5. Geometric programing in convex form

monomial $f(x) = cx_1^{a_1} \dots \cancel{x}_n^{a_n}, \quad x \in R_{++}^n$ Replace x_i with e^{y_i}

$\log f(e^{y_1}, \dots, e^{y_n}) = a^T y + b, \quad b = \log c$

$c e^{y_1 a_1} e^{y_2 a_2} \cdots e^{y_n a_n}$

polynomial $f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} \dots x_n^{a_{nk}}$

$\log f(e^{y_1} \dots e^{y_n}) = \log \sum_{k=1}^{K} e^{a_k^T y + b_k}, \quad b_k = \log c_k$

Geometric program transform

$$\min \log\left(\sum_{k=1}^{K_0} e^{a_{ok}^T y + b_{ok}}\right)$$

$$subject\ to\ \ \log \sum_{k=1}^{K_i} e^{a_{ik}^T y + b_{ik}} \le 0, \ i = 1, \dots, m$$

$$Gy + d = 0$$

Remark: The origonal problem may not be convex.

 W5B

6. Generalized Inequality Constraints

$$\min f_o(x)$$
$$s.t.\ \ f_i(x) \preccurlyeq_{K_i} 0$$
$$Ax = b$$
$$(x \preccurlyeq_K y \ \to y - x \in K)$$

Semidefinite Programming (SDP)

$$\min c^T x$$
$$s.t.\ \ x_1 F_1 + \cdots + x_n F_n + G \preccurlyeq 0$$

negative semidefinite.

$$Ax = b$$
$$G, F_1, \dots, F_n \in S^k, A \in R^{p \times n}$$

Standard Form SDP

$$\min tr(CX) = \sum_{ij} C_{ij} X_{ij}$$
$$s.t.\ \ tr(A_i X) = b_i, \ i = 1, \dots, p$$
$$X \succcurlyeq 0$$
$$C, A_1, \dots, A_p \in S^n, X \in S^n$$

Summary

(1). $LP \subset QP \subset QCQP \subset SOCP \subset SDP$
(2). Software Tools (Examples)
CVX: Matlab software for disciplined convex (Boyd)
CPLEX: IP, LP, QP, SOCP (IBM)
Gurobi: LP, QP, MILP, MIQP, MIQCP (Gu, Rothberg, Bixby)
(3). Check if the problem is convex

QC

$x^T Q_i x + P_i x \le b_i \quad Q_i \in S_+^{n \times n}$

$\| Ax + b \|_2 \le c^T x + d$

$x^T A^T A x + 2b^T A x + b^T b \le x^T c c^T x + d^T d + 2 d^T c^T x$

$x^T (A^T A - c c^T) x$

Summary

(1). Format of the formulation
  a. Follow the format of the solver (software package)
  b. Find equivalent formulation for simpler approaches
   (coding, complexity, accuracy)
(2). Feasibility of the solution
  Check if the feasible set is not empty.
(3). Boundness of the solution
  Check if the solution is bounded
  (reasonable, not $-\infty$)
(4). Optimality of the solution
  Check the supporting hyperplane of object function

CSE203B Convex Optimization: Chapter 5 Duality

Project: Problem Statement: Convexity, Duality

One key concept of the textbook

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Extension of dual cones and conjugate functions

Chapter 5 Duality

- Primal and Dual Problem Mechanism
  - Primal Problem
  - Lagrangian Function
  - Lagrange Dual Problem
- Examples (Primal Dual Conversion Procedure) Examples
  - Linear Programming
  - Quadratic Programming
  - Conjugate Functions (Duality)
  - Entropy Maximization
- Interpretation (Duality) Concepts + Theory
  - Saddle-Point Interpretation
  - Geometric Interpretation
  - Slater's Condition
  - Shadow-Price Interpretation
- KKT Conditions (Optimality Conditions) Optimality
- Sensitivity (Shadow-Price)
- Generalized Inequalities

Duality

<u>Primal Problem (Feasible Solution)</u>

$$\left.\begin{array}{ll} \min\ f_0(x) & x \in R^n \\ s.t.\ f_i(x) \le 0 & i = 1, \dots, m \\ \quad h_i(x) = 0 & i = 1, \dots, p \end{array}\right] \begin{array}{l} domain\ D \\ = dom\ f_0 \cap_i\ dom\ f_i\ \cap_i dom\ h_i \end{array}$$

Opt: $x^*, p^* = f_0(x^*)$ notation

Lagrangian: $L: R^n \times R^m \times R^p \to R$

$$L(x, \lambda, v) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} v_i h_i(x)$$

convert constraints to costs

λ_i, v_i: *Lagrange multiplier*, $\lambda_i, \in R_+, v_i \in R$.

Lagrange dual function or shadow price

$g(\lambda, v) = \inf_{x \in D} L(x, \lambda, v)$ (x may not be feasible)

Not relevant to x.

Duality

<u>Dual Problem (Infeasible Solution)</u>

$max_{\lambda,v}\ g(\lambda, v)\quad s.t. \lambda \succcurlyeq 0$

1. $g(\lambda, v)$ *is concave*
2. $g(\lambda, v) \le p^*$ an optimal value where $\lambda \succcurlyeq 0$

Proof 1: By definition of $g(\lambda, v)$ and the convexity of pointwise max operation on convex functions.

Proof 2: For any feasible $\tilde{x}$ and $\lambda \succcurlyeq 0$

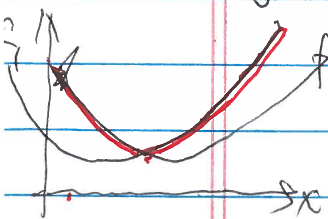
$f_0(\tilde{x}) \ge L(\tilde{x}, \lambda, v)$ (Because $\sum \lambda_i f_i(\tilde{x}) + \sum v_i h_i(\tilde{x}) \le 0$)

$L(\tilde{x}, \lambda, v) \ge g(\lambda, v)$ by definition of $g(\lambda, v)$

Thus $p^* = f_0(x^*) \ge g(\lambda, v)$

$f(x) = \max(f_1(x), f_2(x))$ If f_1 & f_2 are convex, then $f(x)$ is convex

$g(x) = \max_{y \in C} f(x,y)$ $f(x,y)$ is convex with respect to x then $g(x)$ is convex



$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i$$

$$g(\lambda, \nu) = \min_{x \in D} L(x, \lambda, \nu)$$

$$-g(\lambda, \nu) = \max_{x \in D} -L(x, \lambda, \nu)$$

$$= \max_{x \in D} \underbrace{-f_0(x) - \sum_{i=1}^{m} \lambda_i f_i(x) - \sum_{i=1}^{p} \nu_i h_i(x)}$$

An affine function of λ & ν with respect to each given x

Pointwise max operation on convex functions

$\Rightarrow -g(\lambda, \nu)$ is convex

$\Rightarrow g(\lambda, \nu)$ is concave.

Remark. the convexity of $-g(\lambda, \nu)$ is not relevant to the convexity of $L(x, \lambda, \nu)$

A4

(1). For any feasible $\tilde{x}$, $\lambda_i \geq 0$

$$f_0(\tilde{x}) \geq f_0(\tilde{x}) + \underbrace{\sum_{i=1}^{m} \lambda_i f_i(\tilde{x})}_{\leq 0} + \underbrace{\sum_{i=1}^{p} \nu_i h_i(\tilde{x})}_{0}$$

$$= L(\tilde{x}, \lambda, \nu)$$

(2). Feasible region $\subset$ Domain D

$$L(\tilde{x}, \lambda, \nu) \geq \min_{x \in D} L(x, \lambda, \nu) = g(\lambda, \nu)$$

From (1) & (2)

$$f_0(\tilde{x}) \geq g(\lambda, \nu) \quad \text{for any feasible } \tilde{x}$$

Thus, $p^* = f_0(x^*) \geq g(\lambda, \nu)$

To reduce the gap

$$\max_{\lambda \geq 0, \nu} g(\lambda, \nu)$$