2.2 Optimality Criterion for Differentiable $f_0(x)$

Theorem: If $\nabla f_0(x)^T(y-x) \geq 0,$ for a given $x \in$Feasible Set and for all $y \in$ Feasible Set, then x is optimal.
(i. e. $K = \{y - x | y \in feasible\ set\}, \nabla f_0(x) \in K^*$)

Proof: From the first order condition of convex function, we have $f_0(y) \geq f_0(x) + \nabla f_0(x)^T(y-x).$
Given the condition that $\nabla f_0^T(x)(y-x) \geq 0,\ \forall y$ in feasible set.
We have $f_0(y) \geq f_0(x),\ \forall y$ in feasible set, which implies that x is optimal.

Remark: $\nabla f_0^T(x)(y-x) = 0$ is a supporting hyperplane to feasible set at $x.$

2.2.1 Optimality Criterion without Constraints

Theorem: For problem min $f_0(x), x \in R^n$, where f_0 is convex, the optimal condition is $\nabla f_0(x) = 0$.

Proof: ($\nabla f_0(x) = 0 \Rightarrow$ Optimality)
Since $f_0(y) \geq f_0(x) + \nabla f_0(x)^T(y-x), \forall x, y \in R^n$ (first order condition of convex function)
We have $f_0(y) \geq f_0(x)$.
Therefore, x is an optimal solution.
($\nabla f_0(x) = 0 \Leftarrow$ Optimality) By contradiction

2.2.2 Opt. with Inequality Constraints

Problem: Min $f_0(x)$ $\longrightarrow$ $\nabla f_0(x)$

$s.t.\ Ax \le b, A \in R^{m\times n}$

Kone $\to K^*$

Suppose that $A\bar{x} = b$ (one particular case).

Let $x = \bar{x} + u$.

We can write $\begin{cases} \min f_0(\bar{x}+u) \\ \quad Au \le 0 \end{cases}$

Opt. condition: $\nabla f_0(x)^T u \ge 0,\ \forall \{u \mid Au \le 0\} \equiv K$

In other words,

$$\nabla f_0(\bar{x}) \in K^*\ of\ K = \{u \mid Au \le 0\}\ and\ K^* = \{-A^T v \mid v \ge 0\}$$

i.e. $\nabla f_0(\bar{x}) = -A^T v, \exists v \in R_+^m$

$$\nabla f_0(\bar{x}) + A^T v = 0,\ v \ge 0.$$

2.2.3 Opt. with Equality Constraints

$\begin{cases} \min f_0(x) \\ s.t.\ Ax = b \end{cases}$

Let $x = \bar{x} + u$ and $A\bar{x} = b$,

we have $\begin{cases} \min f_0(\bar{x}+u) \\ \quad Au = 0 \end{cases}$, $K = \{u \mid Au = 0\}$

$$\nabla f_0(\bar{x}) \in K^*,\ K^* = \{A^T v \mid v \in R^p\}$$

$$\nabla f_0(\bar{x}) + A^T v = 0$$

$\tilde{A}$

Let $K_1 = \{u \mid Au \ge 0\}$

$K_2 = \{u \mid -Au \ge 0\}$

$K_1 \cap K_2 = \{u \mid Au \ge 0, -Au \ge 0\}$

$\left\{u \mid \begin{bmatrix} A \\ -A \end{bmatrix} u \ge \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}\right\}$

$\{\tilde{A}^T v \mid v \ge 0\}$

We have

$$(K_1 \cap K_2)^* = \{A^T v_1 + (-A)^T v_2 \mid v_1, v_2 \ge 0\}$$
$$= \{A^T v \mid v \in R^p\}$$

$\left\{[A^T\ -A^T]\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \mid v_1 \ge 0, v_2 \ge 0\right\}$

$\times\ (K_1^* \cup K_2^*)$

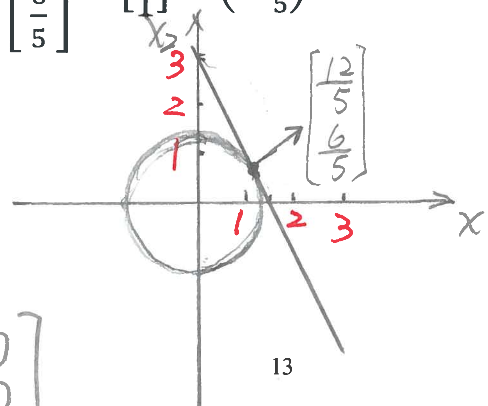
2.2.3 Opt. with Equality Constraints: Example

$$\min_x f(x) = x_1^2 + x_2^2$$

$$s.t.\ [2\ 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3$$

We can derive $x^* = (x_1^*, x_2^*) = (\frac{6}{5}, \frac{3}{5})$

$$\nabla f(x^*) = \begin{bmatrix} 2x_1^* \\ 2x_2^* \end{bmatrix} = \begin{bmatrix} \frac{12}{5} \\ \frac{6}{5} \end{bmatrix},\ \nabla f(x^*) + A^T v = \begin{bmatrix} \frac{12}{5} \\ \frac{6}{5} \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \times \left(-\frac{6}{5}\right) = 0$$



New Problem:

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{matrix} \nabla f(x) + A^T v = 0 \\ Ax = b \end{matrix} \Rightarrow \begin{matrix} \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} v = 0 \\ [2\ 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3 \end{matrix}$$

$\nabla^2 f(x)$

$$\begin{bmatrix} \boxed{\begin{matrix} 2 & 0 \\ 0 & 2 \end{matrix}} & \boxed{\begin{matrix} 2 \\ 1 \end{matrix}} \\ \boxed{\begin{matrix} 2 & 1 \end{matrix}} & 0 \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} x_1 \\ x_2 \end{matrix}} \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

A

A^T

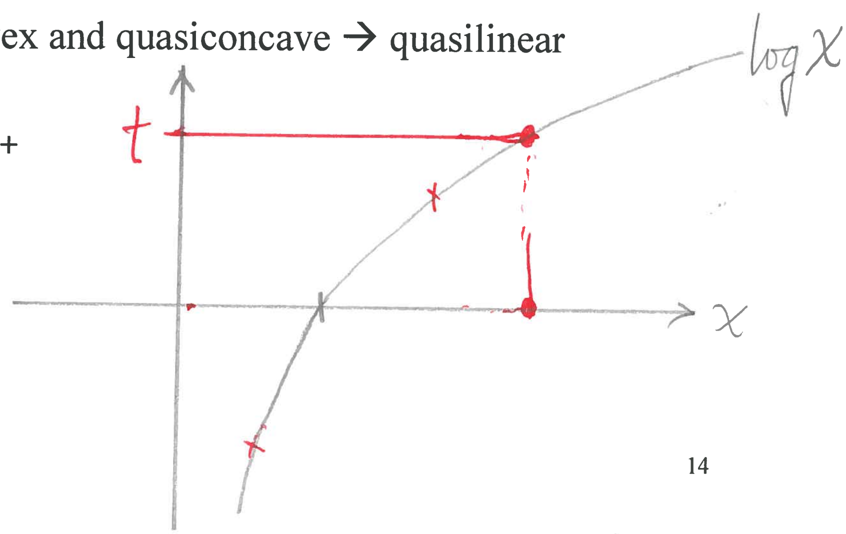
2.3 Quasiconvex Functions

$f: R^n \to R$ is called quasiconvex (unimodal)
sublevel set $S_t = \{x | x \in dom\, f,\ f(x) \le t\}$
if its domain and all sublevel sets $S_t, \forall t \in R$ are convex,
$f: R^n \to R$ is called quasiconcave if $-f$ is quasiconvex.

$f(x)$ quasiconvex and quasiconcave → quasilinear

Ex: $\log x, x \in R_{++}$

$\{x \mid \log x \le t, x \in R_{++}\}$



2.3 Quasiconvex Functions

$1.1 \to 2$
$1 \to 1$
$0.1 \to 1$

Ex: Ceiling function

$Ceil(x) = \inf\{z \in Z | z > x\}$: quasilinear

Ex: $f(x_1, x_2) = x_1 x_2 = \frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

is quasiconcave in R_+^2, $S_t = \{x \in R_+^2 | x_1 x_2 \geq t\}$

Ex: $f(x) = \frac{a^T x + b}{c^T x + d}$ for $c^T x + d > 0$

$S_t = \{x | c^T x + d > 0, a^T x + b \leq t(c^T x + d)\}$

open halfspace closed halfspace

→ S_t is convex (t is given here)

→ $f(x)$ is $\left.\begin{matrix} \text{quasiconvex} \\ \text{quasiconcave} \end{matrix}\right\}$ → quasilinear

2.3 Quasiconvex Optimization

min $f_o(x)$ ($f_o(x)$ is quasiconvex, f_i's are convex.)

$s.t.$ $f_i(x) \leq 0, i = 1, \dots, m$

$Ax = b$

Remark: A locally opt. solution $(x, f_0(x))$ may not be globally opt.

Algorithm: Bisection method for quasiconvex optimization.

Given $l \leq p^* \leq u, \epsilon > 0$

Repeat 1. $t = (l + u)/2$

2. Find a feasible solution x:

Find a convex function

$s.t. \Phi_t(x) \leq 0$ ($f_0(x) \leq t \Leftrightarrow \Phi_t(x) \leq 0$)

$f_i(x) \leq 0$

$Ax = b$

3. If solution is feasible, $u = t, else\ l = t$

Until $u - l \leq \epsilon$

Ex: $f(x) = \frac{p(x)}{q(x)} \leq t \to p(x) - tq(x) \leq 0$ (p is convex & q is concave)

Quasiconvex function

function

$$f : R^n \to R, \quad \text{domain } f \text{ is convex.}$$

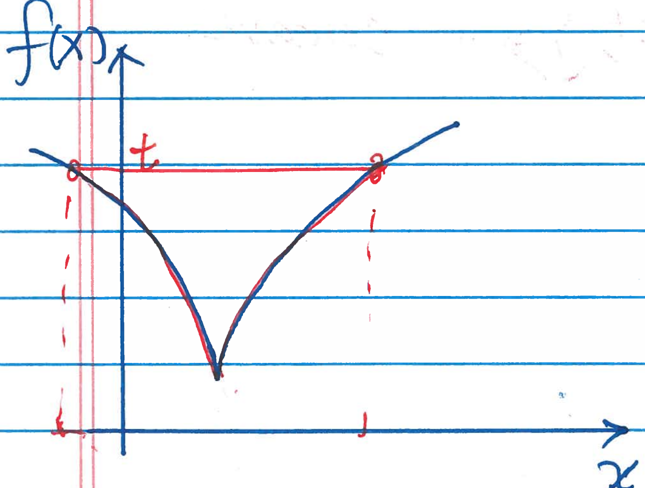
Set

$$S_t = \{x \mid f(x) \le t\} \quad \text{is convex.}$$

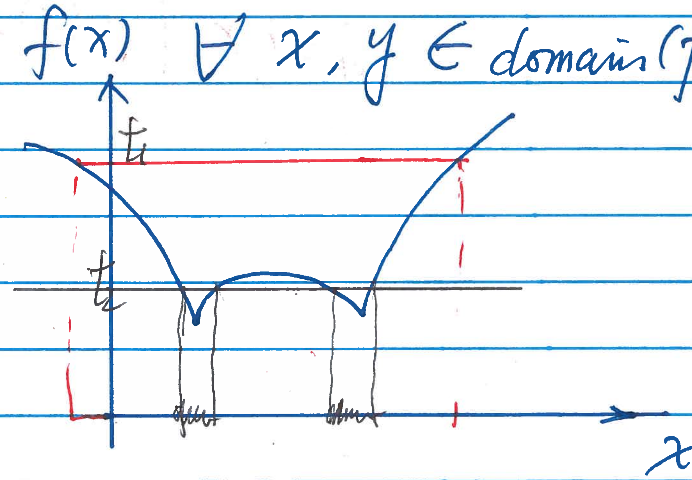
Or

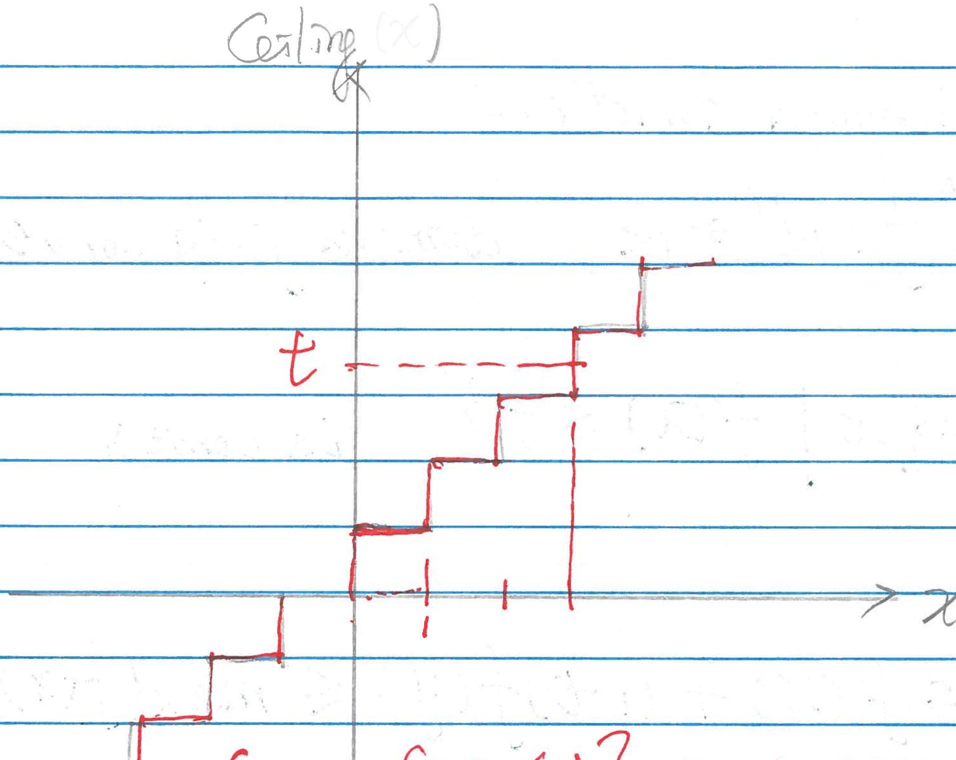
$$\theta f(x) + (1-\theta) f(y) \le \max(f(x), f(y))$$

$\forall \theta \ge 0, \theta \le 1,$



$\forall x, y \in \text{domain}(f).$

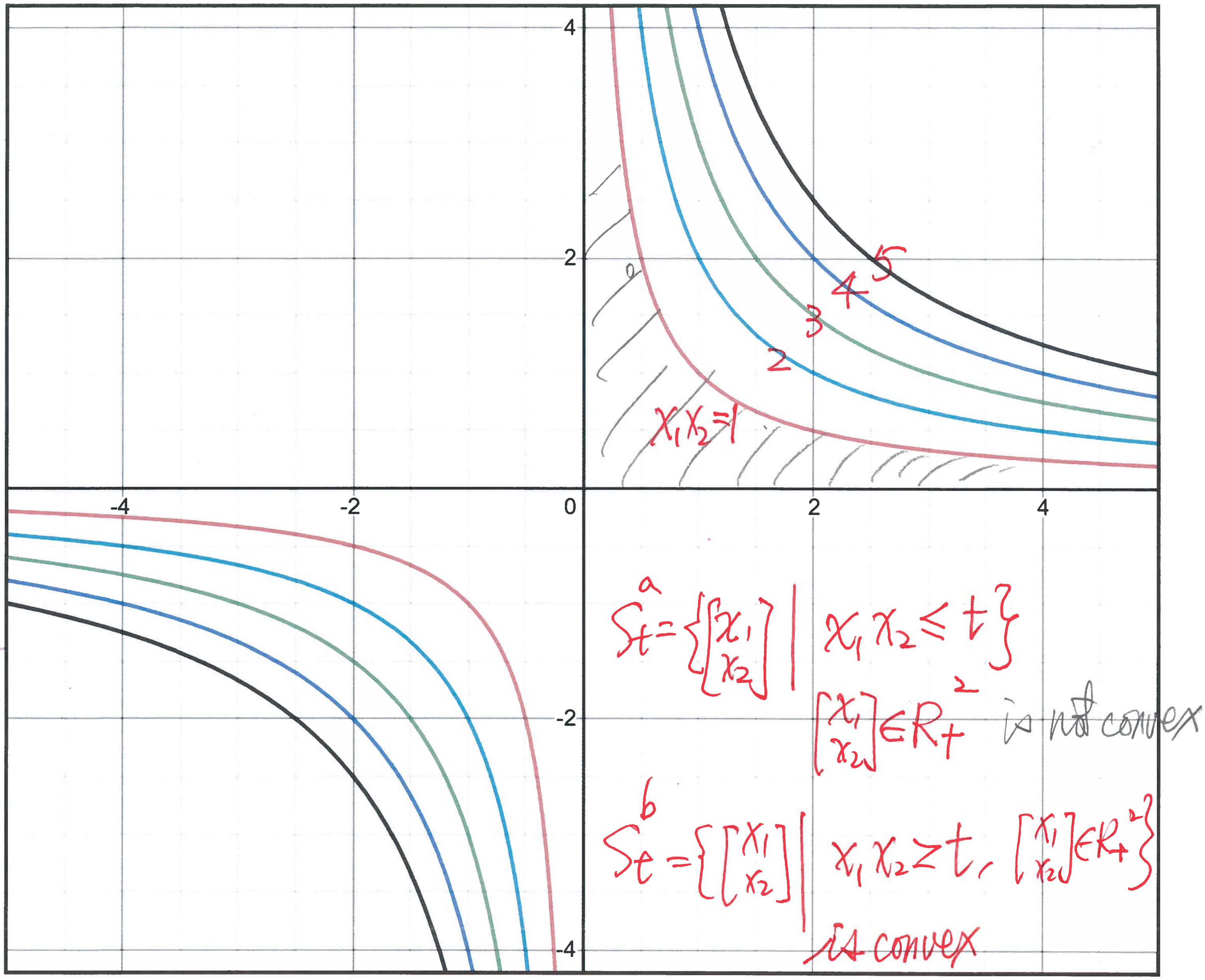




$S_t = \{x \mid f(x) \leq t\}$ quasi conex

$S_t = \{x \mid f(x) \geq t\} \Rightarrow$ " concave.

A15.1


4
2
-2
-4
-4
-2
0
2
4
5
4
3
2
$x_1 x_2 = 1$
$S_t^a = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \; x_1 x_2 \leq t, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in R_+^2 \right\}$ is not convex
$S_t^b = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \; x_1 x_2 \geq t, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in R_+^2 \right\}$ is convex

A15.2

1 $xy = 1$

2 $xy = 2$

3 $xy = 3$

4 $xy = 4$

5 $xy = 5$

3. Linear Programming: Format

General Form :

$$\min c^T x$$
$$s.t.\ \ Gx \leq h,\ \ G \in R^{m*n}, A \in R^{p*n}$$
$$Ax = b$$

Standard Form :

$$\min c^T x$$
$$s.t.\ \ Ax = b$$
$$x \geq 0$$

Remark: Figure out three possible situations

1. No feasible solutions
2. Unbounded solutions
3. Bounded solutions

3. Linear Programming: Cases

$\min c^T x$

$s.t.\ \ Ax = b$

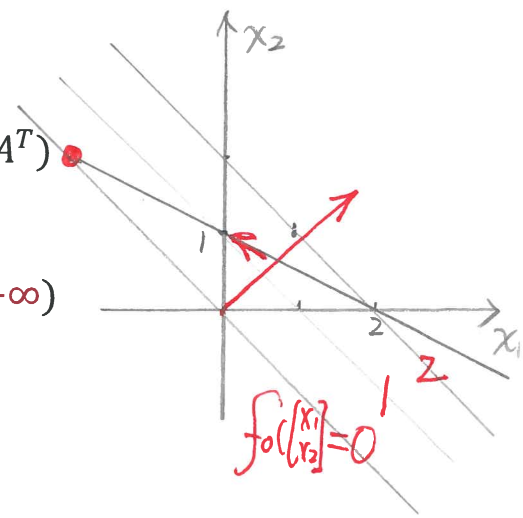
(1) No feasible solutions: $b \notin R(A)$ (b is not in the range of A)

e.g. $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

(2) Unbounded solutions: $b \in R(A)$ but $c \notin R(A^T)$

e.g. $\min \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2$ (The solution → $-\infty$)



(3) Bounded solutions: b$\in R(A), c \in R(A^T)$

e.g. $\min \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

Thus $x^* = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, f(x^*) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2$

3. Linear Fractional Programming

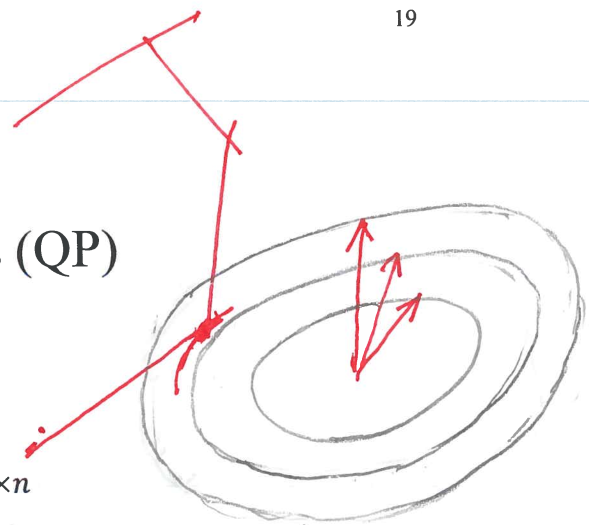
P1: $\min f_o(x) = \frac{c^T x + d}{e^T x + f}$, $dom\ f_o = \{x | e^T x + f > 0\}$

$$\begin{aligned} s.t.\ \ & Gx \preccurlyeq h \\ & Ax = b \end{aligned}$$

P1⇒P2: Let $y = \frac{x}{e^T x + f}$, $\quad z = \frac{1}{e^T x + f}$

P2: $\min c^T y + dz$

$$\begin{aligned} s.t.\ \ & Gy - hz \leq 0 \\ & Ay - bz = 0 \\ & e^T y + fz = 1 \\ & z \geq 0 \end{aligned}$$



4. Quadratic Opt. Problems (QP)

QP : $\min \frac{1}{2} x^T P x + q^T x + r$

$$\begin{aligned} s.t.\ \ & Gx \preccurlyeq h \\ & Ax = b \\ & P \in S_+^n,\ G \in R^{m \times n},\ A \in R^{p \times n} \end{aligned}$$

QCQP : (Quadratically Constrained Quadratic Program)

$$\min \frac{1}{2} x^T P_o x + q_o^T x + r_o$$

$$\begin{aligned} s.t.\ \ & \frac{1}{2} x^T P_i x + q_i^T x + r_i \leq 0, i = 1, \ldots, m \\ & Ax = b \\ & P_i \in S_+^n, i = 0,1, \ldots, m \end{aligned}$$

4. Quadratic Opt. Problems (SOCP)

SOCP : (Second-Order Cone Program)

QCQP ⊂ SOCP

$$\min f^T x$$
$$s.t.\ \ \left\|A_i x + b_i\right\|_2 \le c_i^T x + d_i, i = 1, \dots, m$$
$$Fx = g$$

$SOCP$: $(Ax + b, c^T x + d)$ lies in the second order cone

$$\{(y,t) \mid \|y\|_2 \le t, y \in R^k\}$$

QCQP viewed as SOCP

QCQP constraint: $x^T A^T A x + b^T x + c \le 0$

$\|x\|_2 = (\sum x_i^2)^{1/2}$

can be expressed as a SOCP constraint:

$$\left\| \begin{matrix} \frac{1 + b^T x + c}{2} \\ Ax \end{matrix} \right\|_2 \le (1 - b^T x - c)/2$$

$(Ax+b)^T(Ax+b) \le (c^Tx+d)^T(c^Tx$

$\boxed{\|x\|_2 < t}$

$x^TA^TAx + 2b^TAx + b^Tb$

$\le x^Tc^Tc^Tx + d^Td$

$+ 2d^Tc^Tx$

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4. Quadratic Opt. Problems (SOCP)

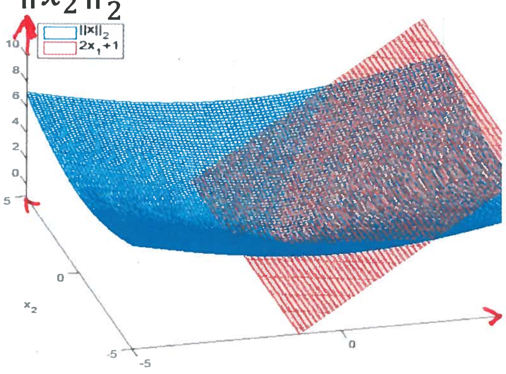
SOCP : (Second-Order Cone Program)

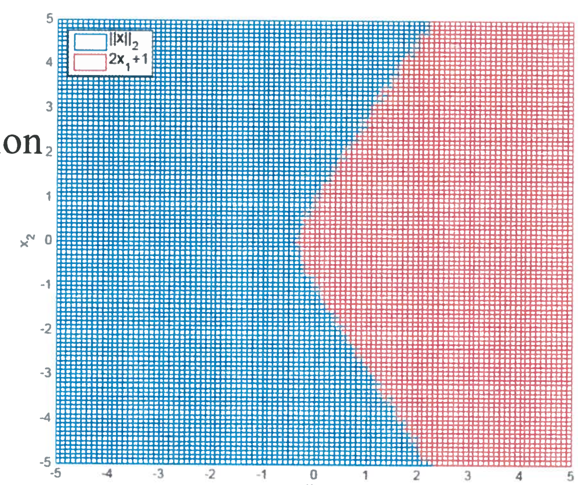
$$\min f^T x$$
$$s.t.\ \ \left\|A_i x + b_i\right\|_2 \le c_i^T x + d_i, i = 1, \dots, m$$
$$Fx = g$$

Example: SOCP constraint:

$\left\| \begin{matrix} x_1 \\ x_2 \end{matrix} \right\|_2 \le 2x_1 + 1$, feasible region

h





4. Quadratic Opt. Problems (SOCP)

SOCP : (Second-Order Cone Program)

$$\min f^T x$$
$$s.t. \quad \|A_i x + b_i\|_2 \le c_i^T x + d_i, i = 1, \dots, m$$
$$Fx = g$$

$SOCP$: $(Ax + b, c^T x + d)$ lies in the second order cone

$$\{(y,t) \mid \|y\|_2 \le t, y \in R^k\}$$

SOCP viewed as a Semidefinite Program Problem

SOCP constraint: $\|Ax + b\|_2 \le c^T x + d$

can be expressed as a Semidefinite Program constraint:

$$\begin{bmatrix} (c^T x + d)I & Ax + b \\ (Ax + b)^T & c^T x + d \end{bmatrix} \succcurlyeq 0$$

$$R^T \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} R = \begin{bmatrix} A & 0 \\ 0 & \underbrace{C - B^T A^{-1} B} \end{bmatrix} \ge 0$$

$$\uparrow \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix}$$

$$c^T x + d - (Ax+b)^T [(c^T x + d)I]^{-1} (Ax+b) \ge 0$$

$$(c^T x + d)^2 - (Ax+b)^T (Ax+b) \ge 0$$

5. Geometric Programming

$$f(x) = \sum_{k=1}^{\cancel{k}\,K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \dots x_n^{a_{nk}}, \qquad c_k > 0, a_{ik} \in R, x \in R_{++}^n$$

Each term is called monomial

$f(x)$ is called posynomial

Geometric Program:

$\min f_o(x)$

s.t.

$f_i(x) \le 1, i = 1, \dots, m$

$h_i(x) = 1, i = 1, \dots, p$

$x > 0$

f_is are posynomials

h_is are monomials