Interpretation: Saddle-point

$$\text{Example}: f(w,z) \quad w = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$
$$z = 1, \quad 2, \quad 3$$

$$\min_{w \in W} f(w,1) = 1 \qquad \max_{z \in Z} f(1,z) = 3$$
$$\min_{w \in W} f(w,2) = 1 \qquad \max_{z \in Z} f(2,z) = 3$$
$$\min_{w \in W} f(w,3) = 1 \qquad \max_{z \in Z} f(3,z) = 3$$

$$\max_{z \in Z} \min_{w \in W} f(w,z) = 1 \qquad \min_{w \in W} \max_{z \in Z} f(w,z) = 3$$

Interpretation: Saddle-point

$$\text{Example}: f(w,z) \quad w = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}\begin{bmatrix} 4 & 6 & 3 \\ 2 & 3 & 1 \\ 3 & 5 & 2 \end{bmatrix}$$
$$z = 1, \quad 2, \quad 3$$

$w=2 \quad f(w, \underset{=2}{z}):$

$\max_{z=2} \min f(w,z)$

$\min_{w=2} \max_{z=2} f(2,z) = 3$

$\min_{w=2} f(w, \underset{=2}{z}) : 3$

W7A midterm

W7B

Interpretation: Saddle-point

Claim : Result of II $\geq$ Result of I

Given an arbitrary pair $(\tilde{w}, \tilde{z}) \in D$

$$\min_{w \in W} f(w, \tilde{z}) \leq f(\tilde{w}, \tilde{z}) \leq \max_{z \in Z} f(\tilde{w}, z) \quad \forall \tilde{w}, \tilde{z} \in D$$

$$\min_{w \in W} f(w, \tilde{z}) \leq \max_{z \in Z} f(\tilde{w}, z)$$

Thus $\max_{z \in Z} \min_{w \in W} f(w, z) \leq \min_{w \in W} \max_{z \in Z} f(w, z)$

Interpretation: Saddle-point

Example : $f(w, z)$ $\quad w = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

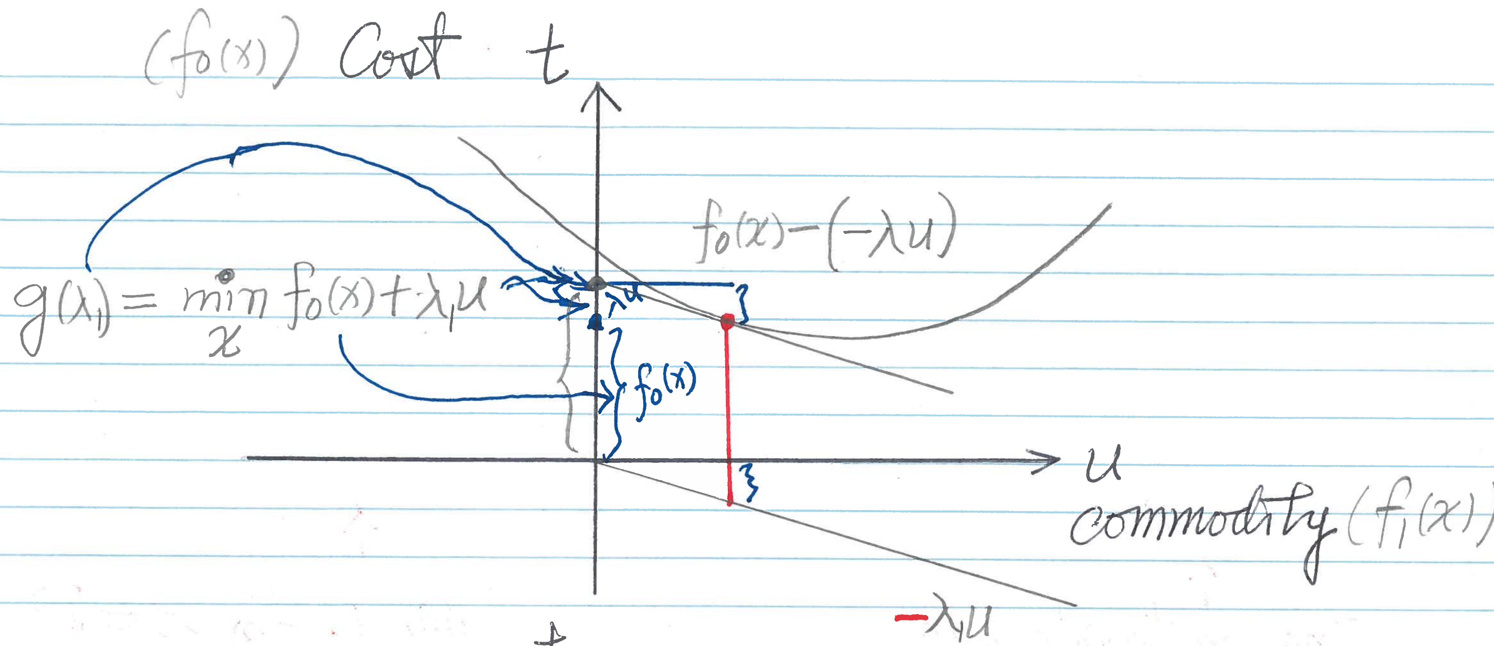
$z = 1, \quad 2, \quad 3$

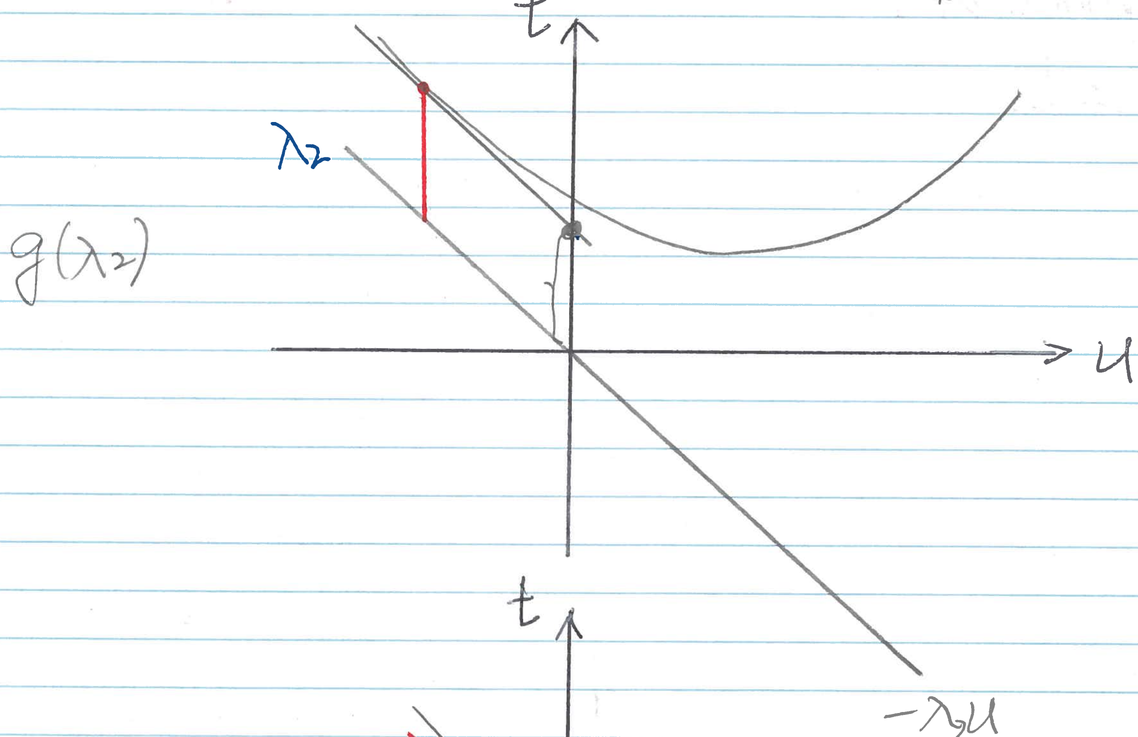
$$\min_{w \in W} f(w, 1) = 1 \qquad \max_{z \in Z} f(1, z) = 1$$

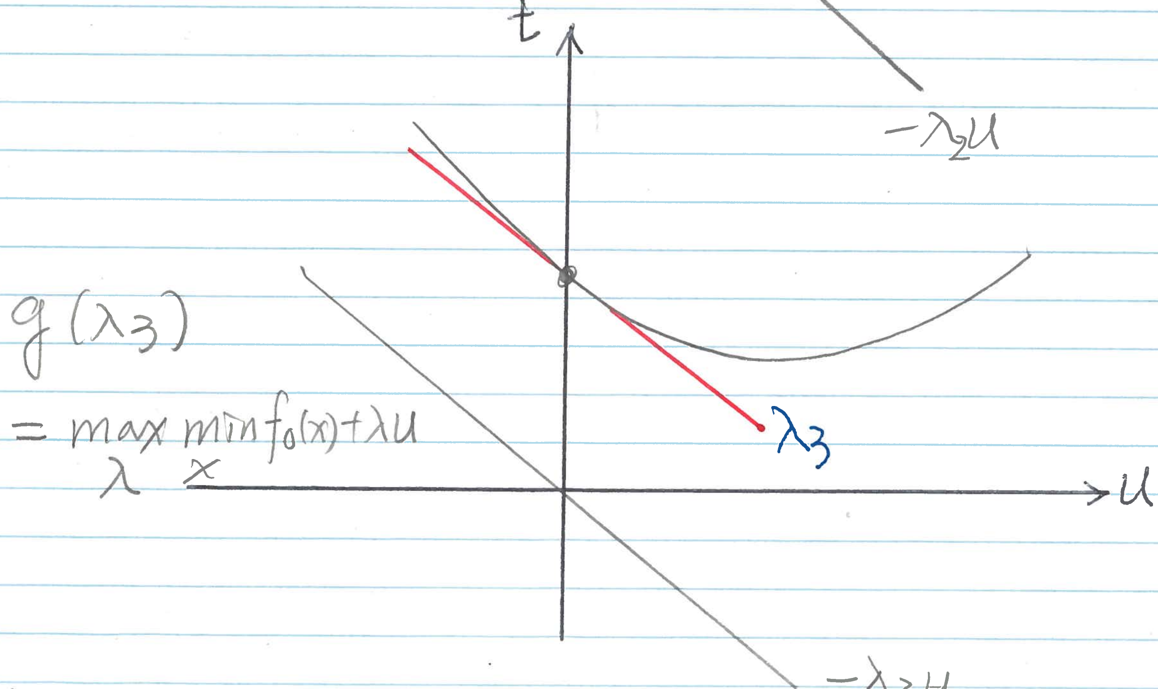
$$\min_{w \in W} f(w, 2) = 1 \qquad \max_{z \in Z} f(2, z) = 2$$

$$\min_{w \in W} f(w, 3) = 1 \qquad \max_{z \in Z} f(3, z) = 3$$

$$\max_{z \in Z} \min_{w \in W} f(w, z) = 1 \qquad \min_{w \in W} \max_{z \in Z} f(w, z) = 1$$


$(f_0(x))$ Cost
t
$f_0(x) - (-\lambda u)$
$g(\lambda_1) = \min_x f_0(x) + \lambda_1 u$
λu
$f_0(x)$
u
commodity $(f_1(x))$
$-\lambda_1 u$


t
λ_2
$g(\lambda_2)$
u
$-\lambda_2 u$


t
$g(\lambda_3)$
$= \max_\lambda \min_x f_0(x) + \lambda u$
λ_3
u
$-\lambda_3 u$

A.21.1

Primal

$$\min f_0(x)$$

s.t.

$$f_i(x) \leq 0$$

$$h_i(x) = 0$$

$$\min_{x} \max_{\lambda, \nu} L(x, \lambda, \nu)$$

The constraints are enforced.

Dual

$$\max g(\lambda)$$

$$\min_{x} L(x, \lambda, \nu)$$

$$\max_{\lambda, \nu} \min_{x} L(x, \lambda, \nu)$$

min L for $x \in D$

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

Let

$$f(a,b) = \min_{\omega} \max_{z} f(w,z), \quad f(c,d) = \max_{z} \min_{\omega} f(w,z)$$

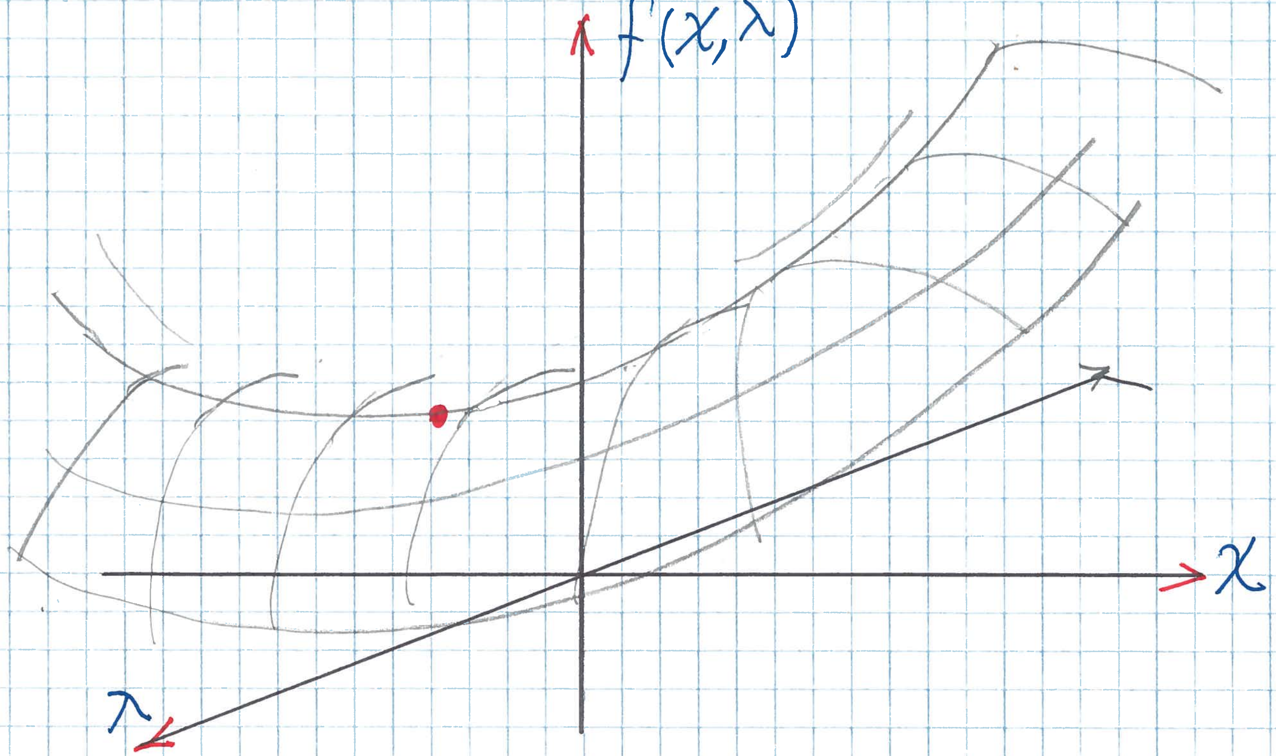
Then

$$f(a,b) \geq f(a,d) \geq f(c,d)$$

$f(w,z)$ is convex w.r.t. ω

concave w.r.t. z

A.21.0



Definition

I. Saddle Point

Given function $f(w,z)$

$(\tilde{w},\tilde{z})$ is a saddle point of f

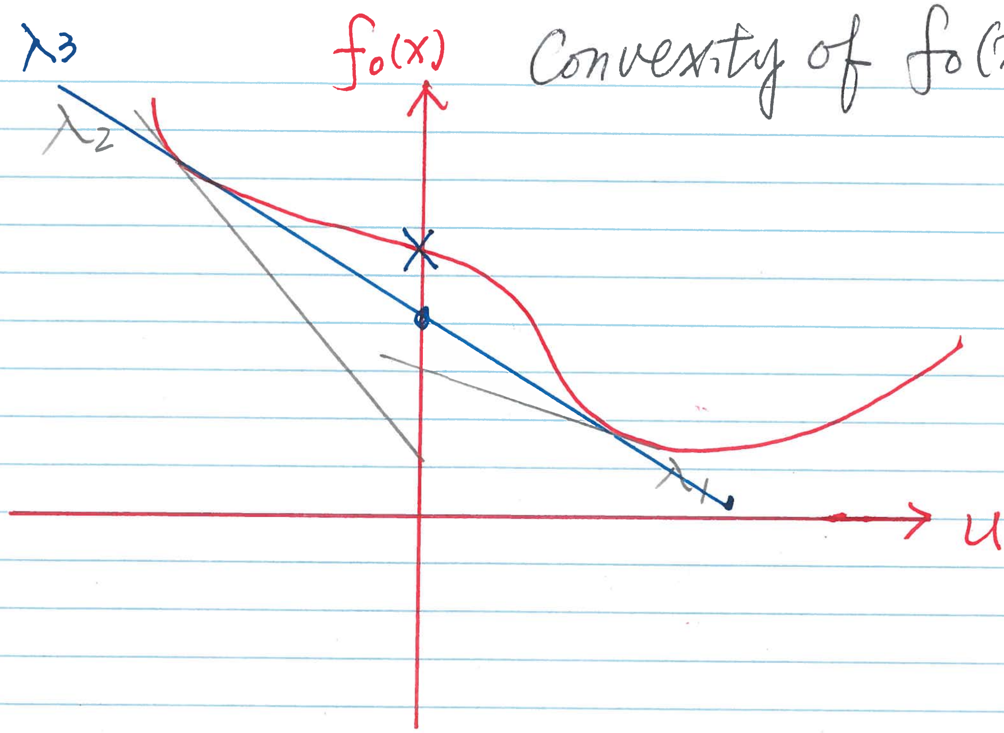
if $$\max_{z} f(\tilde{w},z) = f(\tilde{w},\tilde{z})$$

$$\min_{w} f(w,\tilde{z}) = f(\tilde{w},\tilde{z})$$

II. Theorem: $\max_{z} \min_{w} f(w,z) = \min_{w} \max_{z} f(w,z)$

iff a saddle point of f exists.

Convexity of $f_0(x)$



A.21.2

Proof: Necessity:

Assume that

$$\min_{\omega} \max_{z} f(\omega, z) = \max_{z} \min_{\omega} f(\omega, z)$$

Let $\tilde{\omega} = \underset{\omega}{\arg} \min_{\omega} \max f(\omega, z)$

$\tilde{z} = \underset{z}{\arg} \max_{z} \min_{\omega} f(\omega, z)$

We have

$$f(\tilde{\omega}, \tilde{z}) \leq \max_{z} f(\tilde{\omega}, z) = \min_{\omega} f(\omega, \tilde{z}) \leq f(\tilde{\omega}, \tilde{z})$$

By definition

$(\tilde{\omega}, \tilde{z})$ is a saddle point.

Sufficiency

Assume that $(\tilde{\omega}, \tilde{z})$ is a saddle point

We have

$$\max_{z} \min_{\omega} f(\omega, z) \geq \min_{\omega} f(\omega, \tilde{z}) = f(\tilde{\omega}, \tilde{z})$$

$$\min_{\omega} \max_{z} f(\omega, z) \leq \max_{z} f(\tilde{\omega}, z) = f(\tilde{\omega}, \tilde{z})$$

Thus, $\max_{z} \min_{\omega} f(\omega, z) = \min_{\omega} \max_{z} f(\omega, z)$

Formulation: The row & column selection is formulated as a bilinear optimization problem.

$$\min_{\omega} \max_{z} f(\omega, z) = \sum_i \sum_j a_{ij} \omega_i z_j \quad \Big| \quad \max_{z} \min_{\omega} f(\omega, z)$$

I. row & column selection constraints

where $\omega_i, z_j \in \{0,1\}$ $\quad \sum \omega_i = 1$ & $\sum z_j = 1$

II. relaxed constraints

$$\sum \omega_i = 1 \ \& \ \sum z_j = 1 \quad \omega_i \geq 0, z_j \geq 0. \ \forall i, j.$$

A. The optimization problem with relaxed constraints can be solved with algorithms Dantzig

$$\min_{\omega} \max_{z} f(\omega, z) = \max_{z} \min_{\omega} f(\omega, z)$$

B. Since $f(\omega, z)$ is convex w.r.t ω
concave z.

the solution can reduce to constraint I (row & column selection)

$$f(\tilde{\omega}, \tilde{z})$$

C. From B, $(\tilde{\omega}, \tilde{z})$ is a saddle point.