Interpretation: Saddle-point

Claim : Result of II $\geqslant$ Result of I

Given an arbitrary pair $(\tilde{w}, \tilde{z}) \in D$

$$\min_{w \in W} f(w, \tilde{z}) \leq f(\tilde{w}, \tilde{z}) \leq \max_{z \in Z} f(\tilde{w}, z) \quad \forall \tilde{w}, \tilde{z} \in D$$

$$\min_{w \in W} f(w, \tilde{z}) \leq \max_{z \in Z} f(\tilde{w}, z)$$

Thus $\max_{z \in Z} \min_{w \in W} f(w, z) \leq \min_{w \in W} \max_{z \in Z} f(w, z)$

Interpretation: Saddle-point

Example : $f(w, z)$ $w = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

$z = 1, \quad 2, \quad 3$

$$\min_{w \in W} f(w, 1) = 1 \qquad \max_{z \in Z} f(1, z) = 1$$

$$\min_{w \in W} f(w, 2) = 1 \qquad \max_{z \in Z} f(2, z) = 2$$

$$\min_{w \in W} f(w, 3) = 1 \qquad \max_{z \in Z} f(3, z) = 3$$

$$\max_{z \in Z} \min_{w \in W} f(w, z) = 1 \qquad \min_{w \in W} \max_{z \in Z} f(w, z) = 1$$

Interpretation: Saddle-point

$$\text{Example}: f(w,z) \quad w = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$
$$z = 1, \quad 2, \quad 3$$

$$\min_{w\in W} f(w,1) = 1 \qquad \max_{z\in Z} f(1,z) = 3$$
$$\min_{w\in W} f(w,2) = 1 \qquad \max_{z\in Z} f(2,z) = 3$$
$$\min_{w\in W} f(w,3) = 1 \qquad \max_{z\in Z} f(3,z) = 3$$

$$\max_{z\in Z} \min_{w\in W} f(w,z) = 1 \qquad \min_{w\in W} \max_{z\in Z} f(w,z) = 3$$

$$[w_1\ w_2\ w_3]\begin{bmatrix} \times & & \\ \times & & \\ & & \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$[\tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{1}{3}]\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = [2\ 2\ 2]\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 2$$

Interpretation: Saddle-point

$$\text{Example}: f(w,z) \quad w = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}\begin{bmatrix} 4 & 6 & 3 \\ 2 & 3 & 1 \\ 3 & 5 & 2 \end{bmatrix}$$
$$z = 1, \quad 2, \quad 3$$

$w=2 \quad f(w, \underset{=2}{z}) :$

$$\min_{w=2} \max_{z=2} f(2,z) = 3$$

$$\max_{z=2} \min f(w,z)$$

$$\min_{w=2} f(w, \underset{=2}{z}) : 3$$

W7A midterm

W7B

Proof: Neccessity:

Assume that

$$\min_{\omega} \max_{z} f(\omega, z) = \max_{z} \min_{\omega} f(\omega, z)$$

Let $\tilde{\omega} = \arg_{\omega} \min_{\omega} \max f(\omega, z)$

$\tilde{z} = \arg_{z} \max_{z} \min_{\omega} f(\omega, z)$

We have

$$f(\tilde{\omega}, \tilde{z}) \underset{1}{\leq} \max_{z} f(\tilde{\omega}, z) \underset{3}{=} \min_{\omega} f(\omega, \tilde{z}) \underset{2}{\leq} f(\tilde{\omega}, \tilde{z})$$

By defination

$(\tilde{\omega}, \tilde{z})$ is a saddle point.

Sufficiency

Assume that $(\tilde{\omega}, \tilde{z})$ is a saddle point

We have

$$\max_{z} \min_{\omega} f(\omega, z) \geq \min_{\omega} f(\omega, \tilde{z}) = f(\tilde{\omega}, \tilde{z})$$

$$\min_{\omega} \max_{z} f(\omega, z) \leq \max_{z} f(\tilde{\omega}, z) = f(\tilde{\omega}, \tilde{z})$$

Thus, $\max_{z} \min_{\omega} f(\omega, z) = \min_{\omega} \max_{z} f(\omega, z)$

Convexity $\Rightarrow$ Saddle Point

Formulation: The row & column selection is formulated as a bilinear optimization problem.

$$\min_{\omega}\max_{z} f(\omega, z) = \sum_i \sum_j a_{ij} w_i z_j \quad \Big| \quad \max_{z}\min_{\omega} f(\omega, z)$$

I. row & column selection constraints
where $\quad w_i, z_j \in \{0,1\} \quad \sum w_i = 1 \ \& \ \sum z_j = 1$

II. relaxed constraints

$$\sum w_i = 1 \ \& \ \sum z_j = 1 \quad w_i \geq 0, z_j \geq 0. \ \forall i, j.$$

A. The optimization problem with relaxed constraints can be solved with algorithms Dantzig

$$\min_{\omega}\max_{z} f(\omega, z) = \max_{z}\min_{\omega} f(\omega, z)$$

B. Since $f(\omega, z)$ is convex w.r.t ω
concave w.r.t z.

the solution can reduce to constraint I (row & column selection)

$$f(\tilde{\omega}, \tilde{z})$$

C. From B, $(\tilde{\omega}, \tilde{z})$ is a saddle point.

Geometric Interpretation

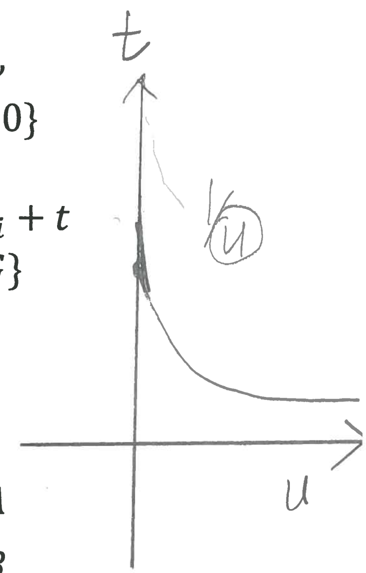
$min\, f_o(x) \qquad (t)$
$s.t.\, f_1(x) \le 0 \quad (u \le 0)$
$g(\lambda) = \min_{(u,t)\in G} t + \lambda u \qquad G = \{(f_1(x), f_o(x))|x \in D\}$

$g(\lambda) = \lambda u + t$
supporting hyperplane to G
that intersects t axis at $t = g(\lambda)$

u

8WA

Duality via Separating Hyperplane



Set $G = \{\big(f_1(x), \ldots, f_m(x), h_1(x), \ldots, h_p(x), f_o(x)\big)\,|\mathrm{x} \in D\}$,
$G \in R^m \times R^p \times R, p^* = \inf\{t|(u, w, t) \in g, u \le 0, w = 0\}$

Lagrangian $L = (\lambda, v, 1)^T(u, w, t) = \sum_{i=1}^m \lambda_i u_i + \sum_{i=1}^p v_i w_i + t$
Dual Problem $g(\lambda, v) = \inf_{u,w,t}\{(\lambda, v, 1)^T(u, w, t)|(u, w, t) \in G\}$

Separating hyperplane: Example

$$\{(u,t)|f_o(x) \le t, f_1(x) \le u, \exists x \in D\}$$

$$(\tilde{\lambda}, \tilde{v}, \tilde{\mu})^T(u, w, t) \ge \alpha, \qquad \forall(u, w, t) \in A$$
$$(\tilde{\lambda}, \tilde{v}, \tilde{\mu})^T(u, w, t) \le \alpha, \qquad \forall(u, w, t) \in B$$

Since $\tilde{\mu} \ne 0$, we can have $(\lambda, v, 1) = (\frac{\tilde{\lambda}}{\tilde{\mu}}, \frac{\tilde{v}}{\tilde{\mu}}, 1)$

$A = \{(u, w, t)|\exists x \in D, f_i(x) \le u_i, i = 1, \ldots, m,$
$\qquad h_i(x) = w_i, i = 1, \ldots, p, f_o(x) \le t\}$

Lagrange dual problem

$$\max g(\lambda, v)$$
$$s.t. \quad \lambda \geq 0$$

min $1/u$

$u \leqslant 0$

$u \in R_+$

Properties

This is a convex problem.

The opt. solution is denoted as d^*

$p^* - d^* = gap \geq 0$

If $gap > 0$, it is a weak duality.

If $gap = 0$, it is a strong duality.

relint: relative interior of set D

Slater's condition

Given that the primal problem is convex,

If $f_i(x) < 0, i = 1, \dots, m, \exists x \in relint\ D$

Then strong duality holds.

$$relint\ C = \{x \in C | B(x, r) \cap aff\ C \subseteq C\ for\ some\ r > 0\}$$
$$B(x, r) = \{y | \|y - x\| \leq r\}$$

any norm

Shadow Price Interpretation: Food vs. Vitamin

Flour	protein powder
Veg.	vitamins A,B,D,E,K
Fruits	minerals

Primal $\min c^T x$, $s.t.\ Ax \succcurlyeq b$, $x \succcurlyeq 0$ $\qquad$ $\min c^T x$, $s.t.\ -Ax + b \preccurlyeq 0$, $-x \preccurlyeq 0$

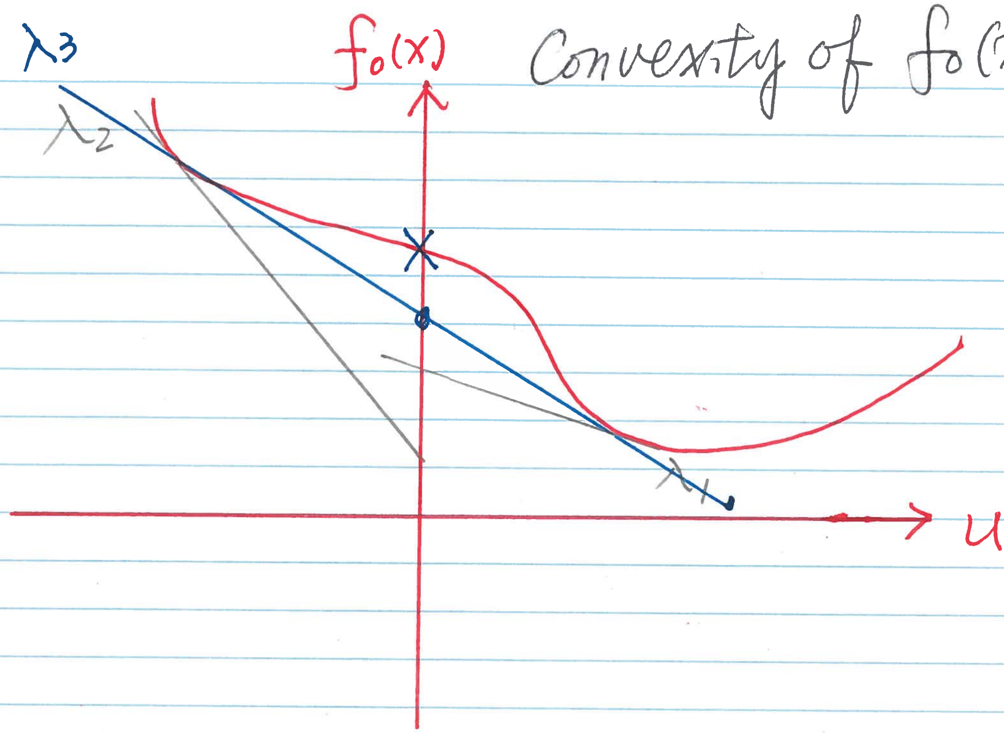
$$\min c_1x_1 + c_2x_2 + c_3x_3 \quad \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \succcurlyeq \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, x_i \geq 0, \forall i$$

Dual $\max \lambda^T b$, $s.t.\ A^T \lambda \preccurlyeq c$, $\lambda \succcurlyeq 0$

$$\max \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3$$
$$\text{s.t.} \begin{bmatrix} v_{11} & v_{21} & v_{31} \\ v_{12} & v_{22} & v_{32} \\ v_{13} & v_{23} & v_{33} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \preccurlyeq \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Lagrangian
$$L(x, \lambda) = c^T x + \lambda_1^T(-Ax + b) + \lambda_2^T(-x)$$
$$= [c^T + \lambda_1^T(-A) - \lambda_2^T]x + \lambda_1^T b$$
$$c^T = \lambda_1^T(A) + \lambda_2^T, \ or\ A^T\lambda_1 \leq c$$

Convexity of $f_0(x)$



A.21.2

Primal

$$\min f_0(x)$$

s.t.

$$f_i(x) \leq 0$$

$$h_i(x) = 0$$

$$\min_{x} \max_{\lambda, \nu} L(x, \lambda, \nu)$$

The constraints are enforced.

Dual

$$\max g(\lambda)$$

$$\min_{x} L(x, \lambda, \nu)$$

$$\max_{\lambda, \nu} \min_{x} L(x, \lambda, \nu)$$

min L for $x \in D$

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

Let

$$f(a,b) = \min_{w} \max_{z} f(w,z), \quad f(c,d) = \max_{z} \min_{w} f(w,z)$$

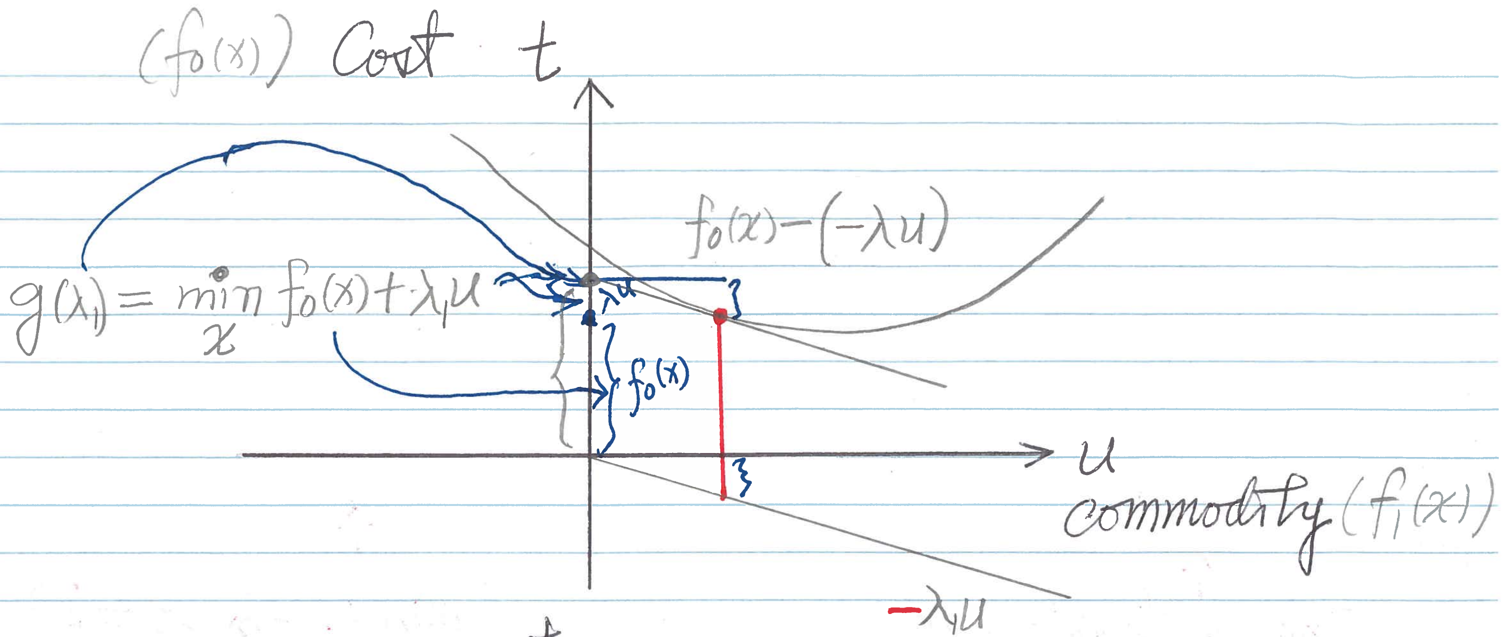
Then

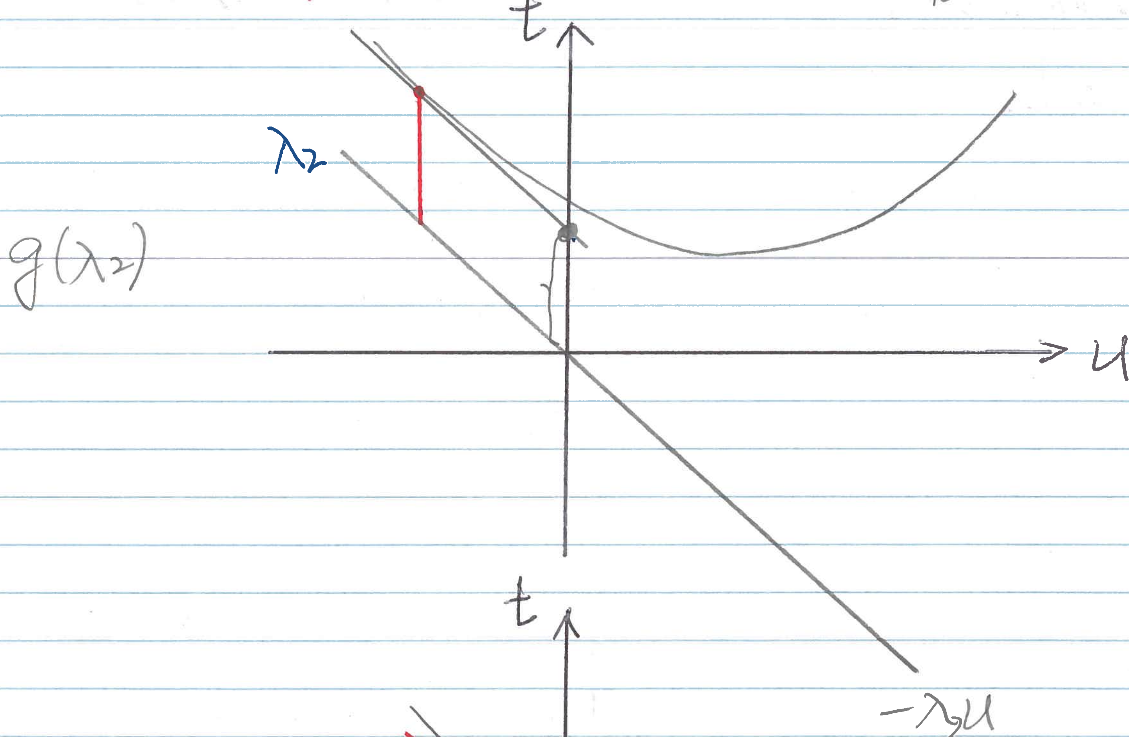
$$f(a,b) \geq f(a,d) \geq f(c,d)$$

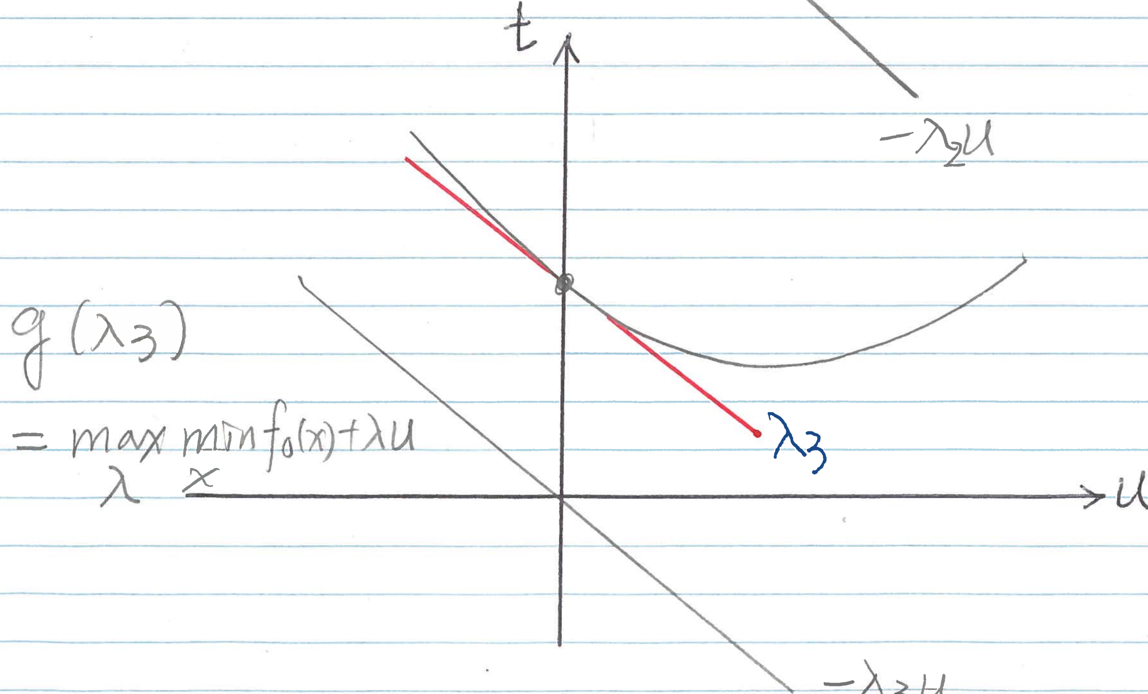
$f(w,z)$ is convex w.r.t. w

concave w.r.t. z

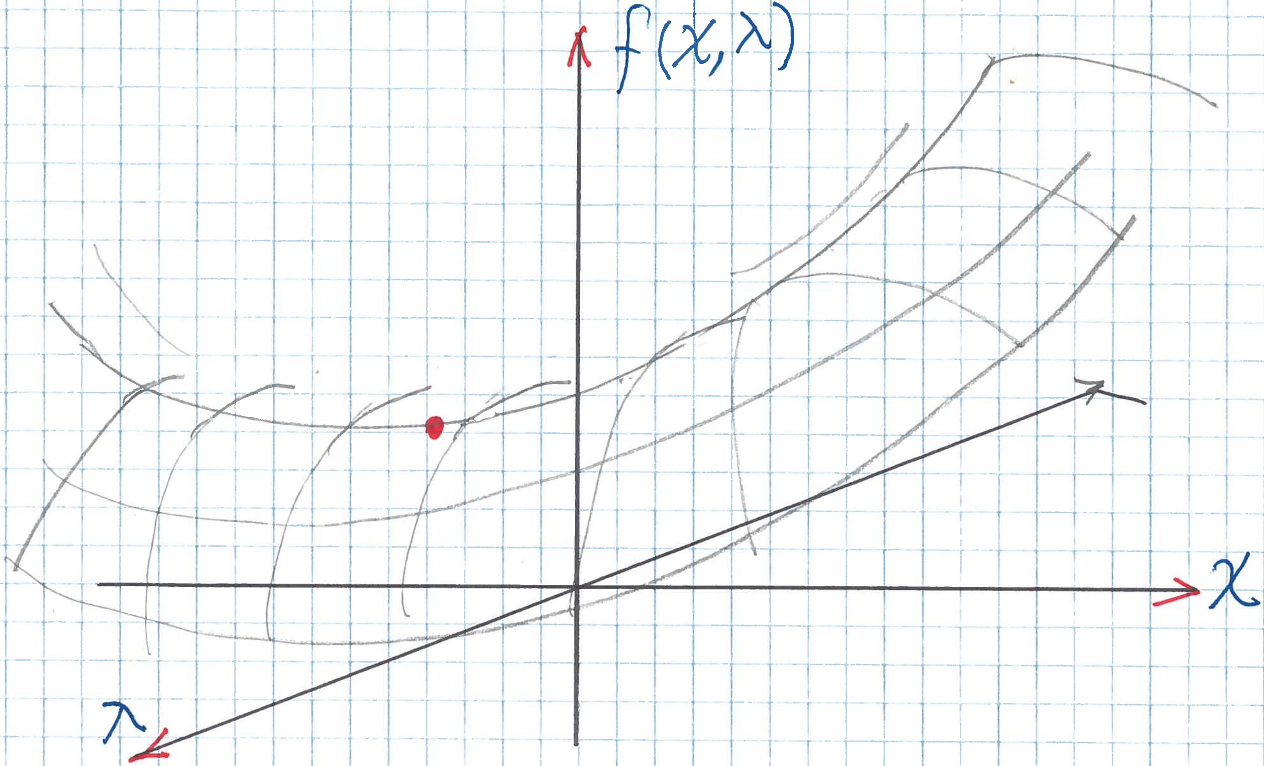
A.21.0


($f_0(x)$) Cost
t
$f_0(x) - (-\lambda u)$
$g(\lambda_1) = \min_x f_0(x) + \lambda_1 u$
$\lambda_1 u$
$f_0(x)$
ξ
u
commodity ($f_1(x)$)
$-\lambda_1 u$


t
λ_2
$g(\lambda_2)$
u
$-\lambda_2 u$


t
$g(\lambda_3)$
$= \max_\lambda \min_x f_0(x) + \lambda u$
λ_3
u
$-\lambda_3 u$

A.21.1



Definition

I. Saddle Point

Given function $f(w, z)$

$(\tilde{w}, \tilde{z})$ is a saddle point of f

if
$$\max_{z} f(\tilde{w}, z) = f(\tilde{w}, \tilde{z})$$
$$\min_{w} f(w, \tilde{z}) = f(\tilde{w}, \tilde{z})$$

II. Theorem: $\max_{z} \min_{w} f(w, z) = \min_{w} \max_{z} f(w, z)$

iff a saddle point of f exists

Primal Dual Interpretation

Flour, Vegetable, Fruits Protein, Vitamin A-K, Mineral

x_1, x_2, x_3 λ_1, λ_2, λ_3, λ_4, λ_5

$$\text{Obj: } \overset{\min}{C_1 x_1 + C_2 x_2 + C_3 x_3}$$

$$\begin{array}{ll} \text{Protein} & \lambda_1 \\ \text{Vit A} & \lambda_2 \\ \text{Vit B} & \lambda_3 \\ \text{Vit E} & \lambda_4 \\ \text{Mineral} & \lambda_5 \end{array} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

$$x_1, x_2, x_3 \geq 0$$

$$\text{Obj: max } \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 + \lambda_4 b_4 + \lambda_5 b_5$$

$$\begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{41} & a_{51} \\ a_{12} & a_{22} & a_{32} & a_{42} & a_{52} \\ a_{13} & a_{23} & a_{33} & a_{43} & a_{53} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{bmatrix} \leq \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

$$\lambda_i \geq 0 \quad \forall i$$