CSE203B Convex Optimization: Chapter 4: Problem Statement

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Convex Optimization Formulation

1. Introduction
   I. Eliminating equality constants
   II. Slack variables
   III. Absolute values, softmax
2. Optimality Conditions
   I. Local vs. global optimum
   II. Optimality criterion for differentiable f_0
      i. Optimization without constraints
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      iii. Opt. with equality constraints
   III. Quasi-convex optimization
3. Linear Optimization
4. Quadratic Optimization
5. Geometric Programming
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variations → Dual Problems
Same origonial problems

1. Introduction

Formulation: One of the most critical processes to conduct a project.

$$
\begin{aligned}
&\min\ f_0(x)\\
&s.t.\ f_i(x) \le 0 \ \ i = 1, \dots, m\\
&\qquad h_i(x) = 0\ i = 1, \dots, p\ (Ax = b \text{ Affine set})
\end{aligned}
$$

$x \in R^n$

$D_{f_0}\ f_0: R^n \to R$

$D_{f_i}\ f_i: R^n \to R$

$D_{h_i}\ h_i: R^n \to R$

$f_0, f_i, \dots, f_m$ *are convex*

$D = \cap_{i=0,m} D_f \cap_{i=0,p} D_{h_i}$ Domain of functions, but not the feasible set.

Feasible Set: The set which satisfies the constraints (is convex for convex problems).

1.1 Introduction: Eliminating Equality Constraints

$\{x \mid Ax \le b\} \Rightarrow \{U\theta \mid \mathbf{1}^T\theta = 1, \theta \ge 0\}$

$$
\begin{aligned}
&\min\ f_0(x)\\
&s.t.\ f_i(x) \le 0 \ \ i = 1, \dots, m\\
&\qquad Ax = b
\end{aligned}
$$

(note that this is an equality relation)

a. Convert $\{x \mid Ax = b\}$ *to* $\{Fz + x_0 \mid z \in R^k\}$ $Ax_0 = b$

b. We have a equivalent problem

$$
\begin{aligned}
&\min\ f_0(Fz + x_0)\\
&s.t.\ f_i(Fz + x_0) \le 0
\end{aligned}
$$

$h = Fz + x_0$

Remark: Matrix F contains columns of null space basis

elimination

$Az = 0$

$p \times n$

$$
\begin{array}{c c}
 & n \\
p & \\
n &
\end{array}
\begin{bmatrix} A \\ \hline I \end{bmatrix}_{n\times n}
\underset{n\times n}{P} = \left[\begin{array}{c|c} B & \\ \hline C & \end{array}\right] \to F
$$

$AP = B$

$IP = C \Rightarrow P = C \to AC = B$

1.2 Introduction: Slack Variables

$$\min \ f_0(x)$$
$$s.t. \ f_i(x) \leq 0, i = 1, \dots, m$$
$$Ax = b$$

Add slack variables to convert to an equivalent problem

a. Convert the objective function with variable t

$$\min \ t$$
$$s.t. f_0(x) - t \leq 0$$
$$f_i(x) \leq 0, i = 1, \dots, m$$
$$A^T x = b$$

b. Convert the inequality with variables s_i

$$\min \ f_0(x)$$
$$s.t. f_i(x) + s_i = 0$$
$$A^T x = b$$
$$s_i \in R_+, i = 1, \dots, m$$

1.3 Introduction: Absolute values and Softmax

a. Absolute values

$$|f_i(x)| \leq b$$
$$\Rightarrow f_i(x) \leq b \ and$$
$$-f_i(x) \leq b$$

b. Maximum values

$$\max\{f_1, f_2, \dots, f_m\}$$

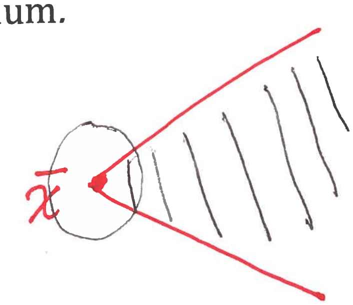
Softmax: $\frac{1}{\alpha}\log\left(e^{\alpha f_1} + e^{\alpha f_1} + \cdots + e^{\alpha f_m}\right)$

$Example$: $\max\{1, 5, 10, 2, 3\} \Rightarrow$ Softmax

$$\frac{1}{\alpha}\log(e^{\alpha} + e^{5\alpha} + e^{10\alpha} + e^{2\alpha} + e^{3\alpha}) \approx 10$$

2.1 Optimality Conditions: Local vs. Global Optima

Definition: Local Optima
Given a convex optimization problem and a point $\bar{x} \in R^n$
If there exists a $r > 0$
$s.t.\quad f_0(z) \geq f_0(\bar{x})$ for all $z \in$ Feasible Set, and $\|z - \bar{x}\|_2 \leq r$
Then $\bar{x}$ is a local optimum.



2.2 Optimality Conditions

Theorem: Given a convex opt. problem
If $\bar{x}$ is a local optimum, then $\bar{x}$ is a global optimum
Proof: By contradiction
Suppose that $\exists y \in \mathit{Feasible\ Set}$
$s.t.\quad f_0(\bar{x}) > f_0(y)$
We have $f_0(\bar{x}) > (1-\theta)f_0(\bar{x}) + \theta f_0(\bar{y})$ (*by assumption*) I
$\qquad\qquad > f_0((1-\theta)\bar{x} + \theta\bar{y})$ (f_0 *is convex*) II

$(1-\theta)+\theta$

And $(1-\theta)\bar{x} + \theta\bar{y}$ is feasible (Feasible set is convex)
The inequality contradicts to the assumption of local optima.

Null space

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad \begin{bmatrix} A \\ I \end{bmatrix} P = \begin{bmatrix} B \\ C \end{bmatrix}$$

$$\left[\begin{array}{cc} 1 & 2 \\ 2 & 4 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] P = \begin{bmatrix} 1 & \boxed{0} \\ 2 & \boxed{0} \\ 1 & \boxed{-2} \\ 0 & \boxed{1} \end{bmatrix} \qquad AC = B$$

$$A \begin{bmatrix} 1 & \boxed{-2} \\ 0 & \boxed{1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$$

$C_1 \quad C_2$

$C_2 - 2 \times C_1$

$\Rightarrow$ Null space $\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \theta \;\middle|\; \theta \in R \right\}$

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