Example: Quadratic Program

Quadratic Program

$$\min x^T P x \qquad P \in S^n_{++}$$
$$s.t.\ Ax \preccurlyeq b$$

Lagrange Dual Function:

$$g(\lambda) = \min_x x^T P x + \lambda^T (Ax - b)$$
$$= -\frac{1}{4}\lambda^T A P^{-1} A^T \lambda - b^T \lambda, \quad \lambda \succcurlyeq 0$$

$$\nabla L(x,\lambda) = 2Px + A^T\lambda = 0$$
$$x = -\frac{1}{2} P^{-1} A^T \lambda$$

Dual Problem:

$$\max -\frac{1}{4}\lambda^T A P^{-1} A^T \lambda - b^T \lambda$$
$$s.t.\ \lambda \succcurlyeq 0$$

Example: Quadratic Program (nonconvex prob.)

$$\min x^T A x + 2b^T x$$
$$s.t.\ x^T x \leq 1 \qquad A \in S^n, A \not\succcurlyeq 0$$

<u>Dual Function:</u>

$$L(x,\lambda) = x^T A x + 2b^T x + \lambda(x^T x - 1)$$
$$= x^T A x + 2b^T x + \lambda(x^T I x - 1)$$

$$g(\lambda) = \min_x x^T (A + \lambda I) x + 2b^T x - \lambda$$

Unbounded below if $A + \lambda I \not\succcurlyeq 0$ or if $A + \lambda I \succcurlyeq 0$ & $b \notin R(A + \lambda I)$

Minimized by $x = -(A + \lambda I)^+ b$

Otherwise $g(\lambda) = -b^T (A + \lambda I)^+ b - \lambda$

<u>Dual Problem:</u>

$\max\ -b^T (A + \lambda I)^+ b - \lambda$ $s.t.\ A + \lambda I \succcurlyeq 0$ $b \in R(A + \lambda I)$	$\max\ -t - \lambda$ $s.t.\ \begin{bmatrix} A + \lambda I & b \\ b^T & t \end{bmatrix} \succcurlyeq 0$

$$\begin{bmatrix} I & 0 \\ -((A + \lambda I)^+ b)^T & I \end{bmatrix} \begin{bmatrix} A + \lambda I & b \\ b^T & t \end{bmatrix} \begin{bmatrix} I & -(A + \lambda I)^+ b \\ 0 & I \end{bmatrix} \succcurlyeq 0$$

$$\begin{bmatrix} \underline{A + \lambda I} & 0 \\ 0 & \underline{-b^T (A + \lambda I)^+ b + t} \end{bmatrix} \succcurlyeq 0$$

Example: Discrete Problem

Two-Way Partitioning Problem

Primal:

$$\min x^T W x \qquad x \in R^n, w_{ij} \in R$$
$$s.t.\; x_i^2 = 1 \qquad i = 1, \dots, m$$
$$i.e.\; x_i \in \{-1, 1\},\; x^T W x = \textstyle\sum_{ij} x_i x_j w_{ij}$$

Not a convex opt problem (Partitioning is an NP complete problem)

We can assume that

$$W \in S^n \quad (x^T W x = \tfrac{1}{2} x^T W x + \tfrac{1}{2} x^T W^T x = \tfrac{1}{2} x^T (W + W^T) x)$$

Lagrangian:

$$L(x, v) = x^T W x + \textstyle\sum_{i=1}^n v_i (x_i^2 - 1) = x^T \big(W + diag(v)\big) x - I^T v$$

Lagrange dual function:

$$g(v) = \inf_x x^T \big(W + diag(v)\big) x - I^T v = \begin{cases} -I^T v, & W + diag(v) \succcurlyeq 0 \\ -\infty, & otherwise \end{cases}$$

$$\sum_i \nu_i x_i^2 = x^T \begin{bmatrix} \nu & & & \\ & \nu & & \\ & & \ddots & \\ & & & \nu \end{bmatrix} x = \sum_i \nu x_i^2$$

W6B

Example: Discrete Problem

Dual:

$$\max g(v) = -I^T v$$
$$s.t.\; W + diag(v) \succcurlyeq 0$$

Solution $\quad v = -\lambda_{min}(W) 1$

$p^* \geq d^* = -1^T v = n \lambda_{min}(W)$

$\downarrow$ ~~$W \in R S_+^n$~~

$$W \in S^n$$
$$\downarrow$$
$$D \Sigma D^T$$
$$\downarrow$$
$$\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Examples: Conjugate Function

$$\min f_0(x)$$
$$s.t.\ Ax \preccurlyeq b$$
$$Cx = d$$

<u>Dual function</u>

$$g(\lambda, v) = \inf_{x \in dom f_0} (f_0(x) + \lambda^T(Ax - b) + v^T(Cx - d))$$
$$= \inf_{x \in dom f_0} (f_0(x) + (A^T\lambda + C^T v)^T x - b^T\lambda - d^T v)$$
$$= -f_0^*(-A^T\lambda - C^T v) - b^T\lambda - d^T v$$

Conjugate function

Where $f_0^*(y) = \max_{x \in dom f} y^T x - f_0(x)$

$$\min c^T x$$
$$s.t.\ Ax = b$$
$$x \succcurlyeq 0$$

$$\max -b^T v$$
$$s.t.\ A^T v + c \succcurlyeq 0$$

$$\min c^T x$$
$$s.t.\ Ax \preccurlyeq b$$

$$\max -b^T \lambda$$
$$s.t.\ A^T\lambda + c = 0$$
$$\lambda \succcurlyeq 0$$

Examples: Entropy Maximization

$$\min f_0(x) = \sum_{i=1}^n x_i log x_i, \quad x \in R_{++}^n$$
$$s.t.\ Ax \preccurlyeq b$$
$$1^T x = 1$$

Since $f_0^*(y) = \sum_{i=1}^n e^{y_i - 1}, y_i \in R$

Thus, the dual function is

$$g(\lambda, v) = -b^T\lambda - v - \sum_{i=1}^n e^{\boxed{-a_i^T\lambda - v} - 1}$$
$$= -b^T\lambda - v - e^{-v-1}\sum_{i=1}^n e^{-a_i^T\lambda},\ \ \mathrm{a_i}\text{: the } \mathrm{i^{th}} \text{column of A.}$$

To maximize $g(\lambda, v)$, we set $v = log\sum_{i=1}^n e^{-a_i^T\lambda} - 1$

<u>Dual Problem</u>

$$\max\ -b^T\lambda - \log(\textstyle\sum_{i=1}^n e^{-a_i^T\lambda})$$
$$s.t.\ \ \lambda \succcurlyeq 0$$

Examples: Minimum Volume Covering Ellipsoid

$$\min f_0(x) = \log\det X^{-1}, \quad X \in S^n_{++}$$
$$s.t.\ a_i^T X a_i \le 1, i = 1, \dots, m$$

<u>Lagrangian</u>

$$L(x,\lambda) = logdetX^{-1} + \sum_{i=1}^m \lambda_i a_i^T X a_i - 1^T\lambda, \quad \lambda \in R^m_+$$

<u>Lagrange dual function</u>

$$g(\lambda) = \min_x L(x,\lambda), \lambda \in R^m_+$$

<u>Dual Problem</u>

$$\max \quad logdet\left(\sum_{i=1} \lambda_i a_i a_i^T\right) - 1^T\lambda + n$$
$$s.t. \quad \sum_{i=1} \lambda_i a_i a_i^T \succ 0, \lambda \in R^m_+$$

Interpretation: Saddle-point

$$\max_{z\in Z} g(z) \qquad \max_{z\in Z}\min_{w\in W} f(w,z) \le \min_{w\in W}\max_{z\in Z} f(w,z)$$

$$g(z) = \min_w f(w,z)$$

Example : $f(w,z)$
$$\begin{array}{cc} & \\ w = & \begin{array}{c}1\\2\\3\end{array} \left[\begin{array}{ccc} 1 & -1 & 3 \\ 2 & 2 & -1 \\ 3 & 1 & -2 \end{array}\right] \\ & z = 1, \quad 2, \quad 3 \end{array}$$

I. z decides first
$$\begin{cases} \min_{w\in W} f(w,1) = 1 \\ \min_{w\in W} f(w,2) = -1 \\ \min_{w\in W} f(w,3) = -2 \end{cases} \quad \max_{z\in Z}\min_{w\in W} f(w,z) = 1$$

II. w decides first
$$\begin{cases} \max_{z\in Z} f(1,z) = 3 \\ \max_{z\in Z} f(2,z) = 2 \\ \max_{z\in Z} f(3,z) = 3 \end{cases} \quad \min_{w\in W}\max_{z\in Z} f(w,z) = 2$$

$$f^*(y) = \max_x x^T y - f(x)$$

$$f_0^*(-A^T y - C^T v)$$

$$= \max_x (-A^T y - C^T v)^T x - f_0(x)$$

Or

$$-f_0^*(-A^T y - C^T v) = \min_x (A^T y + C^T v)^T x + f_0(x)$$