KKT (Karush-Kuhn-Tucker) Conditions

1. Primal, Lagrangian, and Dual

$$\max_{\lambda,v}\min_{x}: (x,\lambda,v) = \max_{\lambda,v} g(\lambda,v)$$

$$\begin{aligned} &\min f_o(x) && L(x,\lambda,v)\\ &f_i \le 0 && \\ &h_i = 0 && = f_o(x) + \sum_{i=1}^{m}\lambda_i f_i(x) + \sum_{i=1}^{p} v_i h_i(x), \lambda_i \ge 0 \end{aligned}$$

$$\min_{x}\max_{\lambda,v} \qquad \max_{\lambda,v}\min_{x} \ (dual)$$

1. Feasibility (x,λ,v)
2. $L(x,\lambda,v) = f_o(x)$ $\begin{cases}\lambda_i > 0 \ if \ f_i = 0\\ \lambda_i = 0 \ if \ f_i < 0\end{cases}$
3. $g(\lambda,v) = \min_{x} L(x,\lambda,v) = f_o(x)$

Necessary condition for local optimality
Sufficient when the problem is convex & satisfy regularity conditions (Slater condition)

Sensitivity

Perturbed Problem

$$\begin{aligned} &\min f_o(x)\\ &s.t. \ \ f_i \le u_i\\ &\qquad h_i(x) = w_i \end{aligned}$$

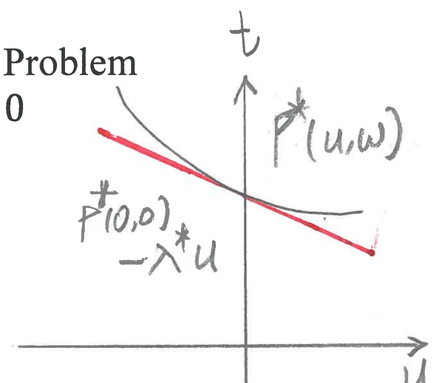
$L(x,\lambda,v) = f_o(x) + \sum \lambda_i f_i - \sum \lambda_i u_i + \sum v_i h_i - \sum v_i w_i$

$$\begin{aligned} &\max \tilde{g} = g(\lambda,v) - u^T\lambda - w^T v\\ &s.t. \quad \lambda \ge 0\\ &p^*(u,w) = \max_{\lambda,v} g(\lambda,v) - u^T\lambda - w^T v \end{aligned}$$

Unperturbed Problem

$$u_i = w_i = 0$$

$$\begin{aligned} &\max g(\lambda,v)\\ &s.t. \quad \lambda \ge 0\\ &p^*(0,0), \lambda^*, v^* \end{aligned}$$



$$p^*(u,w) \ge g(\lambda^*,v^*) - u^T\lambda^* - w^T v^* = p^*(0,0) - u^T\lambda^* - w^T v^*$$

$$\lambda_i^* = -\frac{\partial p^*(u,w)}{\partial u_i}\bigg| \underline{(u,w) = (0,0)}, \quad v_i^* = -\frac{\partial p^*(u,w)}{\partial w_i}\bigg| \underline{(u,w) = (0,0)}$$

Proof: $\dfrac{\partial p^*(0,0)}{\partial u_i} = \lim_{t \searrow 0}\dfrac{p^*(te_i,0) - p^*(0,0)}{t} \ge -\lambda_i^*$

$\dfrac{\partial p^*(0,0)}{\partial u_i} = \lim_{t \nearrow 0}\dfrac{p^*(te_i,0) - p^*(0,0)}{t} \le -\lambda_i^*$

hence, equality

$$\frac{\partial p^*(0,0)}{\partial u_i} = -\lambda_i^*$$

8WB

Shadow Price Interpretation: Spring Energy & Force

$$min\, f_o(x_1, x_2) = \frac{1}{2}k_1 x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2 + \frac{1}{2}k_3(l - x_2)^2$$

f_o : potential energy $\quad k_i > 0$: stiffness constant of spring i

$w/2 - x_1 \leq 0$

$w + x_1 - x_2 \leq 0$

$w/2 - l + x_2 \leq 0$

$$min \frac{1}{2}(k_1 x_1^2 + k_2(x_2 - x_1)^2 + k_3(l - x_2)^2)$$

$\lambda_1 \quad w/2 - x_1 \leq 0$

$\lambda_2 \quad w + x_1 - x_2 \leq 0$

$\lambda_3 \quad w/2 - l + x_2 \leq 0$

$\lambda_1(w/2 - x_1) = 0, \lambda_2(w - x_2 + x_1) = 0, \lambda_3(w/2 - l + x_2) = 0$

zero gradient condition

$$\begin{bmatrix} k_1 x_1 - k_2(x_2 - x_1) \\ k_2(x_2 - x_1) - k_3(l - x_2) \end{bmatrix} + \lambda_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

λ_i : contact forces between the walls & blocks

KKT (Karush-Kuhn-Tucker) Conditions

2. $f_i(x), i = 1, \dots, m,\ h_i(x),\ i = 1, \dots, p$ *are differentiable*

1. Primal constraints : $f_i(x) \leq 0, i = 1, \dots, m.$
   $h_i(x) = 0, i = 1, \dots, p.$

2. Dual constraints : $\lambda \succcurlyeq 0$

$L(x, \lambda, \nu) = f_0(x) + \sum_i \lambda_i f_i(x) + \sum_i \nu_i h_i(x)$

3. Complementary slackness : $\lambda_i f_i(x) = 0, i = 1, \dots, m.$

① $f_i(x) = 0,\ \lambda_i \geq 0.$

② $f_i(x) < 0,\ \lambda_i = 0.$

4. Gradient of Lagrangian with respect to x variables
   $\nabla_x f_o(x) + \sum_{i=1}^{m} \lambda_i \nabla_x f_i(x) + \sum_{i=1}^{p} v_i \nabla_x h_i(x) = 0$

Generalized Inequalities: SOCP

<u>Primal</u>

$$\min f^T x$$
$$\|A_i x + b_i\|_2 \le c_i^T x + d, i = 1, \dots, m$$

i.e. $(A_i x + b_i, c_i^T + d_i) \in K_i, \; i = 1, \dots, m$

<u>Lagrangian</u>

$$L(x, \lambda, v) = f^T x - \sum (z_i^T (A_i x + b_i) + w_i (c_i^T + d_i))$$

$$(z_i, w_i) \in K_i^*, \; i = 1, \dots, m$$

<u>Lagrange Dual</u>

$$\max - \textstyle\sum_i (b_i^T z_i + d_i w_i)$$

$$\|z_i\|_2 \le w_i, \qquad i = 1, \dots, k$$

$$\textstyle\sum_i A_i^T z_i + c_i w_i = f$$

Generalized Inequalities: Semidefinite Program

<u>Primal</u>

$$\min c^T x$$
$$x_1 F_1 + \dots + x_n F_n + G \preccurlyeq 0, \text{where } F_1, \dots, F_n, G \in S^k$$

<u>Lagrangian</u>

$$L(x, \lambda, v) = \inf_x (c^T x + tr((\underline{x_1 F_1 + \dots + x_n F_n + G}) \underline{Z})),$$

$$Z \in S_+^k$$

<u>Lagrange Dual</u> $\quad g(\lambda, v) = \inf_x L(x, \lambda, v)$

$tr(\underbrace{W \times Z})$

$= \sum_{i,j} w_{ij} z_{ij}$

<u>Dual</u>

$$\max_Z tr(GZ)$$

$$tr(F_i Z) + c_i = 0, i = 1, \dots, n$$

$$Z \succcurlyeq 0$$

Generalized Inequalities

$f_i(x) \in R^{K_i}$

Primal

$$\min f_o(x)$$
$$f_i(x) \preccurlyeq_{K_i} 0, i = 1, \dots, m \Rightarrow 0 - f_i(x) \in K_i$$
$$h_i(x) = 0, i = 1, \dots, p$$

Lagrangian

$$L(x, \lambda, v) = f_o(x) + \sum_{i=1}^{m} \lambda_i^T f_i(x) + \sum_{i=1}^{p} v_i h_i(x),$$

$$\lambda_i \succcurlyeq_{K_i^*} 0, i = 1, \dots, m, \ \lambda_i \in R^{k_i}$$

$\Rightarrow \lambda_i^T f_i(x) \leq 0$

if $-f_i(x) \in K_i$

& $\forall \lambda_i \in K_i^*$.

Lagrange Dual

$$g(\lambda, v) = \inf_x L(x, \lambda, v)$$

$\lambda_i \succeq K_i^*$

$(\|x\|_2 \leq c^T x + b)$

Generalized Inequality: KKT Conditions

$$\min f_o(x)$$
$$s.t\ f_i \preccurlyeq_{K_i} 0, i = 1, \dots, m$$
$$h_i = 0, i = 1, \dots, p$$

$f_i(x), i = 1, \dots, m, \ h_i(x), \ i = 1, \dots, p$ are differentiable

1. Primal constraints : $f_i(x) \preccurlyeq_{K_i} 0, i = 1, \dots, m.$
   $h_i(x) = 0, i = 1, \dots, p.$
2. Dual constraints : $\lambda \succcurlyeq_{K_i^*} 0$
3. Complementary slackness : $\lambda_i^T f_i(x) = 0, i = 1, \dots, m.$
4. Gradient of Lagrangian with respect to x variables
   $\nabla_x f_o(x) + \sum_{i=1}^{m} \lambda_i^T \nabla_x f_i(x) + \sum_{i=1}^{p} v_i \nabla_x h_i(x) = 0$

Chapter 5 Summary

- Primal and Dual Problem
  - Primal Problem
  - Lagrangian Function
  - Lagrange Dual Problem
- Examples (Primal Dual Conversion Procedure)
  - Linear Programming
  - Quadratic Programming
  - Conjugate Functions (Duality)
  - Entropy Maximization
- Interpretation (Duality)
  - Saddle-Point Interpretation
  - Geometric Interpretation
  - Slater's Condition
  - Shadow-Price Interpretation
- KKT Conditions (Optimality Conditions)
- Sensitivity (Shadow-Price)
- Generalized Inequalities

Chapter 5 Summary

- Duality provides a lower bound of the problem even the primal may not be convex.
- KKT conditions convert the minimization problem into equations.
- Lagrange multipliers provide the sensitivity of the constraints.
- Generalized inequality broadens the scope of convex optimization.