the nullspace of  $A$ .  $P$  can be derived from the QR decomposition of  $A^T$ :

$$A^T = QR = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1/\sqrt{5} & 0 & 0 \\ 0 & 0 & -1/\sqrt{2} & 0 \\ 0 & -2/\sqrt{5} & 0 & 0 \\ 0 & 0 & -1/\sqrt{2} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} R$$

$$P = I - QQ^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & -0.4 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & -0.5 & 0 \\ 0 & -0.4 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & -0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

II.1.3. Derive the dual cone of the set  $\{x|Ax \leq 0, x \in \mathbb{R}^6\}$ . (4 pts)

[Solution] Let  $C = \{x | Ax \leq 0, x \in \mathbb{R}^6\}$ , the dual cone is the set

$$\begin{aligned} C^* &= \{y | y^T x \geq 0, \forall x \in C\} \\ &= \{y | x^T y \geq 0, \forall x \in C\} \\ &= \{y | x^T y \geq 0, \forall Ax \leq 0, x \in \mathbb{R}^6\} \end{aligned}$$

Let  $y = A^T \theta$ ,  $x^T y = x^T A^T \theta = (Ax)^T \theta \geq 0$ ,  $\forall Ax \leq 0$ , if  $\theta \leq 0$   
Therefore,  $C^* = \{A^T \theta | \theta \in \mathbb{R}_-^4\}$ .

II. 2. Qualification vs. enumeration of convex sets:

Given

$$U = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix},$$

we describe the convex sets as follows.

II.2.1. Convert set  $\{U\theta | 1^T \theta \leq 1, \theta \in \mathbb{R}_+^3\}$  from an enumeration oriented expression to a qualification oriented expression in the format of  $\{x | Ax \leq b, x \in \mathbb{R}^4\}$ . (8 pts)

[Solution 1] Let set  $C = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ , set  $\{U\theta | 1^T \theta \leq 1, \theta \in \mathbb{R}_+^3\}$  is the convex hull of

$C$ , denoted as  $\text{conv}C$ . The vertices of  $\text{conv}C$  are the four points in  $C$ .

Let  $M = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ ,  $\text{conv}C$  lies in the column space of  $M$  (denoted as  $V$ ). As  $\text{rank}(M) = 3$

and the origin is in  $\text{conv}C$ ,  $\text{conv}C$  lies in a 3-dimensional subspace. Furthermore,  $\text{conv}C$  is a 3-dimensional simplex with 4 vertices and 4 faces. Any three vertices determine one hyperplane

enclosing this simplex, which corresponds to one inequality in the qualification-oriented expression.

For computational convenience, we will first transform the coordinate system from  $\mathbb{R}^4$  to  $V$ , compute the hyperplane expressions in  $V$ , then transform back to  $\mathbb{R}^4$ .

The transformation from  $\mathbb{R}^4$  to  $V$  includes the following two steps:

1. Rotate  $\mathbb{R}^4$  to a four-dimensional vector space  $W$  whose fourth axis aligns with the normal of  $V$ . We can use SVD to get the rotation matrix from  $\mathbb{R}^4$  to  $W$ . Let  $M = U\Sigma V^T$ , the columns of  $U$  give the orthonormal basis of  $W$ . Therefore, for any vector  $v$ , we can transform its representation from  $\mathbb{R}^4$  to  $W$  by

$$v' = U^T v \quad (1)$$

2. Project the four-dimensional vector space  $W$  to three-dimensional subspace  $V$  by a projection matrix, i.e. for any vector  $v$ , we can transform its representation from  $W$  to  $V$  by

$$v' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} v \quad (2)$$

Thus, we can transform the vertices from  $\mathbb{R}^4$  to  $V$  by applying (1) and (2) to  $M$ . These two transformations can be composed as the transpose of the first three columns of  $U$  because the first  $\text{rank}(M)$  columns of  $U$  give the orthonormal basis of  $V$ . We denote  $\text{col}(U)_{1-3}$  as  $R$ .

$$M' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} U^T M = \text{col}(U)_{1-3}^T M = R^T M \quad (3)$$

Now each column of  $M'$  is a vertex represented in  $V$ , for any three of them, say vertex  $u, v, w$ , we can compute a hyperplane  $a^T(x - u) = 0$  they are lying on. And its normal vector is

$$a = \frac{d}{\|d\|_2}$$

where  $d = (v - u) \times (w - v)$ . Note that the direction of  $a$  should always point towards the outside of the simplex.

We can express these 4 hyperplanes as  $3 \times 4$  matrices  $A_1$  and  $P_1$  where the  $i$ th column of  $A_1$  is the normal vector of hyperplane  $i$  and the  $i$ th column of  $P_1$  is one vertex on hyperplane  $i$ .

Then we can transform these hyperplanes back to  $\mathbb{R}^4$  by applying transformation  $R$  to them:

$$\begin{aligned} A &= RA_1 \\ P &= RP_1 \end{aligned}$$

Moreover, the collapse of the fourth dimension gives us an equality constraint in  $W$ :

$$c_0^T(x - x_0) = 0 \quad (4)$$

where  $c_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$  and  $x_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

We can rotate the hyperplane (4) back to  $\mathbb{R}^4$  by applying  $U$  to it:

$$c = Uc_0$$

$$x_1 = Ux_0$$

Next, we convert the hyperplane expression from form  $a^T(x - x_0) = 0$  to  $a^T x = b$  by computing  $b = [b_1, \dots, b_4]^T$  and  $d$  as:

$$b_i = \text{col}(A)_i^T \text{col}(P)_i \quad i = 1, \dots, 4$$

$$d = c^T x_1$$

Finally, we have the qualification-oriented expression of the set as  $\{x | Ax \leq b, c^T x = d, x \in \mathbb{R}^4\}$  where

$$A = \begin{bmatrix} 0 & -1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 0 & -1 & 0 \\ -2/\sqrt{5} & 0 & 1/\sqrt{5} & 0 \\ 0.555 & 0.555 & 0.277 & 0.555 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1.109 \end{bmatrix}$$

$$c = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$d = 0$$

Note that multiplying  $[c^T \quad d]$  with any non-zero scalar or multiplying each row in  $[A \quad b]$  with any positive scalar will result in the same solution set, thus are equivalently correct.

**[Solution 2]** Since the convex set described in the problem is a 3-simplex, we can use the formulas in the textbook [Boyd and Vandenberghe] P32-P33 to convert from the enumeration-oriented expression to a qualification-oriented expression.

With the same notations, let  $v_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ,  $B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$ ,  $A_1$  is the left pseudo inverse of

$B$ , rows of  $A_2$  are the basis of the nullspace of  $B^T$ . Then the qualification-oriented expression  $\{x | Ax \leq b, c^T x = d, x \in \mathbb{R}^4\}$  can be constructed as

$$A = \begin{bmatrix} -A_1 \\ \mathbf{1}^T A_1 \end{bmatrix}$$

$$b = \begin{bmatrix} -A_1 v_0 \\ 1 + \mathbf{1}^T A_1 v_0 \end{bmatrix}$$

$$c = A_2^T$$

$$d = A_2 v_0$$

where

$$A = \begin{bmatrix} -0.5 & 0 & 0.25 & 0 \\ 0 & -0.5 & 0 & -0.5 \\ 0 & 0 & -0.5 & 0 \\ 0.5 & 0.5 & 0.25 & 0.5 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$c = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$d = 0$$

II.2.2. Derive the dual cone of the set  $\{U\theta \mid \theta \in \mathbb{R}_+^3\}$ . (4 pts)

**[Solution]** Let  $C = \{U\theta \mid \theta \in \mathbb{R}_+^3\}$ , the dual cone is the set

$$\begin{aligned} C^* &= \{y \mid y^T x \geq 0, \forall x \in C\} \\ &= \{y \mid y^T U\theta \geq 0, \forall \theta \in \mathbb{R}_+^3\} \\ &= \{y \mid \theta^T (U^T y) \geq 0, \forall \theta \in \mathbb{R}_+^3\} \\ &= \{y \mid U^T y \geq 0, y \in \mathbb{R}^4\} \end{aligned}$$

Therefore,  $C^* = \{y \mid U^T y \geq 0, y \in \mathbb{R}^4\}$ .

II. 3. Given  $p$  hyperplanes

$$a_i^T x = b_i, \text{ for } i = 1, 2, \dots, p, \quad x \in \mathbb{R}^n.$$

List the maximum number of disjoint regions separated by the hyperplanes for the following cases.

(6 pts)

(1)  $n = 2, p = 2$ .

**[Solution]**

The maximum number of disjoint regions separated by 2 hyperplanes in  $\mathbb{R}^2$  is 4.

(2)  $n = 2, p = 3$ .

**[Solution]**

The maximum number of disjoint regions separated by 3 hyperplanes in  $\mathbb{R}^2$  is 7.

(3)  $n = 2, p = 5$ .

[Solution] The maximum number of disjoint regions separated by 5 hyperplanes in  $\mathbb{R}^2$  is 16.

(4)  $n = 3, p = 3$ .

[Solution]

The maximum number of disjoint regions separated by 3 hyperplanes in  $\mathbb{R}^3$  is 8.

(5)  $n = 3, p = 6$ .

[Solution]

The maximum number of disjoint regions separated by 6 hyperplanes in  $\mathbb{R}^3$  is 42.

(6) Generalize the problem to any given  $n$  and  $p$  and write down the equation. For example,  $N(n, p) = 1 + p$  if  $n = 1$ .

[Solution]

The general equation for any given  $n$  and  $p$  is

$$N(n, p) = \sum_{i=0}^{\min(n, p)} C_p^i$$